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The structure of the unit group of the group algebras $\mathbb{F}_{3^k}D_{6n}$ and $\mathbb{F}_qD_{42}$

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Abstract. Let $\mathbb{F}_q$ be a finite field of order $q = p^k$ for some prime p and a positive integer k. In this article, we provide the structure of the unit group $\mathcal{U}(\mathbb{F}_{3^k}D_{6n})$ of the group algebra $\mathbb{F}_{3^k}D_{6n}$ when n is not divisible by 3. Also, a characterization of the unit group $\mathcal{U}(\mathbb{F}_qD_{42})$ of the group algebra $\mathbb{F}_qD_{42}$ has been provided for all the possible cases corresponding to different values of the characteristic p.

Keywords: group algebra, dihedral group, unit group, Wedderburn decomposition

AMS Subject Classification: 16U60, 16S34

1. Introduction

Let $\mathcal{U}(\mathbb{F}_qG)$ be the unit group of the group algebra $\mathbb{F}_qG$ of a group G over a finite field $\mathbb{F}_q$ of order $q = p^k$, for some prime p. For $H \lhd G$, one can extend the canonical homomorphism $\omega\colon G \to G/H$ to form an epimorphism $\omega'\colon \mathbb{F}_qG \to \mathbb{F}_q(G/H)$ which is defined by $\omega'(\sum_{g\in G} \alpha_g g) = \sum_{g\in G} \alpha_g \omega(g)$. Let $\Delta(G, H) = \mathrm{Ker}(\omega')$ and $J(\mathbb{F}_qG)$ be the Jacobson radical of $\mathbb{F}_qG$. The canonical involution $*\colon \mathbb{F}_qG \to \mathbb{F}_qG$ is defined by $(\sum_{g\in G} \alpha_g g)^* = \sum_{g\in G} \alpha_g g^{-1}$. The dihedral group of order $2n$ is represented by $D_{2n} = \langle r, s \mid r^n = s^2 = 1,\ rs = sr^{-1}\rangle$. For basic definitions and results, we

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refer to [12].

The structure of $\mathcal{U}(\mathbb{F}_qG)$ has been presented for many different groups G in [6, 8, 9, 13–16]. In [7], Kaur and Khan studied $\mathcal{U}(\mathbb{F}_{2^k}D_{2p})$ for prime p. Furthermore, the structure of $\mathcal{U}(\mathbb{F}_{2^k}D_{2n})$ for odd integers n was described by Makhijani and Sharma [10]. In [11], authors have provided characterizations of $\mathcal{U}(\mathbb{Z}D_8)$ and $\mathcal{U}(\mathbb{Z}D_{12})$. Creedon and Gildea [3, 4] provided the structures of $\mathcal{U}(\mathbb{F}_{3^k}D_6)$ and $\mathcal{U}(\mathbb{F}_{2^k}D_8)$ in terms of explicit extensions of elementary cyclic groups. The unitary units of some group algebras have been studied in [1, 2]. The description of $\mathcal{U}(\mathbb{F}_qG)$ for a non semi-simple group algebra $\mathbb{F}_qG$ is quite challenging.

In this paper, we aim to establish the structures of the unit groups $\mathcal{U}(\mathbb{F}_{3^k}D_{6n})$ and $\mathcal{U}(\mathbb{F}_qD_{42})$. Some associated useful results are listed in Section 2. The result related to $\mathcal{U}(\mathbb{F}_{3^k}D_{6n})$ is discussed in Section 3. Section 4 of this article identifies the structure of $\mathcal{U}(\mathbb{F}_qD_{42})$ for characteristic 2 by employing the result in [10]. Additionally, the characterization of $\mathcal{U}(\mathbb{F}_qD_{42})$ is established for the other two non semi-simple cases for $p = 3,\ 7$. Finally, we discuss the semi-simple case for $\mathbb{F}_qD_{42}$ and consequently describe $\mathcal{U}(\mathbb{F}_qD_{42})$ by means of the Wedderburn decomposition.

2. Preliminaries

If $p = 2$, then from [10] we get a generalized result given as follows:

Lemma 2.1 ([10], Theorem 3.2). *Let $q = 2^k$ and n be a positive odd integer. Then,*

$$\mathcal{U}(\mathbb{F}_qD_{2n}) \cong C_2^k \times C_{q-1} \times \prod_{d|n, d>1} GL(2, \mathbb{F}_{q^{c_d}})^{\frac{\phi(d)}{2c_d}}$$

where ϕ is the Euler totient function,

$$c_d = \begin{cases} \frac{b_d}{2}, & \text{if } b_d \text{ is even and } q^{\frac{b_d}{2}} \equiv -1 \bmod d; \\ b_d, & \text{otherwise} \end{cases}$$

and b_d is the multiplicative order of q under mod d.

