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Plancherel formula for the attenuated Radon transform*

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Abstract. Motivated by stereology, based on Novikov's inversion formula, we prove a Plancherel-type formula for the attenuated Radon transform.

Keywords: attenuated Radon transform, Plancherel formula

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1. Introduction

In this article we deduct a Plancherel-type formula for the attenuated Radon transform. We shall compute

$$\int_{\mathbb{R}^2} f(x)g(x)\,\mathrm{d}x$$

from the attenuated Radon transform of f, with any g of our choice (known exactly or with arbitrary precision).

The problem is motivated by methods in stereology [1]. Using classical Radon transform [2], it is easy to estimate the area or volume of a body, or an integral of a function with compact support, as the expected value of sections, or that of integrals over straight lines or planes, because the weights are constant. It does not

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work so easily for the attenuated Radon transform due to its complicated weights over the line we integrate on. If the weights are non constants, we have to be satisfied by not too good approximations.

If we know all the weighted sections, as in the case of attenuated Radon-transform, then one would think that by applying a reconstruction formula, we can compute this integral. But in practice we do not have stable reconstructions due to the phenomena of "ghost images" and the unstability of the derivatives in the reconstruction formula. (See [3–5, 7] for the case of the classical Radon-transform.) This makes the integral of the reconstructed function very unstable. Also, the computation is very costly, and we need much higher precision of data for using a recontsruction formula, than necessary with our technique.

Our Plancherel type formula solves these issues: we can use less precise data because we do not have to calculate (generalized) derivatives of the attenuated Radon-transform $R_a f$, just that of the exactly known test function g. So we can apply Monte-Carlo methods for computing the integrals.

For deriving a Plancherel-type formula for the attenuated Radon transform, we will use Natterer's version [6] of Novikov's inversion formula [8].

Definition 1.1. Let $a\colon \mathbb{R}^2 \to \mathbb{R}$ be a sufficiently smooth and sufficiently decaying function (as in [8]), e.g. an integrable C^1 function whose partial derivatives are also integrable. For $\omega \in \mathbb{S}^1$ and $x \in \mathbb{R}^2$ we define

$$(Da)(x,\omega) = \int_0^\infty a(x + t\omega)\,\mathrm{d}t.$$

We will use $(Da)(x,\omega^\perp)$, with $\omega^\perp$ defined below.

Remark 1.2. Depending on the context, the symbol ω can be understood as the angle of the vector ω with the first coordinate axis, or the vector itself, that is, $\omega = (\cos\omega, \sin\omega)$. There is a one-to-one correspondence between the vector $\omega \in \mathbb{S}^1$ and $\omega \in [0, 2\pi)$ (geometrically equivalent to an angle). When the argument of the function to be differentiated is $x \in \mathbb{R}^2$, then ∂_ω (or $\partial_{\underline{\omega}}$) means directional derivative in the direction ω. When one of the arguments of the function to be differentiated is ω, then ∂_ω means partial derivative with respect to the angle ω. Occasionally we underline the name of vectors for easier understanding.

Definition 1.3. The *attenuated Radon transform* $R_a\colon [0, 2\pi) \times \mathbb{R} \to \mathbb{R}$ (or equivalently, $R_a\colon (-\pi, \pi] \times \mathbb{R} \to \mathbb{R}$) is defined as

$$(R_a f)(\omega, p) = \int_{\langle x,\omega\rangle = p} e^{-Da(x,\omega^\perp)} f(x)\,\mathrm{d}x,$$

where $\mathrm{d}x$ is the Lebesgue measure on the line $l(\omega, p) := \{x \in \mathbb{R}^2 : \langle x, \omega\rangle = p\}$, and $\omega^\perp$ is $(\cos\omega, \sin\omega)^\perp = (\cos(\omega + \frac{\pi}{2}), \sin(\omega + \frac{\pi}{2})) = (-\sin\omega, \cos\omega)$.

With $a = 0$ we get the classical Radon transform of f. We can rewrite this definition as

$$(R_a f)(\omega, p) = \int_{-\infty}^{\infty} e^{-Da(p\omega + u\omega^\perp, \omega^\perp)} f(p\omega + u\omega^\perp)\, \mathrm{d}u.$$

Note that the line $l(\omega, p)$ is the same set as $l(-\omega, -p)$, but admits different orientation. Also note that although defined on the same set, the integral $(R_a f)(\omega, p)$ usually differs from $(R_a f)(-\omega, -p)$, unlike the special case of the classical Radon transform where $Rf(\omega, p) = Rf(-\omega, -p)$.

