Annales Mathematicae et Informaticae
Accepted manuscript
DOI: 10.33039/ami.2024.03.006
URL: https://ami.uni-eszterhazy.hu

An identity for two sequences and its combinatorial interpretation

Yahia Djemmada[a], **Abdelghani Mehdaoui**[a], **László Németh**[b*†], **László Szalay**[c†‡]

[a]National Higher School of Mathematics, Sidi Abdellah, Algiers, Algeria
yahia.djem@gmail.com and mehabdelghani@gmail.com

[b]Institute of Basic Sciences, Departement of Mathematics
University of Sopron, Hungary
nemeth.laszlo@uni-sopron.hu

[c]Institute of Basic Sciences, Departement of Mathematics
University of Sopron, Hungary
and Department of Mathematics, J. Selye University, Slovakia
szalay.laszlo@uni-sopron.hu

Abstract. We recall a theorem on linear recurrences that we have already proved earlier, and we use it to provide new identities. The nature of the new result allows us to combine two linear recurrences of distinct order in the identity if they satisfy some prescribed conditions about their similarity. For example, we found a rule including consecutive k- and ℓ-generalized Fibonacci numbers. In addition, a combinatorial interpretation is explained if the coefficients of the recurrences are positive integers.

Keywords: linear recurrence, combinatorial interpretation, generalized Fibonacci number, generalized Pell number

AMS Subject Classification: 11B37, 11B39, 05A19, 05B45

*Corresponding author.

†For these authors the research was supported in part by National Research, Development and Innovation Office Grant 2019-2.1.11-TÉT-2020-00165.

‡For the author the research was also supported by Hungarian National Foundation for Scientific Research Grant No. 128088, and No. 130909.

Submitted: July 23, 2023
Accepted: March 4, 2024
Published online: April 19, 2024

1. Introduction

Suppose k is a positive integer, and $f_0, f_1, \ldots, f_{k-1}$ are complex numbers. Define

$$f_n = A_1 f_{n-1} + A_2 f_{n-2} + \cdots + A_k f_{n-k} \quad (n \geq k), \tag{1.1}$$

where the coefficients $A_1, \ldots, A_{k-1}$, and $A_k \neq 0$ are fixed complex numbers. Moreover, suppose that $(w_n)_{n\geq 0} \in \mathbb{C}^\infty$ is an arbitrary sequence. Based on the notation above, we construct the linear recurrence

$$G_n = A_1 G_{n-1} + A_2 G_{n-2} + \cdots + A_k G_{n-k} + w_{n-k} \quad (n \geq k), \tag{1.2}$$

assuming that the complex initial values $G_0, G_1, \ldots, G_{k-1}$ are also given. Note that formulae (1.1) and (1.2) differ essentially only in the term w_n.

Belbachir et al. [1] studied the connection between the sequences $(G_n)_{n\geq 0}$, $(f_n)_{n\geq 0}$ and $(w_n)_{n\geq 0}$, and proved the following general result.

Theorem 1.1. *For $n \geq k$, the terms of the sequences (f_n), (w_n) and (G_n) satisfy the identity*

$$\begin{aligned}\sum_{j=0}^{k-1} f_j G_{n+k-j} = &\sum_{j=0}^{k-1}\sum_{i=0}^{k-1-j} f_{n-j} A_{j+1+i} G_{k-1-i}\\ &+ \sum_{j=0}^{k-2}\sum_{i=1}^{k-1-j} f_j A_i G_{n+k-j-i} + \sum_{j=0}^{n} f_{n-j} w_j.\end{aligned} \tag{1.3}$$

The theorem is valid also for $k = 1$ (with an empty sum of the three on the right-hand side), but this case is not of much interest. Hence, we may suppose $k \geq 2$. Observe that the terms $A_{j+1+i}G_{k-1-i}$ and $f_j A_i$ on the right-hand side of (1.3) can take only finitely many values. Moreover, note that Theorem 1.1 is obviously true for arbitrary initial values of the sequence (f_n). Coefficients $A_1, \ldots, A_k$ in the definition of (f_n) are important in the sense that they, together with (w_n) also establish the sequence (G_n). But, generally, the initial values $f_0, \ldots, f_{k-1}$ can be chosen arbitrarily. Therefore, it is natural, if there is no other reason, to put $f_0 = \cdots = f_{k-1} = 0$, $f_{k-1} = 1$. The next corollary simplifies Theorem 1.1 as it describes this situation. (See [1] again.)