We recall a useful result from [12, Proposition 3.6.11] to determine the Wedderburn decomposition of semi-simple group algebras which states that if $\mathbb{F}_qG$ is semi-simple, then

$$\mathbb{F}_qG \cong \mathbb{F}_q(G/G') \bigoplus \Delta(G, G')$$

where $\mathbb{F}_q(G/G')$ is the sum of all the commutative simple components of $\mathbb{F}_qG$ and $\Delta(G, G')$ is the sum of all others.

In order to describe the structure of $\mathbb{F}_qG/J(\mathbb{F}_qG)$, we utilize some results given by Ferraz [5]. Let G be a finite group. An element $g \in G$ is said to be a p'-element if the order of g is not divisible by p. Let A be the set of all p'-elements in G and e be the l.c.m. of the orders of all the elements in A. Let ξ be the primitive e-th root of unity over $\mathbb{F}_q$. Define the set

$$B = \{t \mid \xi \to \xi^t \text{ is an automorphism of } \mathbb{F}_q(\xi) \text{ over } \mathbb{F}_q\}.$$

Then $B = \{1, q, \ldots, q^{x-1}\} \bmod e$, where x is the multiplicative order of $q \bmod e$. Let $g \in G$ be a p'-element and β_g be the sum of all conjugates of g. The cyclotomic $\mathbb{F}_q$-class of β_g is defined by

$$S(\beta_g) = \{\beta_{g^t} \mid t \in B\}.$$

We use the above description and the following two results to characterize $\mathcal{U}(\mathbb{F}_q D_{42})$ when $\mathbb{F}_q D_{42}$ is semi-simple.

Lemma 2.2 ([5])**.** *The number of cyclotomic $\mathbb{F}_q$-classes in G is equal to the number of simple components of $\mathbb{F}_q G/J(\mathbb{F}_q G)$.*

Lemma 2.3 ([5])**.** *Let t be the number of cyclotmic $\mathbb{F}_q$-classes in G and ξ be the same as defined above. If $S_1, \ldots, S_t$ are the cyclotomic $\mathbb{F}_q$-classes in G and $P_1, \ldots, P_t$ are the simple components of the center of $\mathbb{F}_q G/J(\mathbb{F}_q G)$, then an appropriate ordering of the indices gives $|S_i| = [P_i : \mathbb{F}_q]$.*

3. The structure of $\mathcal{U}(\mathbb{F}_{3^k} D_{6n})$

Theorem 3.1. *Let $\mathbb{F}_q$ be a finite field of order $q = 3^k$ and n be a positive integer not divisible by 3. Then,*

$$\mathcal{U}(\mathbb{F}_q D_{6n}) \cong ((\cdots(C_3^{3nk} \rtimes \underbrace{C_3^k) \rtimes C_3^k) \rtimes \cdots \rtimes C_3^k}_{n \text{ times}}) \rtimes \mathcal{U}(\mathbb{F}_q D_{2n}).$$

Proof. Let $G = D_{6n}$ and $N = \langle r^n \rangle$. Then, $N \lhd G$ and $G/N \cong \langle r^3, s \rangle$. Let $K = \langle r^3, s \rangle$ and define a ring epimorphism $\phi \colon \mathbb{F}_q G \to \mathbb{F}_q K$ by

$$\phi\left(\sum_{j=0}^{n-1} \sum_{i=0}^{2} r^{ni+3j}(x_{i+3j} + x_{i+3j+3n}s)\right) = \sum_{j=0}^{n-1} \sum_{i=0}^{2} r^{3j}(x_{i+3j} + x_{i+3j+3n}s).$$

By restricting the map ϕ, we find a group epimorphism $\phi' \colon \mathcal{U}(\mathbb{F}_q G) \to \mathcal{U}(\mathbb{F}_q K)$. The inclusion map from $\mathbb{F}_q K \to \mathbb{F}_q G$ is a ring monomorphism. Restricting this map, we get a group monomorphism $\theta \colon \mathcal{U}(\mathbb{F}_q K) \to \mathcal{U}(\mathbb{F}_q G)$ given by

$$\theta\left(\sum_{i=0}^{n-1} r^{3i}(z_i + z_{i+n}s)\right) = \sum_{i=0}^{n-1} r^{3i}(z_i + z_{i+n}s).$$

Observe that $\phi' \circ \theta = 1_{\mathcal{U}(\mathbb{F}_q K)}$ and hence, $\mathcal{U}(\mathbb{F}_q G) \cong S \rtimes \mathcal{U}(\mathbb{F}_q K)$ where $S = \mathrm{Ker}(\phi')$.