Our goal is to compute

$$\int_{\mathbb{R}^2} f(x) g(x)\, \mathrm{d}x$$

from the attenuated Radon transform $R_a f$, knowing g with arbitrary precision.

2. The Plancherel formula

Let us define the function

$$h(\omega, p) = \frac{1}{2}(I + i\mathcal{H}) Ra(\omega, p),$$

where $Ra(\omega, p)$ is the classical Radon transform of the function a along the line $l(\omega, p)$, and $\mathcal{H}$ is the Hilbert transform, defined by

$$\mathcal{H} g(p) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(t)}{p - t}\, \mathrm{d}t,$$

where the integral is understood as a Cauchy principal value.

Theorem 2.1. *With the conditions of Novikov's inversion formula, the Plancherel formula for the complete attenuated Radon transform R_a is as follows:*

$$\int_{\mathbb{R}^2} f(x) g(x)\, \mathrm{d}x$$
$$= \frac{1}{4\pi} \int_{\mathbb{S}^1} \int_{-\infty}^{\infty} (R_a f)(\omega, p) \operatorname{Re}\Big[e^{h(\omega,p)} \mathcal{H}_p \Big\{ e^{-h(\omega,p)} (R_a\, \partial_\omega g)(\omega, p) \Big\} \Big]\, \mathrm{d}p\, \mathrm{d}\omega.$$

3. Proof of Theorem 2.1

First we prove the following lemma.

Lemma 3.1.

$$f(x) = \frac{1}{4\pi}\int\limits_{\mathbb{S}^1} \partial_{\underline{\omega}}(e^{Da(x,\omega^\perp)})(\mathrm{Re}\, e^{-h}\mathcal{H}e^h R_a f)(\omega, \langle x,\omega\rangle)\,\mathrm{d}\omega$$
$$+ \frac{1}{4\pi}\int\limits_{\mathbb{S}^1} e^{Da(x,\omega^\perp)}\,\partial_p(\mathrm{Re}\, e^{-h}\mathcal{H}e^h R_a f)(\omega, \langle x,\omega\rangle)\,\mathrm{d}\omega.$$

Proof. We have the following reconstruction formula due to Natterer [6] based on Novikov's [8], if a and f are sufficiently smooth functions decaying sufficiently fast at infinity:

$$\begin{aligned}
f(x) &= \frac{1}{4\pi}\,\mathrm{div}\int\limits_{\mathbb{S}^1} \underline{\omega}\Big\{e^{Da(x,\omega^\perp)}(\mathrm{Re}\, e^{-h}\mathcal{H}e^h R_a f)(\omega, \langle x,\omega\rangle)\Big\}\,\mathrm{d}\omega \\
&= \frac{1}{4\pi}\frac{\partial}{\partial x_1}\int\limits_{\mathbb{S}^1} \cos\omega\Big\{e^{Da(x,\omega^\perp)}(\mathrm{Re}\, e^{-h}\mathcal{H}e^h R_a f)(\omega, \langle x,\omega\rangle)\Big\}\,\mathrm{d}\omega \\
&\quad+ \frac{1}{4\pi}\frac{\partial}{\partial x_2}\int\limits_{\mathbb{S}^1} \sin\omega\Big\{e^{Da(x,\omega^\perp)}(\mathrm{Re}\, e^{-h}\mathcal{H}e^h R_a f)(\omega, \langle x,\omega\rangle)\Big\}\,\mathrm{d}\omega \\
&= \frac{1}{4\pi}\int\limits_{\mathbb{S}^1} \partial_{\underline{\omega}}(e^{Da(x,\omega^\perp)})(\mathrm{Re}\, e^{-h}\mathcal{H}e^h R_a f)(\omega, \langle x,\omega\rangle)\,\mathrm{d}\omega \\
&\quad+ \frac{1}{4\pi}\int\limits_{\mathbb{S}^1} e^{Da(x,\omega^\perp)}\,\partial_p(\mathrm{Re}\, e^{-h}\mathcal{H}e^h R_a f)(\omega, \langle x,\omega\rangle)\,\mathrm{d}\omega,
\end{aligned}$$

