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Approximation by de La Vallée Poussin type matrix transform of Vilenkin-Fourier series

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Abstract. In the present paper, we discuss the rate of the approximation by de La Vallée Poussin type matrix transform means of Vilenkin-Fourier series in $L_p(G_m)$ spaces $(1 \leq p < \infty)$ and in $C(G_m)$. Moreover, we present an application for functions in Lipschitz classes Lip (α, p, G_m) $(\alpha > 0,\ 1 \leq p < \infty)$ and Lip $(\alpha, C(G_m))$ $(\alpha > 0)$.

Keywords: Vilenkin group, Vilenkin system, Vilenkin-Fourier series, rate of approximation, modulus of continuity, Lipschitz function, de La Vallée Poussin summation, matrix transform means

AMS Subject Classification: 42C10

1. Introduction

First, we give a brief introduction to the theory of Vilenkin-Fourier analysis (for more details see [1, 27]). Denote by $\mathbb{P}$ the set of the positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$. Let $m := (m_0, m_1, \dots)$ be a sequence of the positive integers not less than 2. Denote by $\mathbb{Z}_{m_n} := \{0, 1, \dots, m_n - 1\}$ the additive group of integers modulo m_n for any $n \in \mathbb{N}$. Define the group G_m as the complete direct product of the groups $\mathbb{Z}_{m_n}$ with the product of the discrete topologies of $\mathbb{Z}_{m_n}$'s.

The direct product μ of the measures

$$\mu_n(\{j\}) := \frac{1}{m_n}$$

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is the Haar measure on G_m with $\mu(G_m) = 1$, for any $j \in \mathbb{Z}_{m_n}$.

If the sequence m is bounded, then G_m is called a bounded Vilenkin group, otherwise, it is called an unbounded one. In the case of $m = (2, 2, \dots)$ we get G_2 (or simply G), the so-called Walsh group. The elements of G_m are represented by sequences

$$x := (x_0, x_1, \dots, x_n, \dots),$$

where $x_n \in \mathbb{Z}_{m_n}$. It is easy to give a base for the neighbourhoods of G_m:

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}$$

if $x \in G_m$ and $n \in \mathbb{P}$. Let us denote $I_n := I_n(0)$ for $n \in \mathbb{N}$. We define the so-called generalized number system based on m in the following way:

$$M_0 := 1,\ M_{n+1} := m_n M_n$$

where $n \in \mathbb{N}$. Then every $n \in \mathbb{N}$ can be uniquely expressed as

$$n = \sum_{k=0}^{\infty} n_k M_k,$$

where $n_k \in \mathbb{Z}_{m_k}$ $(k \in \mathbb{N})$ and only a finite number of n_k's differ from zero. Let us denote by $|n|$ a natural number such that

$$M_{|n|} \leq n < M_{|n|+1}.$$

Let $L_p(G_m)$ denote the usual Lebesgue spaces on G_m with corresponding norms $\|.\|_p$ and $C(G_m)$ denote the space of continuous functions on G_m with the norm

$$\|f\|_\infty := \sup\{|f(x)| : x \in G_m\}.$$

Next, we define the modulus of continuity in $L_p(G_m)$ for $1 \leq p < \infty$ of a function $f \in L_p(G_m)$ by

$$\omega_p(f, \delta) := \sup_{|x|<\delta} \|F(\cdot, x)\|_p, \quad \delta > 0,$$

with the notation

$$|x| := \sum_{i=0}^{\infty} \frac{x_i}{M_{i+1}} \quad \text{for all } x \in G_m$$

and

$$F(x, u) := f(x - u) - f(x).$$

Analogously, we can define the modulus of continuity in $C(G_m)$, it is denoted by $\omega_\infty(f, \delta)$.

The Lipschitz classes in $L_p(G_m)$ for each $\alpha > 0$ are defined by

$$\mathrm{Lip}(\alpha, p, G_m) := \{f \in L_p(G_m) : \omega_p(f, \delta) = O(\delta^\alpha) \text{ as } \delta \to 0\}.$$

Moreover,

$$\mathrm{Lip}(\alpha, C(G_m)) := \{f \in C(G_m) : \ |F(x,y)| \leq C|y|^{\alpha}, \ x, y \in G_m\}.$$

Further, for the simplicity we write $\mathrm{Lip}(\alpha, \infty, G_m) := \mathrm{Lip}(\alpha, C(G_m))$.

Next, we introduce on G_m an orthonormal system which is called Vilenkin system. At first, we define the complex-valued functions $r_k(x)\colon G_m \to \mathbb{C}$, the generalized Rademacher functions, by

$$r_k(x) := \exp(2\pi \imath x_k/m_k),$$

where $\imath^2 = -1, \ x \in G_m$ and $k \in \mathbb{N}$.

Let us define the Vilenkin system $\varphi := \{\varphi_n : n \in \mathbb{N}\}$ on G_m as

$$\varphi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x).$$

Specifically, we call this system the Walsh-Paley system, when $m = (2, 2, \dots)$. The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see [31]). Exactly, the elements of the Vilenkin system are the characters of G_m. Namely, $f\colon G_m \to \mathbb{C}$ continuous,

$$f(x+y) = f(x)f(y)$$

and $|f(x)| = 1$ for all $x, y \in G_m$. Moreover, it holds if and only if $f(x) = \varphi_n(x)$ for some $n \in \mathbb{N}$ (see [27]).

Let $\mathcal{P}_n$ be the collection of Vilenkin polynomials of order less than n, that is, functions of the form

$$P(x) = \sum_{k=0}^{n-1} a_k \varphi_k(x),$$

where $n \in \mathbb{P}$ and $\{a_n : n \in \mathbb{N}\}$ is a sequence of complex numbers and let

$$\mathcal{P} := \cup_{n=1}^{\infty} \mathcal{P}_n.$$

The kth Vilenkin-Fourier-coefficient, the ith partial sum of the Vilenkin-Fourier series, the nth Vilenkin-Fejér mean and the nth Vilenkin-Dirichlet kernel is defined by

$$\hat{f}(k) := \int_{G_m} f \bar{\varphi}_k d\mu, \ S_i(f) := \sum_{k=0}^{i-1} \hat{f}(k)\varphi_k, \ \sigma_n(f) := \frac{1}{n}\sum_{i=1}^{n} S_i(f),$$

$$D_n := \sum_{k=0}^{n-1} \varphi_k, \ D_0 := 0$$

where $n, i \in \mathbb{P}$ and $k \in \mathbb{N}$. The M_kth Dirichlet kernel for any $k \in \mathbb{N}$ has a closed form

$$D_{M_k}(x) = \begin{cases} 0, & \text{if } x \notin I_k, \\ M_k, & \text{if } x \in I_k. \end{cases} \tag{1.1}$$

Fejér kernels are defined as the arithmetical means of Dirichlet kernels, that is,

$$K_n := \frac{1}{n}\sum_{k=1}^{n} D_k.$$

It is well-known [1] that if G_m is a bounded Vilenkin group, the $L_1(G_m)$ norm of the Fejér kernels are uniformly bounded. Namely, there exists a positive constant c such that

$$\int_{G_m} |K_n(x)| d\mu(x) \leq c \tag{1.2}$$

holds for any $n \in \mathbb{P}$.

Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-negative numbers. The nth Nörlund mean of the Vilenkin-Fourier series is defined by

$$\frac{1}{Q_n}\sum_{k=1}^{n} q_{n-k} S_k(f; x),$$

where $Q_n := \sum_{k=0}^{n-1} q_k$, where $n \in \mathbb{P}$. It is always assumed that $q_0 > 0$ and

$$\lim_{n\to\infty} Q_n = \infty.$$

In this case, the summability method generated by $\{q_k : k \in \mathbb{N}\}$ is regular (see [24, 35]) if and only if

$$\lim_{n\to\infty} \frac{q_{n-1}}{Q_n} = 0.$$

In paper [24], the rate of the approximation by Nörlund means of Walsh-Fourier series of a function f in $L_p(G_2)$ and in $C(G_2)$ (in particular, in $\mathrm{Lip}(\alpha, p, G_2)$, where $\alpha > 0$ and $1 \leq p \leq \infty$) was studied. As special cases Móricz and Siddiqi obtained the earlier results given by Yano [34], Jastrebova [20] and Skvortsov [29] on the rate of the approximation by Cesàro means. The approximation properties of the Walsh-Cesàro means of negative order were studied by Goginava [17], Vilenkin case was investigated by Shavardenidze [28] and Tepnadze [30]. In 2008, Fridli, Manchanda and Siddiqi generalized the result of Móricz and Siddiqi for homogeneous Banach spaces and dyadic Hardy spaces [16]. Recently, Baramidze, Gát, Goginava, Memić, K. Nagy, Persson, Tephnadze, Wall and the author presented some results with respect to this topic [3, 5, 11, 18, 22]. See [15, 33], as well. For the two-dimensional results see [8, 10, 25, 26].