Let $u = \sum_{j=0}^{n-1} \sum_{i=0}^{2} r^{ni+3j}(x_{i+3j} + x_{i+3j+3n}s) \in S$. Then, $\phi'(u) = 1$. Solving this, we obtain the following equations:

$$\begin{aligned} &x_0 + x_1 + x_2 = 1,\ x_{3m} + x_{3m+1} + x_{3m+2} = 0 && \text{for } m = 1, \ldots, 2n-1. \\ \Longrightarrow\ &x_0 = 1 - x_1 - x_2,\ x_{3m} = -x_{3m+1} - x_{3m+2} && \text{for } m = 1, \ldots, 2n-1. \end{aligned}$$

In view of this, the set S can be equivalently written as

$$S=\left\{1+\sum_{j=0}^{n-1}\sum_{i=1}^{2}(r^{ni}-1)r^{3j}(y_{i+2j}+y_{i+2j+2n}s)\mid y_i\in\mathbb{F}_q\right\}.$$

It is trivial to check that S is a non-abelian group and that $S^3=1$. Since $q=3^k$, therefore $|S|=3^{4nk}$. Assume that $C(r^n)$ is the centralizer of r^n in S. Then,

$$C(r^n)=\{u\in S\mid ur^n=r^nu\}.$$

Let $u=1+\sum_{j=0}^{n-1}\sum_{i=1}^{2}(r^{ni}-1)r^{3j}(y_{i+2j}+y_{i+2j+2n}s)\in C(r^n)$. Then,

$$ur^n-r^nu=\sum_{j=0}^{n-1}\sum_{i=1}^{2}(r^{ni}-1)r^{3j+2n}y_{i+2j+2n}s-\sum_{j=0}^{n-1}\sum_{i=1}^{2}(r^{ni}-1)r^{3j+n}y_{i+2j+2n}s.$$

We get,

$$ur^n-r^nu=\hat{r^n}\sum_{i=0}^{n-1}r^{3i}(y_{2i+2n+1}-y_{2i+2n+2})s.$$

This results in the following condition

$$ur^n-r^nu=0 \text{ if and only if } y_{2i+2n+1}=y_{2i+2n+2}\quad \text{for } i=0,1,\dots,n-1.$$

In conclusion,

$$C(r^n)=\left\{1+\sum_{j=0}^{n-1}\sum_{i=1}^{2}(r^{ni}-1)r^{3j}h_{i+2j}+\hat{r^n}\sum_{i=0}^{n-1}r^{3i}h_{i+2n+1}s\mid h_i\in\mathbb{F}_q\right\}.$$

Let us consider some subgroups of S which are given by:

$$N_m=\{1+a_1\hat{r^n}+a_2(r^n+2r^{2n})r^{3m}s\mid a_i\in\mathbb{F}_q\}\quad \text{for } m=0,1,\dots,n-1,$$

and $W_0=C(r^n)$, $W_n=S$,

$$\begin{aligned}W_m=\Bigg\{1&+\sum_{i=1}^{2}(r^{ni}-1)\left(\sum_{j=0}^{n-1}r^{3j}h_{i+2j}+\sum_{j=0}^{m-1}r^{3j}h_{i+2j+2n}s\right)\\&+\hat{r^n}\sum_{i=m}^{n-1}r^{3i}h_{i+m+2n+1}s\mid h_i\in\mathbb{F}_q\Bigg\}\quad \text{for } m=1,\dots,n-1.\end{aligned}$$