as $\mathrm{div}(\underline{\omega}c(\underline{x})) = \cos\omega\frac{\partial}{\partial x_1}c(\underline{x}) + \sin\omega\frac{\partial}{\partial x_2}c(\underline{x})$, and $\partial_{\underline{\omega}} = \cos\omega\frac{\partial}{\partial x_1} + \sin\omega\frac{\partial}{\partial x_2}$ is the directional derivative of a function (defined in $\mathbb{R}^2$) in the direction ω, which is now acting on the first variable of $e^{Da(x,\omega^\perp)}$, and ∂_p is the partial derivative of a function defined on $\mathbb{P}^2 = \mathbb{S}^1 \times \mathbb{R}_+$, the space of all straight lines, with respect to its second variable (the distance from the origin). The latter comes from $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ acting on the scalar product $\langle x,\omega\rangle$ as an inside function, with derivatives $\cos\omega$ and $\sin\omega$, respectively, multiplying $\cos\omega$ and $\sin\omega$ that were already there, so adding up to 1. □

Note that if p is changing with the direction ω fixed, that has the same geometric meaning as the directional derivative in the direction ω, as shown in Figure 1.

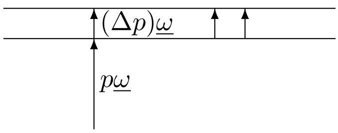

Figure 1

Now we will compute $\int_{\mathbb{R}^2} f(x)g(x)\,\mathrm{d}x$ from the attenuated Radon transform $R_a f$, and an other function g that is known.

We multiply the above expression of $f(x)$ by $g(x)$ and integrate it on $\mathbb{R}^2$, then change the order of integration. The differentiation with respect to p when u and ω are fixed, and the directional derivative in the direction ω, are exactly the same, so the two terms of our previous expression for $f(x)$ can be contracted to a total derivative (using the substitutions $\langle x,\omega\rangle = p$ and $\langle x,\omega^\perp\rangle = u$):

$$\begin{aligned}
&\int\limits_{\mathbb{R}^2} f(x)g(x)\,\mathrm{d}x \\
&= \int\limits_{\mathbb{R}^2} \Bigg\{ \frac{1}{4\pi} \int\limits_{\mathbb{S}^1} \partial_{\underline{\omega}} e^{Da(x,\omega^\perp)} (\mathrm{Re}\, e^{-h}\mathcal{H} e^h R_a f)(\omega, \langle x,\omega\rangle) \\
&\qquad + e^{Da(x,\omega^\perp)}\, \partial_p (\mathrm{Re}\, e^{-h}\mathcal{H} e^h R_a f)(\omega, \langle x,\omega\rangle)\,\mathrm{d}\omega \Bigg\} g(x)\,\mathrm{d}x \\
&= \frac{1}{4\pi} \int\limits_{\mathbb{S}^1} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \Bigg\{ \partial_p \{ e^{\int_0^\infty a(p\omega+u\omega^\perp+t\omega^\perp)\mathrm{d}t} \} (\mathrm{Re}\, e^{-h}\mathcal{H} e^h R_a f)(\omega,p) \\
&\qquad + e^{\int_0^\infty a(p\omega+u\omega^\perp+t\omega^\perp)\mathrm{d}t}\, \partial_p (\mathrm{Re}\, e^{-h}\mathcal{H} e^h R_a f)(\omega,p) \Bigg\} g(p\omega+u\omega^\perp)\,\mathrm{d}p\,\mathrm{d}u\,\mathrm{d}\omega \\
&= \frac{1}{4\pi} \int\limits_{\mathbb{S}^1} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \partial_p \Bigg\{ e^{\int_0^\infty a(p\omega+u\omega^\perp+t\omega^\perp)\mathrm{d}t}\, \mathrm{Re}\, e^{-h}\mathcal{H} e^h R_a f(\omega,p) \Bigg\} \\
&\qquad \cdot g(p\omega+u\omega^\perp)\,\mathrm{d}p\,\mathrm{d}u\,\mathrm{d}\omega \\
&= -\frac{1}{4\pi} \int\limits_{\mathbb{S}^1} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \Bigg\{ e^{\int_0^\infty a(p\omega+u\omega^\perp+t\omega^\perp)\mathrm{d}t}\, \mathrm{Re}\, e^{-h}\mathcal{H} e^h R_a f(\omega,p) \Bigg\} \\
&\qquad \cdot \partial_p \big\{ g(p\omega+u\omega^\perp) \big\}\,\mathrm{d}p\,\mathrm{d}u\,\mathrm{d}\omega \\
&= -\frac{1}{4\pi} \int\limits_{\mathbb{S}^1} \int\limits_{-\infty}^{\infty} \mathrm{Re}\, e^{-h}\mathcal{H} e^h R_a f(\omega,p) \\
&\qquad \cdot \Bigg\{ \int\limits_{-\infty}^{\infty} e^{\int_0^\infty a(p\omega+u\omega^\perp+t\omega^\perp)\mathrm{d}t} (\partial_{\underline{\omega}} g)(p\omega+u\omega^\perp)\,\mathrm{d}u \Bigg\}\,\mathrm{d}p\,\mathrm{d}\omega \\
&= -\frac{1}{4\pi} \int\limits_{\mathbb{S}^1} \int\limits_{-\infty}^{\infty} \mathrm{Re}\, e^{-h}\mathcal{H}\big\{ e^h R_a f(\omega,p) \big\} \cdot R_a(\partial_{\underline{\omega}} g)(\omega,p)\,\mathrm{d}p\,\mathrm{d}\omega. \qquad (3.1)
\end{aligned}$$