Let $\{p_k : k \in \mathbb{P}\}$ be a sequence of non-negative numbers. The nth weighted mean of Vilenkin-Fourier series is defined by

$$\frac{1}{P_n}\sum_{k=1}^{n} p_k S_k(f; x),$$

where $P_n := \sum_{k=1}^{n} p_k$, where $n \in \mathbb{P}$. It is always assumed that $p_1 > 0$ and

$$\lim_{n\to\infty} P_n = \infty,$$

which is the condition for regularity.

In [23] the authors studied the rate of the approximation by weighted means of Walsh-Fourier series of a function in $L_p(G_2)$ and in $C(G_2)$ (in particular, in $\mathrm{Lip}(\alpha, p, G_2)$, where $\alpha > 0$ and $1 \leq p \leq \infty$). As special cases Móricz and Rhoades obtained the earlier results given by Yano [34], Jastrebova [20] on the rate of the approximation by Walsh-Cesàro means. A common generalization of this two results of Móricz and Siddiqi [24] and Móricz and Rhoades [23] was given by K. Nagy and the author in the paper [7].

In particular cases weighted and Nörlund means are the Vilenkin-Fejér means (for all k set $p_k = q_k = 1$).

Let $T := (t_{i,j})_{i,j=1}^{\infty}$ be a doubly infinite matrix of numbers. It is always supposed that matrix T is upper triangular. Let us define the nth de La Vallée Poussin type matrix transform mean determined by the matrix T

$$\sigma_{\tilde{n},n}^{T}(f;x) := \sum_{k=\tilde{n}}^{n} t_{k,n} S_k(f;x),$$

where $S_k(f;x)$ denotes the kth partial sums of the Vilenkin-Fourier series of f. For matrix transform method the conditions of regularity can be found in Zygmund's book [35, p. 74].

Since the nth row of the matrix T determines the linear mean $\sigma_{\tilde{n},n}^{T}$ and its definition contains only finite number of entries, for the simplicity we say $\{t_{k,n} : \tilde{n} \leq k \leq n,\ k \in \mathbb{P}\}$ is a finite sequence of numbers for each $n \in \mathbb{P}$.

In the further part of this paper, let $\{t_{k,n} : \tilde{n} \leq k \leq n,\ k \in \mathbb{P}\}$ be a finite sequence of non-negative numbers for each $n \in \mathbb{P}$. The $(\tilde{n}, n)$th matrix transform kernel is defined by

$$K_{\tilde{n},n}^{T}(x) := \sum_{k=\tilde{n}}^{n} t_{k,n} D_k(x).$$

It is easily seen that

$$\sigma_{\tilde{n},n}^{T}(f;x) = \int_{G_m} f(u) K_{\tilde{n},n}^{T}(x-u)\, d\mu(u).$$

It follows by simple consideration that the Nörlund means and weighted means are matrix transform means.

For matrix transforms means with respect to the trigonometric system see e.g. results of Chandra [13] and Leindler [21], to Walsh system see paper of Blyumin [12].

This paper is motivated by the work of Móricz and Siddiqi [24] on Walsh-Nörlund mean method and the result of Móricz and Rhoades [23] on Walsh weighted mean method.

Our main aim is to investigate the rate of the approximation of de La Vallée Poussin type matrix transform means in terms of modulus of continuity under some general conditions.

The main theorem (Theorem 3.3) gives a kind of common generalization of the two results of Móricz and Siddiqi on Nörlund means [24] and Móricz and Rhoades on weighted means [23]. Moreover, we generalized the system, as well. At the end, we present an application for Lipschitz functions.

Other aspects of these methods regarding Walsh-Fourier series are treated in the papers [15, 33]. Avdispahić and Pepić proved some results also for Vilenkin-system in the paper [2].

We mention, that Iofina and Volosivets obtained similar results on Vilenkin systems with similar assumptions using different methods (independently form technics of Móricz, Rhoades, Siddiqi, Fridli and others) with respect to matrix transform means in [19] and Volosivets in [32]. They used in their proof the definition of the best approximation and the Watari-Efimov inequality. We used it only for the trivial case (if $1 < p < \infty$).

De La Vallée Poussin type matrix transform means for Walsh system were introduced Gát and the author in [5]. In their paper, Blahota and Gát proved similar theorems as in this one, but for the Walsh system. See also [4] and [6]. In the latter article, the authors dealt with Vilenkin systems, but in special cases.

2. Auxiliary results

Lemma 2.1. *Let us set $j, k, l \in \mathbb{N}$, $0 \le k < M_j$ and $0 < l \le m_j$. Then*

$$D_{lM_j-k} = \sum_{s=0}^{l-1} r_j^s D_{M_j} - \varphi_{lM_j-1}\bar{D}_k.$$

Proof. By definition

$$\begin{aligned} D_{lM_j-k} &= \sum_{h=0}^{lM_j-k-1} \varphi_h = \sum_{h=0}^{lM_j-1} \varphi_h - \sum_{h=lM_j-k}^{lM_j-1} \varphi_h \\ &= \sum_{s=0}^{l-1} \sum_{i=0}^{M_j-1} \varphi_{sM_j+i} - \sum_{i=0}^{k-1} \varphi_{lM_j-i-1}. \end{aligned}$$

Since easy to see that $\varphi_{sM_j+i} = \varphi_{sM_j}\varphi_i$ for any $s \in \{1, \dots, m_j - 1\}$, $j \in \mathbb{N}$, where $i \in \{0, \dots, M_j - 1\}$ and $\varphi_{sM_j} = r_j^s$, so we get

$$\sum_{i=0}^{M_j-1} \varphi_{sM_j+i} = r_j^s D_{M_j}.$$

On the other hand, $\varphi_{lM_j-i-1} = \varphi_{lM_j-1}\bar{\varphi}_i$ for any $l \in \{1, \dots, m_j\}$, $j \in \mathbb{N}$, where $i \in \{0, \dots, M_j - 1\}$. It follows

$$\sum_{i=0}^{k-1} \varphi_{lM_j-i-1} = \varphi_{lM_j-1}\bar{D}_k.$$

□

Lemma 2.2 (Blahota and K. Nagy [9]). *Let us set $j,k,l \in \mathbb{N}$, $0 \le k < M_j$ and $0 \le l < m_j$. Then*

$$D_{lM_j+k} = \sum_{s=0}^{l-1} r_j^s D_{M_j} + r_j^l D_k.$$

Lemma 2.3 (Blahota and K. Nagy [9]). *Let $n,q \in \mathbb{P}$, $1 \le q \le m_n - 1$ and $g \in \mathcal{P}_{M_n}$, $f \in L_p(G_m)$, where $1 \le p < \infty$ or $f \in C(G_m)$. Then*

$$\left\| \int_{G_m} r_n^q(u) g(u) F(\cdot,u) d\mu(u) \right\|_p \le m_n \|g\|_1 \omega_p\left(f, \frac{1}{M_n}\right)$$

holds (for $f \in C(G_m)$ we change p by ∞).

Corollary 2.4. *Let $n \in \mathbb{P}$ and $g \in \mathcal{P}_{M_n}$, $f \in L_p(G_m)$, where $1 \le p < \infty$ or $f \in C(G_m)$. Then*

$$\left\| \int_{G_m} \varphi_{M_{n+1}-1}(u) g(u) F(\cdot,u) d\mu(u) \right\|_p \le m_n \|g\|_1 \omega_p\left(f, \frac{1}{M_n}\right)$$

holds (for $f \in C(G_m)$ we change p by ∞).

Proof. This statement is a simple consequence of Lemma 2.3. Since

$$\varphi_{M_{n+1}-1} = \varphi_{(m_n-1)M_n} \varphi_{M_n-1} = r_n^{m_n-1} \varphi_{M_n-1},$$

where $\varphi_{M_n-1} \in \mathcal{P}_n$, so $g' := \varphi_{M_n-1} g \in \mathcal{P}_n$ also. Then we can use Lemma 2.3 for this g' by choosing $q := m_n - 1$. Furthermore it is obvious, that $\|g'\|_1 = \|g\|_1$. □

We introduce the notation $\Delta t_{k,n} := t_{k,n} - t_{k+1,n}$, where $k \in \{1,\dots,n\}$ and $t_{n+1,n} := 0$. In the next Lemma, we give a decomposition of the kernels $K^T_{\tilde{n},n}$.