Clearly N_m and W_m are subgroups of W_{m+1} and $I=N_m\cap W_m=\{1+a_1\hat{r^n}\mid a_1\in\mathbb{F}_q\}\cong C_3^k$, for $m=0,1,\dots,n-1$. Furthermore, N_m is an abelian group and therefore $N_m=I\times Q_m$ for some subgroup Q_m of N_m such that $Q_m\cong C_3^k$, for $m=0,1,\dots,n-1$. We consider the following general elements

$$v_m=1+a_1\hat{r^n}+a_2(r^n+2r^{2n})r^{3m}s\in N_m\quad \text{for } m=0,\dots,n-1,$$

$$u_0 = 1 + \sum_{j=0}^{n-1}\sum_{i=1}^{2}(r^{ni}-1)r^{3j}h_{i+2j} + \hat{r^n}\sum_{i=0}^{n-1} r^{3i}h_{i+2n+1}s \in W_0,$$

$$u_m = 1 + \sum_{i=1}^{2}(r^{ni}-1)(\sum_{j=0}^{n-1} r^{3j}h_{i+2j} + \sum_{j=0}^{m-1} r^{3j}h_{i+2j+2n}s)$$
$$+ \hat{r^n}\sum_{i=m}^{n-1} r^{3i}h_{i+m+2n+1}s \in W_m \quad \text{for } m = 1, \dots, n-1.$$

Let us define

$$H_1 = \sum_{j=0}^{n-1}\sum_{i=1}^{2}(r^{ni}-1)r^{3j}h_{i+2j},$$

$$H_{2,0} = 0,\ H_{2,m} = \sum_{j=0}^{m-1}\sum_{i=1}^{2}(r^{ni}-1)r^{3j}h_{i+2j+2n} \quad \text{for } m = 1, \dots, n-1,$$

$$H_{3,m} = \hat{r^n}\sum_{i=m}^{n-1} r^{3i}h_{i+m+2n+1} \quad \text{for } m = 0, \dots, n-1.$$

Then, we can write

$$u_m = 1 + H_1 + H_{2,m}s + H_{3,m}s \in W_m \quad \text{for } m = 0, \dots, n-1.$$

Since $N_m \subseteq S$, therefore $N_m^3 = 1$. Hence, for $v_m \in N_m$, we have

$$v_m^{-1} = v_m^2 = 1 + 2(a_1 + a_2^2)\hat{r^n} + 2a_2(r^n + 2r^{2n})r^{3m}s \quad \text{for } m = 0, \dots, n-1.$$

The aforementioned information combined with the following steps help to deduce the structure of S.

Step 1: Taking $u_0 \in W_0$ and $v_0 \in N_0$, we have

$$\begin{aligned} u_0^{v_0} &= v_0^{-1}u_0v_0 \\ &= u_0 + a_2(H_1 - H_1^*)(r^n + 2r^{2n})s \in W_0. \end{aligned}$$

In conclusion, N_0 normalizes W_0. It is trivial to show that W_0 is abelian and therefore, $W_0 \cong C_3^{3nk}$. Clearly, $W_0 \cap Q_0 = \{1\}$. Hence, $W_1 \cong W_0 \rtimes Q_0 \cong C_3^{3nk} \rtimes C_3^k$.

Step 2: Taking $u_1 \in W_1$ and $v_1 \in N_1$, we have

$$\begin{aligned} u_1^{v_1} &= v_1^{-1}u_1v_1 \\ &= u_1 + a_2(H_1 - H_1^*)(r^n + 2r^{2n})r^3s \\ &\quad + a_2(H_{2,1}(r^{2n} + 2r^n)r^{-3} - H_{2,1}^*(r^n + 2r^{2n})r^3) \in W_1. \end{aligned}$$

It is concluded that N_1 normalizes W_1. Clearly, $W_1 \cap Q_1 = \{1\}$. Hence, $W_2 \cong W_1 \rtimes Q_1 \cong (C_3^{3nk} \rtimes C_3^k) \rtimes C_3^k$. Consequently, it can be shown that

$$u_m^{v_m} = u_m + a_2(H_1 - H_1^*)(r^n + 2r^{2n})r^{3m}s$$

$$+ a_2(H_{2,m}(r^{2n}+2r^n)r^{-3m} - H^*_{2,m}(r^n+2r^{2n})r^{3m}) \in W_m$$

for $m = 0, \ldots, n-1$.