Here we used integration by parts with respect to p. Now observe the following property of the Hilbert transform:

Lemma 3.2.
$$\int f(x)\mathcal{H}g(x)\,\mathrm{d}x = -\int g(r)\mathcal{H}f(r)\,\mathrm{d}r.$$

Proof.

$$\int f(x)\mathcal{H}g(x)\,\mathrm{d}x = \int f(x)\left\{\int \frac{g(r)}{x-r}\,\mathrm{d}r\right\}\mathrm{d}x = \int\int \frac{f(x)g(r)}{x-r}\,\mathrm{d}r\,\mathrm{d}x$$
$$= -\int\int \frac{g(r)f(x)}{r-x}dx\,\mathrm{d}r = -\int g(r)\left\{\int \frac{f(x)}{r-x}\,\mathrm{d}x\right\}\mathrm{d}r = -\int g(r)\mathcal{H}f(r)\,\mathrm{d}r. \qquad \square$$

Thus, from the end of (3.1):

$$\begin{aligned}&\int\limits_{\mathbb{R}^2} f(x)g(x)\,\mathrm{d}x\\ &= -\frac{1}{4\pi}\int\limits_{\mathbb{S}^1}\int\limits_{-\infty}^{\infty} \operatorname{Re} e^{-h}\mathcal{H}\{e^{h}R_a f(\omega,p)\}\cdot R_a(\partial_{\underline{\omega}}g)(\omega,p)\,\mathrm{d}p\,\mathrm{d}\omega\\ &= \frac{1}{4\pi}\int\limits_{\mathbb{S}^1}\int\limits_{-\infty}^{\infty} \operatorname{Re} e^{h}R_a f(\omega,p)\mathcal{H}\{e^{-h}R_a(\partial_{\underline{\omega}}g)(\omega,p)\}\,\mathrm{d}p\,\mathrm{d}\omega.\end{aligned}$$

This way both the differentiation and the Hilbert transform was “transferred” from the unknown f to the known g. The proof of Theorem 2.1 is completed.

Remark 3.3. If g is known to arbitrary precision, because for example it is of our choice, e.g. it can be constant on a big circular disc containing the support of f, then this computes the integral of f, which is of major concern in stereology. If $a = 0$, we deal with the classical Radon transform, and through the Fubini theorem, having the integral of f is trivial. But not so, if a is non-zero. To our knowledge, this is the first formula for that problem. Observe that we need no derivatives or Hilbert transform of $R_a f$. If g is known, then $R_a(\partial_\omega g)$ can be computed. So our formula is numerically stable and e.g. Monte Carlo and related methods can be applied.

Although our formula is “ideal” for the needs of stereology, for the same reason, due to its asymmetry in f and g it is not so for theoretical reasons, like extension of the attenuated Radon transform to L^2 spaces. Finding a useful symmetric version (except the obvious one, when we express g from $R_a g$ using Novikov’s formula) is still open.

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