Lemma 2.5. *Let $\tilde{n}, n \in \mathbb{P}$ and suppose that $|\tilde{n}| < |n|$. Then we have*

$$\begin{aligned} K^T_{\tilde{n},n} = & \sum_{k=1}^{M_{|\tilde{n}|+1}-\tilde{n}} t_{M_{|\tilde{n}|+1}-k,n} D_{M_{|\tilde{n}|+1}} - t_{\tilde{n},n} \varphi_{M_{|\tilde{n}|+1}-1} (M_{|\tilde{n}|+1} - \tilde{n}) \bar{K}_{M_{|\tilde{n}|+1}-\tilde{n}} \\ & + \varphi_{M_{|\tilde{n}|+1}-1} \sum_{k=1}^{M_{|\tilde{n}|+1}-\tilde{n}-1} \Delta t_{M_{|\tilde{n}|+1}-k-1,n} k \bar{K}_k \\ & + \sum_{j=|\tilde{n}|+1}^{|n|-1} \sum_{l=1}^{m_j-1} \sum_{s=1}^{l-1} \sum_{k=0}^{M_j-1} t_{lM_j+k,n} r_j^s D_{M_j} + \sum_{j=|\tilde{n}|+1}^{|n|-1} \sum_{s=M_j}^{M_{j+1}-1} t_{s,n} D_{M_j} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=|\tilde{n}|+1}^{|n|-1}\sum_{l=1}^{m_j-1}\sum_{k=1}^{M_j-2}\Delta t_{lM_j+k,n}kr_j^lK_k\\
&+\sum_{j=|\tilde{n}|+1}^{|n|-1}\sum_{l=1}^{m_j-1}t_{(l+1)M_j-1,n}(M_j-1)r_j^lK_{M_j-1}\\
&+\sum_{l=1}^{n_{|n|}-1}\sum_{s=1}^{l-1}\sum_{k=0}^{M_{|n|}-1}t_{lM_{|n|}+k,n}r_{|n|}^sD_{M_{|n|}}+\sum_{l=1}^{n_{|n|}-1}\sum_{k=1}^{M_{|n|}-2}\Delta t_{lM_{|n|}+k,n}kr_{|n|}^lK_k\\
&+\sum_{l=1}^{n_{|n|}-1}t_{(l+1)M_{|n|}-1,n}(M_{|n|}-1)r_{|n|}^lK_{M_{|n|}-1}\\
&+\sum_{k=0}^{n-n_{|n|}M_{|n|}}\sum_{s=1}^{n_{|n|}-1}t_{n_{|n|}M_{|n|}+k,n}r_{|n|}^sD_{M_{|n|}}\\
&+\sum_{k=1}^{n-n_{|n|}M_{|n|}}\Delta t_{n_{|n|}M_{|n|}+k,n}kr_{|n|}^{n_{|n|}}K_k+\sum_{l=M_{|n|}}^{n}t_{l,n}D_{M_{|n|}}=:\sum_{i=1}^{13}K_{i,\tilde{n},n}.
\end{aligned}
$$

Proof. We write

$$
\begin{aligned}
K_{\tilde{n},n}^T&=\sum_{l=\tilde{n}}^{M_{|\tilde{n}|+1}-1}t_{l,n}D_l+\sum_{j=|\tilde{n}|+1}^{|n|-1}\sum_{l=M_j}^{M_{j+1}-1}t_{l,n}D_l+\sum_{l=M_{|n|}}^{n}t_{l,n}D_l\\
&=:K_{\tilde{n},n}^A+K_{\tilde{n},n}^B+K_n^C.
\end{aligned}
$$

Now, we apply Lemma 2.1 and Lemma 2.2 for these expressions. We get

$$
\begin{aligned}
K_{\tilde{n},n}^A&=\sum_{k=1}^{M_{|\tilde{n}|+1}-\tilde{n}}t_{M_{|\tilde{n}|+1}-k,n}D_{M_{|\tilde{n}|+1}-k}\\
&=\sum_{k=1}^{M_{|\tilde{n}|+1}-\tilde{n}}t_{M_{|\tilde{n}|+1}-k,n}D_{M_{|\tilde{n}|+1}}-\sum_{k=1}^{M_{|\tilde{n}|+1}-\tilde{n}}t_{M_{|\tilde{n}|+1}-k,n}\varphi_{M_{|\tilde{n}|+1}-1}\bar{D}_k\\
&=:\sum_{i=1}^{2}K_{\tilde{n},n}^{A,i},
\end{aligned}
$$

$$
\begin{aligned}
K_{\tilde{n},n}^B&=\sum_{j=|\tilde{n}|+1}^{|n|-1}\sum_{l=1}^{m_j-1}\sum_{k=0}^{M_j-1}t_{lM_j+k,n}D_{lM_j+k}\\
&=\sum_{j=|\tilde{n}|+1}^{|n|-1}\sum_{l=1}^{m_j-1}\sum_{k=0}^{M_j-1}t_{lM_j+k,n}\sum_{s=0}^{l-1}r_j^sD_{M_j}+\sum_{j=|\tilde{n}|+1}^{|n|-1}\sum_{l=1}^{m_j-1}\sum_{k=0}^{M_j-1}t_{lM_j+k,n}r_j^lD_k
\end{aligned}
$$

$$= \sum_{j=|\tilde{n}|+1}^{|n|-1} \sum_{l=1}^{m_j-1} \sum_{s=1}^{l-1} \sum_{k=0}^{M_j-1} t_{lM_j+k,n} r_j^s D_{M_j} + \sum_{j=|\tilde{n}|+1}^{|n|-1} \sum_{s=M_j}^{M_{j+1}-1} t_{s,n} D_{M_j}$$
$$+ \sum_{j=|\tilde{n}|+1}^{|n|-1} \sum_{l=1}^{m_j-1} r_j^l \sum_{k=0}^{M_j-1} t_{lM_j+k,n} D_k =: \sum_{i=1}^{3} K_{\tilde{n},n}^{B,i}$$

and

$$K_n^C = \sum_{k=0}^{n-M_{|n|}} t_{M_{|n|}+k,n} D_{M_{|n|}+k}$$
$$= \sum_{l=1}^{n_{|n|}-1} \sum_{k=0}^{M_{|n|}-1} t_{lM_{|n|}+k,n} D_{lM_{|n|}+k} + \sum_{k=0}^{n-n_{|n|}M_{|n|}} t_{n_{|n|}M_{|n|}+k,n} D_{n_{|n|}M_{|n|}+k}$$
$$= \sum_{l=1}^{n_{|n|}-1} \sum_{k=0}^{M_{|n|}-1} t_{lM_{|n|}+k,n} \sum_{s=0}^{l-1} r_{|n|}^s D_{M_{|n|}} + \sum_{l=1}^{n_{|n|}-1} \sum_{k=1}^{M_{|n|}-1} t_{lM_{|n|}+k,n} r_{|n|}^l D_k$$
$$+ \sum_{k=0}^{n-n_{|n|}M_{|n|}} t_{n_{|n|}M_{|n|}+k,n} \sum_{s=0}^{n_{|n|}-1} r_{|n|}^s D_{M_{|n|}} + r_{|n|}^{n_{|n|}} \sum_{k=1}^{n-n_{|n|}M_{|n|}} t_{n_{|n|}M_{|n|}+k,n} D_k$$
$$= \sum_{l=1}^{n_{|n|}-1} \sum_{s=1}^{l-1} \sum_{k=0}^{M_{|n|}-1} t_{lM_{|n|}+k,n} r_{|n|}^s D_{M_{|n|}} + \sum_{l=1}^{n_{|n|}-1} \sum_{k=1}^{M_{|n|}-1} t_{lM_{|n|}+k,n} r_{|n|}^l D_k$$
$$+ \sum_{k=0}^{n-n_{|n|}M_{|n|}} \sum_{s=1}^{n_{|n|}-1} t_{n_{|n|}M_{|n|}+k,n} r_{|n|}^s D_{M_{|n|}} + r_{|n|}^{n_{|n|}} \sum_{k=1}^{n-n_{|n|}M_{|n|}} t_{n_{|n|}M_{|n|}+k,n} D_k$$
$$+ \sum_{l=M_{|n|}}^{n} t_{l,n} D_{M_{|n|}} =: \sum_{i=1}^{5} K_n^{C,i}.$$