The succeeding steps can be concluded by following a similar process to obtain that N_m normalizes W_m and therefore $W_{m+1} \cong W_m \rtimes Q_m$ for $m = 2, \ldots, n-1$. Finally, we get $W_n \cong W_{n-1} \rtimes Q_{n-1}$, that is

$$S \cong ((\cdots(C_3^{3nk} \rtimes \underbrace{C_3^k) \rtimes C_3^k) \rtimes \cdots \rtimes C_3^k}_{n \text{ times}}).$$

Moreover, since $K \cong D_{2n}$, we get

$$\mathcal{U}(\mathbb{F}_q D_{6n}) \cong ((\cdots(C_3^{3nk} \rtimes \underbrace{C_3^k) \rtimes C_3^k) \rtimes \cdots \rtimes C_3^k}_{n \text{ times}}) \rtimes \mathcal{U}(\mathbb{F}_q D_{2n}). \qquad \square$$

With the help of the above theorem, the characterization problem of unit groups of group algebras of dihedral groups is reduced to the unit groups of the group algebras of smaller dihedral groups.

4. The structure of $\mathcal{U}(\mathbb{F}_q D_{42})$

This section deals with the characterization of $\mathcal{U}(\mathbb{F}_q D_{42})$. The characterization is complete except in characteristic 7, for which we have partial results.

Theorem 4.1. *Let $\mathbb{F}_q$ be a finite field of order $q = p^k$ with characteristic p.*

1. *If* **Char** $\mathbb{F}_{\boldsymbol{q}} = \mathbf{2}$, *then* $\mathcal{U}(\mathbb{F}_q D_{42})$ *is isomorphic to*
   (i) $C_2^k \times C_{q-1} \times GL(2, \mathbb{F}_q)^{10}$ *if* $k \equiv 0 \bmod 6$.
   (ii) $C_2^k \times C_{q-1} \times GL(2, \mathbb{F}_q) \times GL(2, \mathbb{F}_{q^3}) \times GL(2, \mathbb{F}_{q^6})$ *if* $k \equiv \pm 1 \bmod 6$.
   (iii) $C_2^k \times C_{q-1} \times GL(2, \mathbb{F}_q) \times GL(2, \mathbb{F}_{q^3})^3$ *if* $k \equiv \pm 2 \bmod 6$.
   (iv) $C_2^k \times C_{q-1} \times GL(2, \mathbb{F}_q)^4 \times GL(2, \mathbb{F}_{q^2})^3$ *if* $k \equiv 3 \bmod 6$.
2. *If* **Char** $\mathbb{F}_{\boldsymbol{q}} = \mathbf{3}$, *then* $\mathcal{U}(\mathbb{F}_q D_{42})$ *is isomorphic to*
   (i) $S \rtimes (C_{q-1}^2 \times GL(2, \mathbb{F}_q)^3)$ *if* $q \equiv \pm 1 \bmod 7$,
   (ii) $S \rtimes (C_{q-1}^2 \times GL(2, \mathbb{F}_{q^3}))$ *if* $q \equiv \pm 2 \bmod 7$ *or* $q \equiv \pm 3 \bmod 7$
   where $S \cong (((((((C_3^{21k} \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k)$.
3. *If* **Char** $\mathbb{F}_{\boldsymbol{q}} = \mathbf{7}$, *then*

$$\mathcal{U}(\mathbb{F}_q D_{42}) \cong S \rtimes (\mathbb{F}_q^* \times \mathbb{F}_q^* \times GL(2, \mathbb{F}_q))$$

where S is a non-abelian group such that $|S| = 7^{36k}$ and $S^7 = 1$.

4. *If* **Char** $\mathbb{F}_q \neq 2, 3, 7$*, then* $\mathcal{U}(\mathbb{F}_q D_{42})$ *is isomorphic to*
   (*i*) $C_{q-1}^2 \times GL(2, \mathbb{F}_q)^{10}$ *if* $q \equiv 1, 41 \bmod 42$.
   (*ii*) $C_{q-1}^2 \times GL(2, \mathbb{F}_q) \times GL(2, \mathbb{F}_{q^3})^3$ *if* $q \equiv 5, 17, 25, 37 \bmod 42$.
   (*iii*) $C_{q-1}^2 \times GL(2, \mathbb{F}_q) \times GL(2, \mathbb{F}_{q^3}) \times GL(2, \mathbb{F}_{q^6})$ *if* $q \equiv 11, 19, 23, 31 \bmod 42$.
   (*iv*) $C_{q-1}^2 \times GL(2, \mathbb{F}_q)^4 \times GL(2, \mathbb{F}_{q^2})^3$ *if* $q \equiv 13, 29 \bmod 42$.

Proof. The structure of the unit group $\mathcal{U}(\mathbb{F}_q D_{42})$ differs based on the values of the characterstic p.