Corollary 1.2. *Assume that $f_0 = \cdots = f_{k-2} = 0$, $f_{k-1} = 1$. Then (1.3) simplifies to*

$$G_{n+1} = \sum_{j=0}^{k-1}\sum_{i=0}^{k-1-j} f_{n-j} A_{j+1+i} G_{k-1-i} + \sum_{j=0}^{n} f_{n-j} w_j \quad (n \geq k). \tag{1.4}$$

If we even specify the initial values $G_0 = G_1 = \cdots = G_{k-1} = 0$ (and keep the former conditions $f_0 = \cdots = f_{k-2} = 0$, $f_{k-1} = 1$), then we have

Corollary 1.3. *Under the condition above,* (1.4) *admits*

$$G_{n+1} = \sum_{j=0}^{n} f_{n-j} w_j \quad (n \geq k). \tag{1.5}$$

This corollary was not mentioned in [1], but it is obviously a direct consequence of Theorem 1.1. An illustration of Corollary 1.3 stands here.

Example 1.4. Recall [1] again. Let $\ell \geq 3$ be an integer. Moreover, let $f_n = F_n$, the n^{th} term of the Fibonacci sequence (see The On-Line Encyclopedia of Integer Sequences [5], sequence A000045). Put $w_n = F_{n-1}^{(\ell)} + \cdots + F_{n-(\ell-2)}^{(\ell)}$. Here $(F_n^{(\ell)})$ is the ℓ-generalized Fibonacci sequence (Fibonacci ℓ-step numbers or shortly ℓ-nacci sequence) defined by the initial values $F_0^{(\ell)} = F_1^{(\ell)} = \cdots = F_{\ell-2}^{(\ell)} = 0$, $F_{\ell-1}^{(\ell)} = 1$, and by the recurrence relation

$$F_n^{(\ell)} = F_{n-1}^{(\ell)} + F_{n-2}^{(\ell)} + \cdots + F_{n-\ell}^{(\ell)} \quad (n \geq \ell).$$

We may also need to extend $(F_n^{(\ell)})$ for some terms with negative subscripts follows from the recurrence rule above when it is applied backward. Hence, $F_{-1}^{(\ell)} = 1$, $F_{-2}^{(\ell)} = -1$, $F_{-3}^{(\ell)} = 0$, and so on. Finally, we fix $G_0 = F_0^{(\ell)} = 0$ and $G_1 = F_1^{(\ell)} = 0$. In this manner, we construct the sequence (G_n), which is obviously the ℓ-generalized Fibonacci sequence itself. Thus, (1.5) provides the identity

$$F_{n+1}^{(\ell)} = \sum_{j=0}^{n} F_{n-j} \left(\sum_{i=1}^{\ell-2} F_{j-i}^{(\ell)} \right). \tag{1.6}$$

Suppose $\ell = 3$ to obtain the terms of the so-called Tribonacci sequence $(T_n) = (F_n^{(3)})$ A000073. Then we have

$$T_{n+1} = \sum_{j=0}^{n} F_{n-j} T_{j-1}.$$

This property is also given in [4], see (2.5) therein; moreover, see Benjamin and Quinn's book [2, p. 47, Exercise 4(a)].

The main purpose of this paper is to extend (1.6), and to give a combinatorial interpretation if the coefficients are positive integers.

2. Results

2.1. New corollaries of Theorem 1.1

Let $k \geq 2$ and $\ell > k$ be positive integers. Assume that the sequence (f_n) is given by the initial values

$$f_0 = \cdots = f_{k-2} = 0, \; f_{k-1} = 1, \tag{2.1}$$

and by the recursive scheme (1.1). The coefficients $A_1, \dots, A_k$ are fixed in (1.1). Define the recurrence (G_n) such that

$$G_n = A_1 G_{n-1} + \dots + A_k G_{n-k} + A_{k+1} G_{n-k-1} + \dots + A_\ell G_{n-\ell}. \tag{2.2}$$

Fix G_i for $i \in \{k-\ell, \dots, 0, \dots, k-1\}$, and put

$$w_n = A_{k+1} G_{n-1} + \dots + A_\ell G_{n+k-\ell} \quad \text{for } n \geq 0.$$

Clearly,

$$G_n = A_1 G_{n-1} + \dots + A_k G_{n-k} + w_{n-k} \quad \text{for } n \geq k,$$

and the situation fits the construction described in the Introduction. According to Corollary 1.2 we obtain

Theorem 2.1. *Using the notation above the identity*

$$G_{n+1} = \sum_{j=0}^{k-1} \sum_{i=0}^{k-1-j} f_{n-j} A_{j+1+i} G_{k-1-i} + \sum_{j=0}^{n} f_{n-j} \left(\sum_{i=1}^{\ell-k} A_{k+i} G_{j-i} \right)$$

follows.