Now, we use Abel's transform for the expressions $K_{\tilde{n},n}^{A,2}, K_{\tilde{n},n}^{B,3}, K_n^{C,2}$ and $K_n^{C,4}$. We have

$$K_{\tilde{n},n}^{A,2} = - \sum_{k=1}^{M_{|\tilde{n}|+1}-\tilde{n}} t_{M_{|\tilde{n}|+1}-k,n} \varphi_{M_{|\tilde{n}|+1}-1} \bar{D}_k$$
$$= - t_{\tilde{n},n} \varphi_{M_{|\tilde{n}|+1}-1} (M_{|\tilde{n}|+1} - \tilde{n}) \bar{K}_{M_{|\tilde{n}|+1}-\tilde{n}}$$
$$+ \varphi_{M_{|\tilde{n}|+1}-1} \sum_{k=1}^{M_{|\tilde{n}|+1}-\tilde{n}-1} \Delta t_{M_{|\tilde{n}|+1}-k-1,n} k \bar{K}_k,$$

$$K_{\tilde{n},n}^{B,3} = \sum_{j=|\tilde{n}|+1}^{|n|-1} \sum_{l=1}^{m_j-1} \sum_{k=1}^{M_j-1} t_{lM_j+k,n} r_j^l D_k$$

$$= \sum_{j=|\tilde{n}|+1}^{|n|-1} \sum_{l=1}^{m_j-1} \left(\sum_{k=1}^{M_j-2} \Delta t_{lM_j+k,n} k r_j^l K_k + t_{(l+1)M_j-1,n}(M_j-1) r_j^l K_{M_j-1} \right),$$

$$\begin{aligned} K_n^{C,2} &= \sum_{l=1}^{n_{|n|}-1} \sum_{k=1}^{M_{|n|}-1} t_{lM_{|n|}+k,n} r_{|n|}^l D_k \\ &= \sum_{l=1}^{n_{|n|}-1} \sum_{k=1}^{M_{|n|}-2} \Delta t_{lM_{|n|}+k,n} k r_{|n|}^l K_k \\ &\quad + \sum_{l=1}^{n_{|n|}-1} t_{(l+1)M_{|n|}-1,n}(M_{|n|}-1) r_{|n|}^l K_{M_{|n|}-1}. \end{aligned}$$

Using Abel-transformation and equality $t_{n+1,n} = 0$ we get that

$$\begin{aligned} K_n^{C,4} &= \sum_{k=1}^{n-n_{|n|}M_{|n|}} t_{n_{|n|}M_{|n|}+k,n} r_{|n|}^{n_{|n|}} D_k \\ &= \sum_{k=1}^{n-n_{|n|}M_{|n|}-1} \Delta t_{n_{|n|}M_{|n|}+k,n} k r_{|n|}^{n_{|n|}} K_k \\ &\quad + t_{n,n}(n - n_{|n|}M_{|n|}) r_{|n|}^{n_{|n|}} K_{n-n_{|n|}M_{|n|}} \\ &= \sum_{k=1}^{n-n_{|n|}M_{|n|}} \Delta t_{n_{|n|}M_{|n|}+k,n} k r_{|n|}^{n_{|n|}} K_k. \end{aligned}$$

It completes the proof of Lemma 2.5. □

Lemma 2.6. *Let $\tilde{n}, n \in \mathbb{P}$ and suppose that $\tilde{n} < n$, but $|\tilde{n}| = |n|$. Then we have*

$$\begin{aligned} K_{\tilde{n},n}^T = &\sum_{k=M_{|n|+1}-n}^{M_{|n|+1}-\tilde{n}} t_{M_{|n|+1}-k,n} D_{M_{|n|+1}} - t_{\tilde{n},n} \varphi_{M_{|n|+1}-1}(M_{|n|+1}-\tilde{n}) \bar{K}_{M_{|n|+1}-\tilde{n}} \\ &+ \varphi_{M_{|n|+1}-1} \sum_{k=M_{|n|+1}-n-1}^{M_{|n|+1}-\tilde{n}-1} \Delta t_{M_{|n|+1}-k-1,n} k \bar{K}_k. \end{aligned}$$

Proof. We write

$$K_{\tilde{n},n}^T = \sum_{l=\tilde{n}}^{n} t_{l,n} D_l = \sum_{l=\tilde{n}}^{M_{|n|+1}-1} t_{l,n} D_l,$$

where $t_{l,n} = 0$, if $l \in \{n+1, \ldots, M_{|n|+1}-1\}$. Using Lemma 2.1 for this expression we get

$$K_{\tilde{n},n}^T = \sum_{k=1}^{M_{|n|+1}-\tilde{n}} t_{M_{|n|+1}-k,n} D_{M_{|n|+1}-k}$$

$$= \sum_{k=1}^{M_{|n|+1}-\tilde{n}} t_{M_{|n|+1}-k,n} D_{M_{|n|+1}} - \sum_{k=1}^{M_{|n|+1}-\tilde{n}} t_{M_{|n|+1}-k,n} \varphi_{M_{|n|+1}-1} \bar{D}_k$$
$$= \sum_{k=M_{|n|+1}-n}^{M_{|n|+1}-\tilde{n}} t_{M_{|n|+1}-k,n} D_{M_{|n|+1}} - \sum_{k=1}^{M_{|n|+1}-\tilde{n}} t_{M_{|n|+1}-k,n} \varphi_{M_{|n|+1}-1} \bar{D}_k.$$

Now, we use Abel's transform

$$\begin{aligned} - \sum_{k=1}^{M_{|n|+1}-\tilde{n}} & t_{M_{|n|+1}-k,n} \varphi_{M_{|n|+1}-1} \bar{D}_k \\ &= - t_{\tilde{n},n} \varphi_{M_{|n|+1}-1} (M_{|n|+1} - \tilde{n}) \bar{K}_{M_{|n|+1}-\tilde{n}} \\ &\quad + \varphi_{M_{|n|+1}-1} \sum_{k=1}^{M_{|n|+1}-\tilde{n}-1} \Delta t_{M_{|n|+1}-k-1,n} k \bar{K}_k \\ &= - t_{\tilde{n},n} \varphi_{M_{|n|+1}-1} (M_{|n|+1} - \tilde{n}) \bar{K}_{M_{|n|+1}-\tilde{n}} \\ &\quad + \varphi_{M_{|n|+1}-1} \sum_{k=M_{|n|+1}-n-1}^{M_{|n|+1}-\tilde{n}-1} \Delta t_{M_{|n|+1}-k-1,n} k \bar{K}_k. \end{aligned}$$

Recall that $t_{n+1,n} = 0$. □

From now, we discuss bounded Vilenkin groups, i.e. the case when

$$\sup_{n\in\mathbb{N}} m_n < c.$$

Throughout our article c denotes a positive absolute constant, which may vary at different appearances.

Lemma 2.7. *Let $\tilde{n}, n \in \mathbb{P}$ and the members of finite sequence $\{t_{k,n} : \tilde{n} \le k \le n\}$ be non-negative numbers. We suppose that*

$$\sum_{k=\tilde{n}}^{n} t_{k,n} = 1. \tag{2.1}$$

If the finite sequence $\{t_{k,n} : \tilde{n} \le k \le n\}$ is non-decreasing for all n, then we suppose

$$t_{n,n} = O\left(\frac{1}{n}\right)$$

and if the finite sequence $\{t_{k,n} : \tilde{n} \le k \le n\}$ is non-increasing for all n, then we suppose

$$t_{\tilde{n},n} = O\left(\frac{1}{\tilde{n}}\right).$$

Then in both case Inequality

$$\left\|K^T_{\tilde{n},n}\right\|_1 \le c \tag{2.2}$$

holds.

Proof. Using Abel transformation we get

$$K_{\tilde{n},n}^{T}=\sum_{k=\tilde{n}}^{n-1}\Delta t_{k,n}kK_{k}+t_{n,n}nK_{n}-t_{\tilde{n},n}(\tilde{n}-1)K_{\tilde{n}-1}.$$

Applying Inequality (1.2) we obtain

$$\begin{aligned}\left\|K_{\tilde{n},n}^{T}\right\|_{1}&\leq\sum_{k=\tilde{n}}^{n-1}|\Delta t_{k,n}|k\|K_{k}\|_{1}+t_{n,n}n\|K_{n}\|_{1}+t_{\tilde{n},n}(\tilde{n}-1)\|K_{\tilde{n}-1}\|_{1}\\&\leq c\left(\sum_{k=\tilde{n}}^{n-1}|\Delta t_{k,n}|k+t_{n,n}n+t_{\tilde{n},n}(\tilde{n}-1)\right).\end{aligned}$$

In non-decreasing case

$$\begin{aligned}\left\|K_{\tilde{n},n}^{T}\right\|_{1}&\leq c\left(-\sum_{k=\tilde{n}}^{n}t_{k,n}-t_{\tilde{n},n}(\tilde{n}-1)+t_{n,n}n+t_{n,n}n+t_{\tilde{n},n}(\tilde{n}-1)\right)\\&\leq c2t_{n,n}n\leq c.\end{aligned}$$

In non-increasing case

$$\begin{aligned}\left\|K_{\tilde{n},n}^{T}\right\|_{1}&\leq c\left(\sum_{k=\tilde{n}}^{n}t_{k,n}+t_{\tilde{n},n}(\tilde{n}-1)-t_{n,n}n+t_{n,n}n+t_{\tilde{n},n}(\tilde{n}-1)\right)\\&\leq c(1+2t_{\tilde{n},n}\tilde{n})\leq c.\end{aligned}$$

□

Remark 2.8. Inequality (2.2) surely is not fulfilled without conditions (besides (2.1)). For example consider $t_{n,n}=1$. Then $K_{\tilde{n},n}^{T}=D_n$ and sequence $\|D_n\|_1$ is not bounded.