1. Char $\mathbb{F}_q = 2$: The structure of $\mathcal{U}(\mathbb{F}_q D_{2n})$ for $q = 2^k$ and an odd integer n has been given by the formula in Lemma 2.1, which depends on the value of q as well. In this article, the structure of $\mathcal{U}(\mathbb{F}_q D_{42})$ is being categorized into four cases based on the values of k upto mod 6. The divisors of 21, which are greater than 1, are 3, 7 and 21. By using Lemma 2.1 for different values of k upto mod 6, we get the following results.
(**a**) If $k \equiv 0 \bmod 6$, then $c_3 = c_7 = c_{21} = 1$ and hence, $\mathcal{U}(\mathbb{F}_q D_{42})$ is isomorphic to

$$C_2^k \times C_{q-1} \times GL(2, \mathbb{F}_q)^{10}.$$

(**b**) If $k \equiv \pm 1 \bmod 6$, then $c_3 = 1,\ c_7 = 3,\ c_{21} = 6$ which gives that $\mathcal{U}(\mathbb{F}_q D_{42})$ is isomorphic to

$$C_2^k \times C_{q-1} \times GL(2, \mathbb{F}_q) \times GL(2, \mathbb{F}_{q^3}) \times GL(2, \mathbb{F}_{q^6}).$$

(**c**) If $k \equiv \pm 2 \bmod 6$, then $c_3 = 1,\ c_7 = c_{21} = 3$ and hence, $\mathcal{U}(\mathbb{F}_q D_{42})$ is isomorphic to

$$C_2^k \times C_{q-1} \times GL(2, \mathbb{F}_q) \times GL(2, \mathbb{F}_{q^3})^3.$$

(**d**) If $k \equiv 3 \bmod 6$, then $c_3 = c_7 = 1,\ c_{21} = 2$ and it can be concluded that $\mathcal{U}(\mathbb{F}_q D_{42})$ is isomorphic to

$$C_2^k \times C_{q-1} \times GL(2, \mathbb{F}_q)^4 \times GL(2, \mathbb{F}_{q^2})^3.$$

2. Char $\mathbb{F}_q = 3$: In particular, using Theorem 3.1 for $n = 7$, we obtain

$$\mathcal{U}(\mathbb{F}_q D_{42}) \cong S \rtimes \mathcal{U}(\mathbb{F}_q D_{14})$$

where

$$S \cong (((((((C_3^{21k} \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k).$$

Moreover, on the lines of [14, Theorem 4.1], we get

$$\mathcal{U}(\mathbb{F}_q D_{14}) \cong \begin{cases} C_{q-1}^2 \times GL(2, \mathbb{F}_q)^3, & \text{if } q \equiv \pm 1 \bmod 7; \\ C_{q-1}^2 \times GL(2, \mathbb{F}_{q^3}), & \text{if } q \equiv \pm 2 \bmod 7 \text{ or } q \equiv \pm 3 \bmod 7. \end{cases}$$

Hence,

$$\mathcal{U}(\mathbb{F}_q D_{42}) \cong \begin{cases} S \rtimes (C_{q-1}^2 \times GL(2,\mathbb{F}_q)^3), & \text{if } q \equiv \pm 1 \bmod 7; \\ S \rtimes (C_{q-1}^2 \times GL(2,\mathbb{F}_{q^3})), & \text{if } q \equiv \pm 2 \bmod 7 \text{ or } q \equiv \pm 3 \bmod 7 \end{cases}$$

where $S \cong (((((((C_3^{21k} \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k)$.

3. Char $\mathbb{F}_q = 7$: Let $G = D_{42}$ and $N = \langle r^3 \rangle$. Then, $N \lhd G$ and $G/N \cong \langle r^7, s\rangle \cong D_6$. Let $K = \langle r^7, s\rangle$ and $\phi\colon \mathbb{F}_q G \to \mathbb{F}_q K$ be the ring epimorphism defined by

$$\phi\Bigg(\sum_{j=0}^{2}\sum_{i=0}^{6} r^{3i+7j}(x_{i+7j} + x_{i+7j+21}s)\Bigg) = \sum_{j=0}^{2}\sum_{i=0}^{6} r^{7j}(x_{i+7j} + x_{i+7j+21}s).$$