In particular, we specify Corollary 1.3 in Theorem 2.2.

Theorem 2.2. *The condition $G_0 = \dots = G_{k-1} = 0$ leads to*

$$G_{n+1} = \sum_{j=0}^{n} f_{n-j} \left(\sum_{i=1}^{\ell-k} A_{k+i} G_{j-i} \right). \tag{2.3}$$

2.2. Combinatorial explanation of Theorem 2.2

Consider the domino tilings of a $1 \times h$ chessboard with 1×1, 1×2, $\dots$, $1 \times \ell$ dominoes having A_1, A_2, $\dots$, A_ℓ different colors, respectively. Let C_h denote the total number of tilings, and K_h the number of tilings if the maximal length of the dominoes we can use is k, where $k < \ell$.

Since the length of the last domino in the tiling is one of $1, 2, \dots, \ell$, the total number of ways to tile is

$$C_h = A_1 C_{h-1} + \dots + A_\ell C_{h-\ell}.$$

The comparison of the initial values of sequences (C_n) and (G_n) provides the equality $C_h = G_{h+l-1}$. Indeed, we extend the initial conditions $G_0 = \dots = G_{k-1} = 0$ with $G_k = \dots = G_{\ell-2} = 0$, $G_{\ell-1} = 1$ in the recurrence (2.2). So $C_0 = G_{\ell-1} = 1$, $C_1 = G_\ell = A_1$, $C_2 = G_{\ell+1} = A_1^2 + A_2$, and so on.

Now we examine another approach to calculate the number of tilings. In order to do that, first, we notify that $K_j = f_{j+k-1}$ gives the number of tilings of a $1 \times j$

chessboard with 1×1, 1×2, $\ldots$, $1\times k$ dominoes using the given colors. (It similarly follows from (1.1) and (2.1).)

Assume that the first j positions of the chessboard are tiled under the restriction of maximal length k, and then the next domino has size either $1\times(k+1)$ or $1\times(k+2)$ or, and so on, or $1\times\ell$. For the remaining part of the chessboard, we can use any dominoes with maximum length ℓ (for illustration, see Figure 1). Thus,

$$C_h = \sum_{j=0}^{h} K_j \left(\sum_{i=k+1}^{\ell} A_i C_{h-j-i} \right).$$

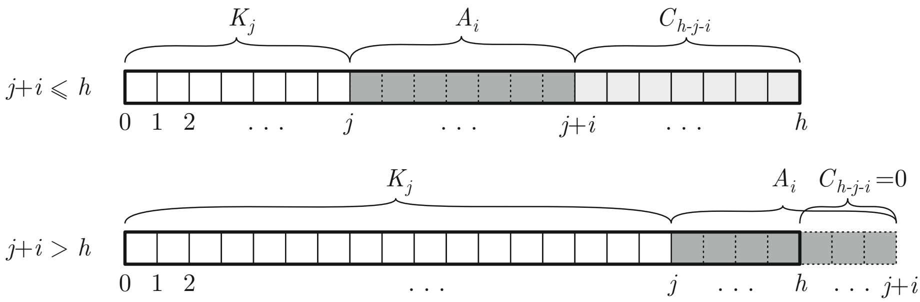

Figure 1. Chessboard and tilings.

Now we return to the sequences (G_n) and (f_n). Clearly,

$$G_{h+\ell-1} = \sum_{j=0}^{h} f_{j+k-1} \left(\sum_{i=k+1}^{\ell} A_i G_{h+\ell-1-j-i} \right).$$

If $h = n-\ell+2$, then we have

$$\begin{aligned} G_{n+1} &= \sum_{j=0}^{n-\ell+2} f_{j+k-1} \left(\sum_{i=k+1}^{\ell} A_i G_{n+1-j-i} \right) \\ &= \sum_{j=0}^{n-\ell+2} f_{n-(\ell-k-1)-j} \left(\sum_{i=k+1}^{\ell} A_i G_{j+(\ell-1)-i} \right) \\ &= \sum_{j=0}^{n-\ell+2} f_{n-(\ell-k-1)-j} \left(\sum_{i=1}^{\ell-k} A_{i+k} G_{j+(\ell-k-1)-i} \right) \\ &= \sum_{j=\ell-k-1}^{n-k+1} f_{n-j} \left(\sum_{i=1}^{\ell-k} A_{i+k} G_{j-i} \right). \end{aligned}$$

In the above equalities, first we used the swap $j \leftrightarrow h-j$, and then certain re-indexing.