The next result is in paper [9]. We use only the half of that statement, only for non-decreasing $t_{k,n}$ sequences and $f\in L_1(G_m)$.

Lemma 2.9 (Blahota and K. Nagy [9])**.** *Let $f\in L_1(G_m)$. For every $n\in\mathbb{P}$, $\{t_{k,n}: 1\leq k\leq n\}$ be a finite sequence of non-negative numbers such that*

$$\sum_{k=1}^{n}t_{k,n}=1$$

is satisfied. If the finite sequence $\{t_{k,n}:1\leq k\leq n\}$ is non-decreasing for a fixed n and the Condition

$$t_{n,n}=O\left(\frac{1}{n}\right)$$

is satisfied, then

$$\|\sigma_{1,n}^{T}(f)-f\|_{1}\leq c\sum_{j=0}^{|n|-1}M_{j}\omega_{1}\left(f,\frac{1}{M_{j}}\right)\sum_{l=1}^{m_{j}-1}t_{(l+1)M_{j}-1,n}+c\omega_{1}\left(f,\frac{1}{M_{|n|}}\right).$$

3. De La Vallée Poussin type approximations in norm

Theorem 3.1. *Let $f \in L_1(G_m)$ and $\tilde{n}, n \in \mathbb{P}$, where $\tilde{n} < n$. Let the finite sequence $\{t_{k,n} : \tilde{n} \leq k \leq n\}$ of non-negative numbers be non-decreasing for all n. We suppose that*

$$\sum_{k=\tilde{n}}^{n} t_{k,n} = 1$$

and

$$t_{n,n} = O\left(\frac{1}{n}\right)$$

hold. Then

$$\left\|\sigma_{\tilde{n},n}^{T}(f) - f\right\|_1 \leq c\omega_1\left(f, \frac{1}{M_{|\tilde{n}|}}\right).$$

Proof. The proof is a simple consequence of Lemma 2.9. Namely, in the statement of Lemma 2.9 let us choose $t_{1,n} := \cdots := t_{\tilde{n}-1,n} = 0$. Then

$$\begin{aligned}
\left\|\sigma_{\tilde{n},n}^{T}(f) - f\right\|_1 &= \left\|\sigma_{1,n}^{T}(f) - f\right\|_1 \\
&\leq c \sum_{j=0}^{|n|-1} M_j \omega_1\left(f, \frac{1}{M_j}\right) \sum_{l=1}^{m_j-1} t_{(l+1)M_j-1,n} + c\omega_1\left(f, \frac{1}{M_{|n|}}\right) \\
&\leq c \sum_{j=0}^{|n|-1} M_j \omega_1\left(f, \frac{1}{M_j}\right)(m_j - 1) t_{M_{j+1}-1,n} + c\omega_1\left(f, \frac{1}{M_{|n|}}\right) \\
&\leq c \sum_{j=|\tilde{n}|}^{|n|-1} M_j \omega_1\left(f, \frac{1}{M_j}\right) c t_{M_{j+1}-1,n} + c\omega_1\left(f, \frac{1}{M_{|n|}}\right) \\
&\leq c n t_{n,n} \omega_1\left(f, \frac{1}{M_{|\tilde{n}|}}\right) + c\omega_1\left(f, \frac{1}{M_{|n|}}\right) \\
&\leq c\omega_1\left(f, \frac{1}{M_{|\tilde{n}|}}\right) + c\omega_1\left(f, \frac{1}{M_{|n|}}\right) \\
&= c\omega_1\left(f, \frac{1}{M_{|\tilde{n}|}}\right)
\end{aligned}$$

□

Remark 3.2. In the proof of Theorem 3.1 we got inequality

$$\left\|\sigma_{\tilde{n},n}^{T}(f) - f\right\|_1 \leq c \sum_{j=|\tilde{n}|}^{|n|-1} M_j \omega_1\left(f, \frac{1}{M_j}\right) c t_{M_{j+1}-1,n} + c\omega_1\left(f, \frac{1}{M_{|n|}}\right).$$

This estimate is more accurate for small $\tilde{n}$ compared to the statement of Theorem 3.1.

The statement of next lemma has mostly analogous form as result of Lemma 2.9, Móricz and Siddiqi [24] and others.

Theorem 3.3. *Let $f \in L_1(G_m)$. For every $n, \tilde{n} \in \mathbb{P}$, $\{t_{k,n} : \tilde{n} \leq k \leq n\}$, where $|\tilde{n}| < |n|$ be a finite sequence of non-negative numbers such that*

$$\sum_{k=\tilde{n}}^{n} t_{k,n} = 1 \tag{3.1}$$

and

$$t_{\tilde{n},n} = O\left(\frac{1}{\tilde{n}}\right) \tag{3.2}$$

are satisfied. If the finite sequence $\{t_{k,n} : \tilde{n} \leq k \leq n\}$ is non-increasing for a fixed n, then

$$\left\|\sigma^T_{\tilde{n},n}(f) - f\right\|_1 \leq c \sum_{j=|\tilde{n}|+1}^{|n|-1} M_j \omega_1\left(f, \frac{1}{M_j}\right) \sum_{l=1}^{m_j-1} t_{lM_j,n} + c\omega_1\left(f, \frac{1}{M_{|\tilde{n}|}}\right).$$

Proof. Using Lemma 2.5, Condition (3.1) and the usual Minkowski inequality we get

$$\begin{aligned}
&\|\sigma^T_{\tilde{n},n}(f) - f\|_1 = \left\| \int_{G_m} K^T_{\tilde{n},n}(u) F(\cdot, u) d\mu(u) \right\|_1 \\
&\leq \sum_{i=1}^{13} \left\| \int_{G_m} K_{i,\tilde{n},n}(u) F(\cdot, u) d\mu(u) \right\|_1 \\
&\leq \sum_{k=1}^{M_{|\tilde{n}|+1}-\tilde{n}} t_{M_{|\tilde{n}|+1}-k,n} \left\| \int_{G_m} D_{M_{|\tilde{n}|+1}}(u) F(\cdot, u) d\mu(u) \right\|_1 \\
&\quad + t_{\tilde{n},n}(M_{|\tilde{n}|+1} - \tilde{n}) \left\| \int_{G_m} \varphi_{M_{|\tilde{n}|+1}-1} \bar{K}_{M_{|\tilde{n}|+1}-\tilde{n}} F(\cdot, u) d\mu(u) \right\|_1 \\
&\quad + \sum_{k=1}^{M_{|\tilde{n}|+1}-\tilde{n}-1} \left|\Delta t_{M_{|\tilde{n}|+1}-k-1,n}\right| k \left\| \int_{G_m} \varphi_{M_{|\tilde{n}|+1}-1} \bar{K}_k F(\cdot, u) d\mu(u) \right\|_1 \\
&\quad + \sum_{j=|\tilde{n}|+1}^{|n|-1} \sum_{l=1}^{m_j-1} \sum_{s=1}^{l-1} \sum_{k=0}^{M_j-1} t_{lM_j+k,n} \left\| \int_{G_m} r_j^s(u) D_{M_j}(u) F(\cdot, u) d\mu(u) \right\|_1 \\
&\quad + \sum_{j=|\tilde{n}|+1}^{|n|-1} \sum_{s=M_j}^{M_{j+1}-1} t_{s,n} \left\| \int_{G_m} D_{M_j}(u) F(\cdot, u) d\mu(u) \right\|_1
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=|\tilde{n}|+1}^{|n|-1}\sum_{l=1}^{m_j-1}\sum_{k=1}^{M_j-2}\left|\Delta t_{lM_j+k,n}\right|k\left\|\int_{G_m} r_j^l(u)K_k(u)F(\cdot,u)d\mu(u)\right\|_1\\
&+\sum_{j=|\tilde{n}|+1}^{|n|-1}\sum_{l=1}^{m_j-1} t_{(l+1)M_j-1,n}(M_j-1)\left\|\int_{G_m} r_j^l(u)K_{M_j-1}(u)F(\cdot,u)d\mu(u)\right\|_1\\
&+\sum_{l=1}^{n_{|n|}-1}\sum_{s=1}^{l-1}\sum_{k=0}^{M_{|n|}-1} t_{lM_{|n|}+k,n}\left\|\int_{G_m} r_{|n|}^s(u)D_{M_{|n|}}(u)F(\cdot,u)d\mu(u)\right\|_1\\
&+\sum_{l=1}^{n_{|n|}-1}\sum_{k=1}^{M_{|n|}-2}\left|\Delta t_{lM_{|n|}+k,n}\right|k\left\|\int_{G_m} r_{|n|}^l(u)K_k(u)F(\cdot,u)d\mu(u)\right\|_1\\
&+\sum_{l=1}^{n_{|n|}-1} t_{(l+1)M_{|n|}-1,n}(M_{|n|}-1)\left\|\int_{G_m} r_{|n|}^l(u)K_{M_{|n|}-1}(u)F(\cdot,u)d\mu(u)\right\|_1\\
&+\sum_{k=0}^{n-n_{|n|}M_{|n|}}\sum_{s=1}^{n_{|n|}-1} t_{n_{|n|}M_{|n|}+k,n}\left\|\int_{G_m} r_{|n|}^s(u)D_{M_{|n|}}(u)F(\cdot,u)d\mu(u)\right\|_1\\
&+\sum_{k=1}^{n-n_{|n|}M_{|n|}}\left|\Delta t_{n_{|n|}M_{|n|}+k,n}\right|k\left\|\int_{G_m} r_{|n|}^{n_{|n|}}(u)K_k(u)F(\cdot,u)d\mu(u)\right\|_1\\
&+\sum_{l=M_{|n|}}^{n} t_{l,n}\left\|\int_{G_m} D_{M_{|n|}}(u)F(\cdot,u)d\mu(u)\right\|_1 =: \sum_{i=1}^{13} I_{i,n}.
\end{aligned}
$$