By restricting the map ϕ, we find a group epimorphism $\phi'\colon \mathcal{U}(\mathbb{F}_q G) \to \mathcal{U}(\mathbb{F}_q K)$. The inclusion map from $\mathbb{F}_q K \to \mathbb{F}_q G$ is a ring monomorphism. A group monomorphism $\theta\colon \mathcal{U}(\mathbb{F}_q K) \to \mathcal{U}(\mathbb{F}_q G)$ is obtained by restricting this inclusion map which is defined by

$$\theta\Bigg(\sum_{i=0}^{2} r^{7i}(z_i + z_{i+3}s)\Bigg) = \sum_{i=0}^{2} r^{7i}(z_i + z_{i+3}s).$$

Observe that $\phi' \circ \theta = 1_{\mathcal{U}(\mathbb{F}_q K)}$ and hence, $\mathcal{U}(\mathbb{F}_q G) \cong S \rtimes \mathcal{U}(\mathbb{F}_q K) \cong S \rtimes \mathcal{U}(\mathbb{F}_q D_6)$ where $S = \mathrm{Ker}(\phi')$.

Let $u = \sum_{j=0}^{2}\sum_{i=0}^{6} r^{3i+7j}(x_{i+7j} + x_{i+7j+21}s) \in S$. Then, $\phi'(u) = 1$. This results in the following equations:

$$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1,$$
$$x_{7m} + x_{7m+1} + x_{7m+2} + x_{7m+3} + x_{7m+4} + x_{7m+5} + x_{7m+6} = 0$$
$$\text{for } m = 1, \dots, 5.$$

Hence, $S = \{1 + \sum_{j=0}^{2}\sum_{i=1}^{6} (r^{3i} - 1) r^{7j}(y_{i+6j} + y_{i+6j+18}s) \mid y_i \in \mathbb{F}_q\}$. It is clear that S is a non-abelian group and that $S^7 = 1$. Since $q = 7^k$, therefore $|S| = 7^{36k}$. From [16, Theorem 2.3] we get that $\mathcal{U}(\mathbb{F}_q D_6) \cong \mathbb{F}_q^* \times \mathbb{F}_q^* \times GL(2,\mathbb{F}_q)$ for $p > 3$. Hence,

$$\mathcal{U}(\mathbb{F}_q G) \cong S \rtimes (\mathbb{F}_q^* \times \mathbb{F}_q^* \times GL(2,\mathbb{F}_q)).$$

4. Char $\mathbb{F}_q \neq 2, 3, 7$: Pertaining to this case, $\mathbb{F}_q D_{42}$ is a semi-simple group algebra by Maschke's theorem and hence $J(\mathbb{F}_q D_{42}) = (0)$. Then,

$$\mathbb{F}_q D_{42} \cong \mathbb{F}_q(D_{42}/D_{42}') \bigoplus \Delta(D_{42}, D_{42}').$$

As $D_{42}/D_{42}' \cong C_2$, then $\mathbb{F}_q(D_{42}/D_{42}') \cong \mathbb{F}_q C_2 \cong \mathbb{F}_q \bigoplus \mathbb{F}_q$. Therefore, the Wedderburn decomposition is

$$\mathbb{F}_q D_{42} \cong \mathbb{F}_q \bigoplus \mathbb{F}_q \bigoplus_{j=1}^{m} M(n_j, R_j)$$

where $n_j \geq 2$ and R_j's are division algebras over the finite field $\mathbb{F}_q$ for $j \in \{1, \ldots, m\}$.

The conjugacy classes of D_{42} are: $\{1\}$, $\{r^{\pm 1}\}$, $\ldots$, $\{r^{\pm 10}\}$, $\{s, rs, \ldots, r^{20}s\}$. Since the class sums form a basis for $Z(\mathbb{F}_q D_{42})$, therefore $\dim(Z(\mathbb{F}_q D_{42}))$ = number of conjugacy classes of $D_{42} = 12$. Hence, $m \leq 10$. Clearly, for the given characterstic p, we obtain e = l.c.m. of the orders of all the p'-elements in $D_{42} = 42$. (**a**) If $q \equiv 1, 41 \bmod 42$, then $B = \{1\} \bmod 42$ or $B = \{1, 41\} \bmod 42$. From this, we get $|S(\beta_g)| = 1$ for all $g \in G$. Then, by Lemma 2.2 and Lemma 2.3, we deduce that

$$\mathbb{F}_q D_{42} \cong \mathbb{F}_q \bigoplus \mathbb{F}_q \bigoplus_{j=1}^{10} M(n_j, \mathbb{F}_q).$$