The generalized Minkowski inequality and Condition (3.1) imply

$$
\begin{aligned}
I_{1,n}\le&\int_{G_m} D_{M_{|\tilde{n}|+1}}(u)\int_{G_m}|F(x,u)|d\mu(x)d\mu(u)\sum_{k=\tilde{n}}^{M_{|\tilde{n}|+1}-1} t_{k,n}\\
\le&\omega_1\big(f,1/M_{|\tilde{n}|+1}\big),
\end{aligned}
$$

$$
\begin{aligned}
I_{5,n}&\le\sum_{j=|\tilde{n}|+1}^{|n|-1}\sum_{s=M_j}^{M_{j+1}-1} t_{s,n}\int_{G_m} D_{M_j}(u)\int_{G_m}|F(x,u)|d\mu(x)d\mu(u)\\
&\le\sum_{j=|\tilde{n}|+1}^{|n|-1}\sum_{k=0}^{M_{j+1}-M_j-1} t_{M_j+k,n}\omega_1(f,1/M_j)\\
&\le\sum_{j=|\tilde{n}|+1}^{|n|-1}\left(\sum_{k=0}^{M_j-1} t_{M_j,n}+\sum_{k=M_j}^{2M_j-1} t_{2M_j,n}+\cdots+\sum_{k=M_{j+1}-M_j}^{M_{j+1}-1} t_{(m_j-1)M_j,n}\right)
\end{aligned}
$$

$$\times\,\omega_1(f,1/M_j)$$
$$=\sum_{j=|\tilde{n}|+1}^{|n|-1} M_j \sum_{l=1}^{m_j-1} t_{lM_j,n}\omega_1(f,1/M_j)$$

and

$$\begin{aligned} I_{13,n} &\leq \int_{G_m} D_{M_{|n|}}(u)\int_{G_m}|F(x,u)|d\mu(x)d\mu(u)\sum_{l=M_{|n|}}^{n} t_{l,n} \\ &\leq \omega_1\big(f,1/M_{|n|}\big). \end{aligned}$$

For expressions $I_{4,n},\ I_{8,n}$ and $I_{11,n}$ we apply Lemma 2.3, equation (1.1) and Condition (3.1). We have

$$\begin{aligned} I_{4,n} &\leq \sum_{j=|\tilde{n}|+1}^{|n|-1} m_j\big\|D_{M_j}\big\|_1\omega_1(f,1/M_j)\sum_{l=1}^{m_j-1}(l-1)\sum_{k=0}^{M_j-1} t_{lM_j+k,n} \\ &\leq \sum_{j=|\tilde{n}|+1}^{|n|-1} m_j^2\omega_1(f,1/M_j)\sum_{l=1}^{m_j-1}\sum_{k=0}^{M_j-1} t_{lM_j+k,n}. \end{aligned}$$

From monotony we obtain

$$I_{4,n} \leq c\sum_{j=|\tilde{n}|+1}^{|n|-1} M_j\omega_1(f,1/M_j)\sum_{l=1}^{m_j-1} t_{lM_j,n}.$$

Similarly,

$$\begin{aligned} I_{8,n} &\leq m_{|n|}\big\|D_{M_{|n|}}\big\|_1\omega_1(f,1/M_{|n|})\sum_{l=1}^{n_{|n|}-1}(l-1)\sum_{k=0}^{M_{|n|}-1} t_{lM_{|n|}+k,n} \\ &\leq m_{|n|}^2\omega_1(f,1/M_{|n|})\sum_{l=1}^{n_{|n|}-1}\sum_{k=0}^{M_{|n|}-1} t_{lM_{|n|}+k,n} \\ &\leq c\omega_1(f,1/M_{|n|})\sum_{k=M_{|n|}}^{n_{|n|}M_{|n|}-1} t_{k,n} \\ &\leq c\omega_1(f,1/M_{|n|}) \end{aligned}$$

and

$$I_{11,n} \leq m_{|n|}(n_{|n|}-1)\big\|D_{M_{|n|}}\big\|_1\omega_1(f,1/M_{|n|})\sum_{k=0}^{n-n_{|n|}M_{|n|}} t_{n_{|n|}M_{|n|}+k,n}$$

$$\leq m_{|n|}^2 \omega_1(f, 1/M_{|n|}) \sum_{k=n_{|n|}M_{|n|}}^{n} t_{k,n}$$
$$\leq c\omega_1(f, 1/M_{|n|}).$$

The usual Minkowski inequality, Inequality (1.2), Corollary 2.4, Condition (3.1) and Condition (3.2) yield estimations of expressions $I_{2,n}$ and $I_{3,n}$. At first

$$\begin{aligned} I_{2,n} &\leq t_{\tilde{n},n}(M_{|\tilde{n}|+1} - \tilde{n}) m_{|\tilde{n}|} \omega_1(f, 1/M_{|\tilde{n}|}) \left\| K_{M_{|\tilde{n}|+1}-\tilde{n}} \right\|_1 \\ &\leq c t_{\tilde{n},n} \tilde{n} \omega_1(f, 1/M_{|\tilde{n}|}) \\ &\leq c\omega_1(f, 1/M_{|\tilde{n}|}). \end{aligned}$$

Then

$$\begin{aligned} I_{3,n} &\leq m_{|\tilde{n}|} \omega_1(f, 1/M_{|\tilde{n}|}) \sum_{k=1}^{M_{|\tilde{n}|+1}-\tilde{n}-1} \left| \Delta t_{M_{|\tilde{n}|+1}-k-1,n} \right| k \| K_k \|_1 \\ &\leq c\omega_1(f, 1/M_{|\tilde{n}|}) \sum_{k=1}^{M_{|\tilde{n}|+1}-\tilde{n}-1} \Delta t_{M_{|\tilde{n}|+1}-k-1,n} k, \end{aligned}$$

and we get

$$\begin{aligned} &\sum_{k=1}^{M_{|\tilde{n}|+1}-\tilde{n}-1} \Delta t_{M_{|\tilde{n}|+1}-k-1,n} k \\ &= \sum_{k=1}^{M_{|\tilde{n}|+1}-\tilde{n}-1} \left(t_{M_{|\tilde{n}|+1}-k-1,n} - t_{M_{|\tilde{n}|+1}-k,n} \right) k \\ &= (M_{|\tilde{n}|+1} - \tilde{n} - 1) t_{\tilde{n},n} - \sum_{k=1}^{M_{|\tilde{n}|+1}-\tilde{n}-1} t_{M_{|\tilde{n}|+1}-k,n} \\ &\leq c\tilde{n} t_{\tilde{n},n} \leq c, \end{aligned}$$

hence

$$I_{3,n} \leq c\omega_1\left(f, 1/M_{|\tilde{n}|}\right).$$

For expressions $I_{6,n}$, $I_{7,n}$, $I_{9,n}$, $I_{10,n}$ and $I_{12,n}$ we apply Inequality (1.2), Lemma 2.3 and Condition (3.1). Estimating $I_{6,n}$ we obtain

$$\begin{aligned} I_{6,n} &\leq \sum_{j=|\tilde{n}|+1}^{|n|-1} m_j \omega_1(f, 1/M_j) \sum_{l=1}^{m_j-1} \sum_{k=1}^{M_j-2} \left| \Delta t_{lM_j+k,n} \right| k \| K_k \|_1 \\ &\leq c \sum_{j=|\tilde{n}|+1}^{|n|-1} \omega_1(f, 1/M_j) \sum_{l=1}^{m_j-1} \sum_{k=1}^{M_j-2} \Delta t_{lM_j+k,n} k. \end{aligned}$$

We write

$$\sum_{k=1}^{M_j-2} \Delta t_{lM_j+k,n}k = \sum_{k=1}^{M_j-2} t_{lM_j+k,n} - (M_j-2)t_{(l+1)M_j-1,n}$$
$$\leq \sum_{k=1}^{M_j-2} t_{lM_j+k,n} \leq M_j t_{lM_j,n}$$

and

$$I_{6,n} \leq c \sum_{j=|\tilde{n}|+1}^{|n|-1} M_j\omega_1(f,1/M_j) \sum_{l=1}^{m_j-1} t_{lM_j,n}.$$

Now, we estimate the expression $I_{7,n}$.

$$I_{7,n} \leq m_j \sum_{j=|\tilde{n}|+1}^{|n|-1} M_j\omega_1(f,1/M_j)\|K_{M_j-1}\|_1 \sum_{l=1}^{m_j-1} t_{(l+1)M_j-1,n}$$
$$\leq c \sum_{j=|\tilde{n}|+1}^{|n|-1} M_j\omega_1(f,1/M_j) \sum_{l=1}^{m_j-1} t_{lM_j,n}.$$

We have

$$I_{9,n} \leq m_{|n|}\omega_1\big(f,1/M_{|n|}\big) \sum_{l=1}^{n_{|n|}-1} \sum_{k=1}^{M_{|n|}-2} \big|\Delta t_{lM_{|n|}+k,n}\big|k\|K_k\|_1$$
$$\leq c\omega_1\big(f,1/M_{|n|}\big) \sum_{l=1}^{n_{|n|}-1} \sum_{k=1}^{M_{|n|}-2} \Delta t_{lM_{|n|}+k,n}k.$$

From monotony we have

$$\sum_{k=1}^{M_{|n|}-2} \Delta t_{lM_{|n|}+k,n}k = \sum_{k=1}^{M_{|n|}-2} t_{lM_{|n|}+k,n} - (M_{|n|}-2)t_{(l+1)M_{|n|}-1,n}$$
$$\leq \sum_{k=1}^{M_{|n|}-2} t_{lM_{|n|}+k,n}$$

and

$$I_{9,n} \leq c\omega_1\big(f,1/M_{|n|}\big) \sum_{l=1}^{n_{|n|}-1} \sum_{k=1}^{M_{|n|}-2} t_{lM_{|n|}+k,n}$$
$$= c\omega_1\big(f,1/M_{|n|}\big) \sum_{k=M_{|n|}+1}^{n_{|n|}M_{|n|}-2} t_{k,n}$$

$$\leq c\omega_1\big(f, 1/M_{|n|}\big).$$

Let us consider the expression $I_{10,n}$.

$$\begin{aligned} I_{10,n} &\leq m_{|n|}M_{|n|}\omega_1\big(f, 1/M_{|n|}\big) \sum_{l=1}^{n_{|n|}-1} t_{(l+1)M_{|n|}-1,n}\|K_{M_{|n|-1}}\|_1 \\ &\leq cM_{|n|}\omega_1\big(f, 1/M_{|n|}\big) \sum_{l=1}^{n_{|n|}-1} t_{(l+1)M_{|n|}-1,n}. \end{aligned}$$

From monotony we write

$$\begin{aligned} I_{10,n} &\leq c\omega_1\big(f, 1/M_{|n|}\big) \sum_{l=1}^{n_{|n|}-1} \sum_{k=0}^{M_{|n|}-1} t_{lM_{|n|}+k,n} \\ &\leq c\omega_1\big(f, 1/M_{|n|}\big) \sum_{k=M_{|n|}}^{n_{|n|}M_{|n|}-1} t_{k,n} \\ &\leq c\omega_1\big(f, 1/M_{|n|}\big). \end{aligned}$$

We discuss expression $I_{12,n}$

$$\begin{aligned} I_{12,n} &\leq m_{|n|} \sum_{k=1}^{n-n_{|n|}M_{|n|}} \big|\Delta t_{n_{|n|}M_{|n|}+k,n}\big| k\omega_1\big(f, 1/M_{|n|}\big)\|K_k\|_1 \\ &\leq c\omega_1\big(f, 1/M_{|n|}\big) \sum_{k=1}^{n-n_{|n|}M_{|n|}} \Delta t_{n_{|n|}M_{|n|}+k,n}k. \end{aligned}$$

We have

$$\begin{aligned} \sum_{k=1}^{n-n_{|n|}M_{|n|}} \Delta t_{n_{|n|}M_{|n|}+k,n}k &= \sum_{k=1}^{n-n_{|n|}M_{|n|}} t_{n_{|n|}M_{|n|}+k,n} - (n - n_{|n|}M_{|n|})t_{n+1,n} \\ &= \sum_{k=1}^{n-n_{|n|}M_{|n|}} t_{n_{|n|}M_{|n|}+k,n} \end{aligned}$$

and

$$\begin{aligned} I_{12,n} &\leq c\omega_1\big(f, 1/M_{|n|}\big) \sum_{k=n_{|n|}M_{|n|}+1}^{n} t_{k,n} \\ &\leq c\omega_1\big(f, 1/M_{|n|}\big). \end{aligned}$$

Because of $|\tilde{n}| < |n|$ we get inequalities

$$\omega_1\big(f, 1/M_{|n|}\big) \leq \omega_1\big(f, 1/M_{|\tilde{n}|+1}\big) \leq \omega_1\big(f, 1/M_{|\tilde{n}|}\big)$$

These complete the proof of Theorem 3.3. □

Corollary 3.4. *Let $f \in L_1(G_m)$. For every $n, \tilde{n} \in \mathbb{P}$, $\{t_{k,n} : \tilde{n} \leq k \leq n\}$, where $|\tilde{n}| < |n|$ be a finite sequence of non-negative numbers such that*

$$\sum_{k=\tilde{n}}^{n} t_{k,n} = 1$$

and

$$t_{\tilde{n},n} = O\left(\frac{1}{\tilde{n}}\right)$$

are satisfied. If the finite sequence $\{t_{k,n} : \tilde{n} \leq k \leq n\}$ is non-increasing for a fixed n, then

$$\left\|\sigma^{T}_{\tilde{n},n}(f) - f\right\|_1 \leq c\omega_1\left(f, \frac{1}{M_{|\tilde{n}|}}\right).$$

Proof. The proof is a consequence of Theorem 3.3. We use inequality

$$\begin{aligned} M_j \sum_{l=1}^{m_j-1} t_{lM_j,n} &= M_j t_{M_j,n} + M_j t_{2M_j,n} + \cdots + M_j t_{(m_j-1)M_j,n} \\ &\leq M_j t_{M_j,n} + \sum_{i=0}^{M_j-1} t_{M_j+i,n} + \cdots + \sum_{i=0}^{M_j-1} t_{(m_j-2)M_j+i,n} \\ &= m_{j-1}M_{j-1}t_{M_j,n} + \sum_{i=M_j}^{M_{j+1}-M_j-1} t_{i,n} \end{aligned}$$

for $j \in \{|\tilde{n}|+1, \ldots, |n|-1\}$ and

$$\sum_{j=|\tilde{n}|+2}^{|n|-1} M_{j-1}t_{M_j,n} \leq \sum_{j=M_{|\tilde{n}|+2}-M_{|\tilde{n}|+1}+1}^{M_{|n|-1}} t_{j,n} < 1.$$