After computing the dimension of both sides, we get the equation $\sum_{j=1}^{10} n_j^2 = 40$, which is only possible when $n_j = 2$ for all $j \in \{1, \ldots, 10\}$. Hence,

$$\mathbb{F}_q D_{42} \cong \mathbb{F}_q \bigoplus \mathbb{F}_q \bigoplus M(2, \mathbb{F}_q)^{10}.$$

(**b**) If $q \equiv 5, 17, 25, 37 \bmod 42$, then $B = \{1, 5, 17, 25, 37, 41\} \bmod 42$ or $B = \{1, 25, 37\} \bmod 42$. This gives $|S(\beta_g)| = 1$ for $g = 1, r^7, s$, and $|S(\beta_g)| = 3$ for $g = r, r^2, r^3$. Then, by Lemma 2.2 and Lemma 2.3, we can conclude that

$$\mathbb{F}_q D_{42} \cong \mathbb{F}_q \bigoplus \mathbb{F}_q \bigoplus M(n_1, \mathbb{F}_q) \bigoplus_{j=2}^{4} M(n_j, \mathbb{F}_{q^3}),$$

with the constraint $n_1^2 + 3n_2^2 + 3n_3^2 + 3n_4^2 = 40$. The only such possibility is $n_j = 2$ for all $j \in \{1, \ldots, 4\}$. Hence,

$$\mathbb{F}_q D_{42} \cong \mathbb{F}_q \bigoplus \mathbb{F}_q \bigoplus M(2, \mathbb{F}_q) \bigoplus M(2, \mathbb{F}_{q^3})^3.$$

(**c**) If $q \equiv 11, 19, 23, 31 \bmod 42$, then $B = \{1, 11, 23, 25, 29, 37\} \bmod 42$ or $B = \{1, 13, 19, 25, 31, 37\} \bmod 42$. Thus, $|S(\beta_g)| = 1$ for $g = 1, r^7, s$, $|S(\beta_g)| = 3$ for $g = r^3$, and $|S(\beta_g)| = 6$ for $g = r$. Then, following Lemma 2.2 and Lemma 2.3, the Wedderburn decomposition is

$$\mathbb{F}_q D_{42} \cong \mathbb{F}_q \bigoplus \mathbb{F}_q \bigoplus M(n_1, \mathbb{F}_q) \bigoplus M(n_2, \mathbb{F}_{q^3}) \bigoplus M(n_3, \mathbb{F}_{q^6}),$$

subject to the constraint $n_1^2 + 3n_2^2 + 6n_3^2 = 40$. The equation is satisfied only when $n_j = 2$ for all $j \in \{1, 2, 3\}$. Hence,

$$\mathbb{F}_q D_{42} \cong \mathbb{F}_q \bigoplus \mathbb{F}_q \bigoplus M(2, \mathbb{F}_q) \bigoplus M(2, \mathbb{F}_{q^3}) \bigoplus M(2, \mathbb{F}_{q^6}).$$

(**d**) If $q \equiv 13, 29 \bmod 42$, then $B = \{1, 13\} \bmod 42$ or $B = \{1, 29\} \bmod 42$. This gives $|S(\beta_g)| = 1$ for $g = 1, r^3, r^6, r^7, r^9, s$, and $|S(\beta_g)| = 2$ for $g = r, r^2, r^4$. Then Lemma 2.2 and Lemma 2.3 guarantees that

$$\mathbb{F}_q D_{42} \cong \mathbb{F}_q \bigoplus \mathbb{F}_q \bigoplus_{j=1}^{4} M(n_j, \mathbb{F}_q) \bigoplus_{j=5}^{7} M(n_j, \mathbb{F}_{q^2}),$$

with the constraint $\sum_{j=1}^{4} n_j^2 + \sum_{j=5}^{7} 2n_j^2 = 40$. The only such possibility is $n_j = 2$ for all $j \in \{1, \ldots, 7\}$. Hence,

$$\mathbb{F}_q D_{42} \cong \mathbb{F}_q \bigoplus \mathbb{F}_q \bigoplus M(2, \mathbb{F}_q)^4 \bigoplus M(2, \mathbb{F}_{q^2})^3.$$

For every case (**a**)–(**d**) discussed above, the structure of $\mathcal{U}(\mathbb{F}_q D_{42})$ is a direct implication of the obtained Wedderburn decomposition of $\mathbb{F}_q D_{42}$. □

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