So

$$\begin{aligned} \sum_{j=|\tilde{n}|+1}^{|n|-1} M_j \sum_{l=1}^{m_j-1} t_{lM_j,n} &\leq \sum_{j=|\tilde{n}|+1}^{|n|-1} m_{j-1}M_{j-1}t_{M_j,n} + \sum_{j=|\tilde{n}|+1}^{|n|-1} \sum_{i=M_j}^{M_{j+1}-M_j-1} t_{i,n} \\ &\leq m_{|\tilde{n}|}M_{|\tilde{n}|}t_{M_{|\tilde{n}|+1},n} + \sum_{j=|\tilde{n}|+2}^{|n|-1} m_{j-1}M_{j-1}t_{M_j,n} + 1 \\ &\leq c\tilde{n}t_{\tilde{n},n} + c + 1 \\ &\leq c. \end{aligned}$$

It implies that

$$\sum_{j=|\tilde{n}|+1}^{|n|-1} M_j \sum_{l=1}^{m_j-1} t_{lM_j,n}\omega_1\left(f, \frac{1}{M_j}\right) \leq c\omega_1\left(f, \frac{1}{M_{|\tilde{n}|}}\right). \qquad \square$$

Theorem 3.5. *Let $f \in L_1(G_m)$. For every $n, \tilde{n} \in \mathbb{P}$, $\{t_{k,n} : \tilde{n} \le k \le n\}$, where $|\tilde{n}| = |n|$ be a finite sequence of non-negative numbers such that*

$$\sum_{k=\tilde{n}}^{n} t_{k,n} = 1$$

and

$$t_{\tilde{n},n} = O\left(\frac{1}{\tilde{n}}\right)$$

are satisfied. Let the finite sequence $\{t_{k,n} : \tilde{n} \le k \le n\}$ be non-increasing for a fixed n. Then

$$\left\|\sigma_{\tilde{n},n}^{T}(f) - f\right\|_1 \le c\omega_1\left(f, \frac{1}{M_{|\tilde{n}|}}\right).$$

Proof. Based on Lemma 2.6, we can prove it similarly, than in cases $I_{1,n}, I_{2,n}$ and $I_{3,n}$ in Theorem 3.3. In the last case

$$\begin{aligned}
&\sum_{k=M_{|n|+1}-\tilde{n}-1}^{M_{|n|+1}-\tilde{n}-1} \Delta t_{M_{|n|+1}-k-1,n} k \\
&= \sum_{k=M_{|n|+1}-\tilde{n}-1}^{M_{|n|+1}-\tilde{n}-1} \left(t_{M_{|n|+1}-k-1,n} - t_{M_{|n|+1}-k,n}\right) k \\
&= (M_{|n|+1} - \tilde{n}) t_{\tilde{n},n} - \sum_{k=M_{|n|+1}-n-1}^{M_{|n|+1}-\tilde{n}-1} t_{M_{|n|+1}-k,n} \\
&\le c\tilde{n} t_{\tilde{n},n} \le c.
\end{aligned}$$

□

Remark 3.6. Theorem 3.1, Corollary 3.4 and Theorem 3.5 (not in this structure) are also reached by Volosivets [32] (see Corollary 3.4.) by a different method.

Theorem 3.7. *Let $f \in L_p(G_m)$ where $1 < p < \infty$ and $\tilde{n}, n \in \mathbb{P}$, where $\tilde{n} < n$. Let the finite sequence $\{t_{k,n} : \tilde{n} \le k \le n\}$ of non-negative numbers. We suppose that*

$$\sum_{k=\tilde{n}}^{n} t_{k,n} = 1 \tag{3.3}$$

holds. Then

$$\left\|\sigma_{\tilde{n},n}^{T}(f) - f\right\|_p \le c_p\omega_p\left(f, \frac{1}{M_{|\tilde{n}|}}\right),$$

where c_p depends only on p.

Proof. Since $M_{|k|} \le k$, from elementary properties of the partial sum we obtain

$$\|S_k(f) - f\|_p \le \left\|S_{M_{|k|}}(f) - f\right\|_p + \left\|S_k(f) - S_{M_{|k|}}(f)\right\|_p$$

$$
\begin{aligned}
&= \left\|S_{M_{|k|}}(f) - f\right\|_p + \left\|S_k(f) - S_k(S_{M_{|k|}}(f))\right\|_p \\
&= \left\|S_{M_{|k|}}(f) - f\right\|_p + \left\|S_k(f - S_{M_{|k|}}(f))\right\|_p.
\end{aligned}
$$

It is known, that if $1 < p < \infty$, then $\|S_k(f)\|_p \le c_p\|f\|_p$, hence

$$\|S_k(f) - f\|_p \le c_p\left\|S_{M_{|k|}}(f) - f\right\|_p.$$

Based on this inequality, Efimov [14] famous result gives us

$$\|S_k(f) - f\|_p \le c_p\omega_p\left(f, \frac{1}{M_{|k|}}\right),$$

so, using Condition (3.3)

$$
\begin{aligned}
\left\|\sigma^T_{\tilde{n},n}(f) - f\right\|_p &\le \sum_{k=\tilde{n}}^{n} t_{k,n}\|S_k(f) - f\|_p \\
&\le c_p \sum_{k=\tilde{n}}^{n} t_{k,n}\omega_p\left(f, \frac{1}{M_{|k|}}\right) \\
&= c_p\omega_p\left(f, \frac{1}{M_{|\tilde{n}|}}\right).
\end{aligned}
$$

□

Remark 3.8. In the proof of Theorem 3.7 we got inequality

$$\left\|\sigma^T_{\tilde{n},n}(f) - f\right\|_1 \le c_p \sum_{k=\tilde{n}}^{n} t_{k,n}\omega_p\left(f, \frac{1}{M_{|k|}}\right).$$

This estimate is better for small $\tilde{n}$ than the statement of Theorem 3.7.

At the end of this, let us summarize our results.

Theorem 3.9. *Let $f \in L_p(G_m)$, where $1 \le p \le \infty$ and $\tilde{n}, n \in \mathbb{P}$, where $\tilde{n} < n$. Let the members of the finite sequence $\{t_{k,n} : \tilde{n} \le k \le n\}$ be non-negative numbers. We suppose that*

$$\sum_{k=\tilde{n}}^{n} t_{k,n} = 1.$$

If the finite sequence $\{t_{k,n} : \tilde{n} \le k \le n\}$ is non-decreasing for all n, then we suppose

$$t_{n,n} = O\left(\frac{1}{n}\right)$$

and if the finite sequence $\{t_{k,n} : \tilde{n} \le k \le n\}$ is non-increasing for all n, then we suppose

$$t_{\tilde{n},n} = O\left(\frac{1}{\tilde{n}}\right).$$

Then in both cases and if $1 < p < \infty$, then without monotony restrictions and supposing boundedness of $nt_{n,n}$ or $\tilde{n}t_{\tilde{n},n}$

$$\left\|\sigma^T_{\tilde{n},n}(f) - f\right\|_p \leq c_p \omega_p\left(f, \frac{1}{M_{|\tilde{n}|}}\right)$$

holds, where c_p depends only on p.

4. Application of the norm estimation

A specific case (for Lipschitz functions) can be formulated as a statement – or rather as an example – one may say, concerning the de La Vallée Poussin means.

Remark 4.1. It is easy to see, that supposing conditions of Theorem 3.9 we get equality

$$\left\|\sigma^T_{\tilde{n},n}(f) - f\right\|_p = O\left(\frac{1}{\tilde{n}^\alpha}\right)$$

for every $f \in \mathrm{Lip}(\alpha, p, G_m)$, $\alpha > 0$ and $1 \leq p \leq \infty$.

Example 4.2. As a special case, let

$$t_{k,n} := \begin{cases} (q\tilde{n})^{-1}, & \text{if } \tilde{n} \leq k \leq n, \\ 0, & \text{otherwise }, \end{cases}$$

where $q > 0$ and $n := \lfloor (q+1)\tilde{n} \rfloor - 1$. In this case we get the rate of the norm convergence of de La Vallée Poussin mean for Lipschiz functions, namely if $f \in \mathrm{Lip}(1, p, G_m)$ for some $1 \leq p \leq \infty$, then

$$\left\|\frac{1}{q\tilde{n}} \sum_{k=\tilde{n}}^{n} S_k(f) - f\right\|_1 = O\left(\frac{1}{\tilde{n}}\right).$$

Proof. This statement is a simple corollary of Remark 4.1. □

Remark 4.3. Statements of Corollary 3.4, Theorems 3.5, 3.7, 3.9 and Remark 4.1 are useful if the $\tilde{n}$ is large. For example, if $n \to \infty$, we will get a good estimate if the $\tilde{n}$ does not remain under a limit.

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