Observe that the range of the first sum can be extended from $n-k+1$ to n. Indeed, the new coefficients $f_{k-2}, \dots, f_0$ are all zero. Similarly, we can modify the sum by reducing the lower value from $\ell-k-1$ to 0 because for such j, the sum

$$\sum_{i=1}^{\ell-k} A_{i+k} G_{j-i}$$

vanishes. Finally, we have obtained

$$G_{n+1} = \sum_{j=0}^{n} f_{n-j} \left(\sum_{i=1}^{\ell-k} A_{i+k} G_{j-i} \right),$$

which is identical to formula (2.3) given in Theorem 2.2.

2.3. Formula for ℓ-generalized Fibonacci sequences

Let all the coefficients A_i be equal to 1. Writing the usual notation of k- and ℓ-generalized Fibonacci sequences (as in Example 1.4) formula (2.3) yields

$$F_{n+1}^{(\ell)} = \sum_{j=0}^{n} F_{n-j}^{(k)} \sum_{i=1}^{\ell-k} F_{j-i}^{(\ell)}, \quad (2 \le k < \ell).$$

This identity extends (1.6) of Example 1.4.

2.4. Formula for ℓ-generalized Pell sequences

Let $(P_n^{(\ell)})$ denote the ℓ-generalized Pell sequence (or shortly ℓ-Pell sequence), where the initial values are $P_0^{(\ell)} = P_1^{(\ell)} = \dots = P_{\ell-2}^{(\ell)} = 0$, $P_{\ell-1}^{(\ell)} = 1$, and the recurrence is given by

$$P_n^{(\ell)} = 2P_{n-1}^{(\ell)} + P_{n-2}^{(\ell)} + \dots + P_{n-\ell}^{(\ell)}. \tag{2.4}$$

If $\ell = 2$, then it gives the Pell sequence ($P_n = 2P_{n-1} + P_{n-2}$, $P_0 = 0$, $P_1 = 1$, A000129 in OEIS [5]). When we apply the recurrence backward rule (2.4) we obtain the terms of the ℓ-Pell sequence with negative subscripts.

Recently, Bravo, Herrera, and Ramírez [3] presented some properties and combinatorial interpretations of the ℓ-generalized Pell sequences.

Lastly we provide a new identity involving the terms of k- and ℓ-generalized Pell sequences as a corollary of formula (2.3) given in Theorem 2.2. For this reason, we put $A_1 = 2$ and $A_2 = A_3 = \dots = A_{\ell-1} = 1$, and refer $(f_n) = (P_n^{(k)})$ and $(G_n) = (P_n^{(\ell)})$. Thus, (2.3) admits

$$P_{n+1}^{(\ell)} = \sum_{j=0}^{n} P_{n-j}^{(k)} \sum_{i=1}^{\ell-k} P_{j-i}^{(\ell)}, \quad (2 \le k < \ell).$$

Conflict of interest. The authors declare that they have no conflict of interest.

References

[1] H. Belbachir, F. Rami, L. Szalay: *A generalization of hyperbolic Pascal triangles*, J. Combin. Theory Ser. A. 188 (2022), p. 105574, issn: 0097-3165, doi: 10.1016/j.jcta.2021.105574.

[2] A. Benjamin, J. Quinn, W. Watkins: *Proofs That Really Count: The Art of Combinatorial Proof*, Mathematical Association of America, 2003, isbn: 9781614442080, doi: 10.5948/9781614442080.

[3] J. J. Bravo, J. L. Herrera, J. L. Ramírez: *Combinatorial Interpretation of Generalized Pell Numbers*, J. Integer Seq. 23.2 (2020), Article 20.2.1, url: https://cs.uwaterloo.ca/journals/JIS/VOL23/Bravo/bravo4.html.

[4] R. Frontczak: *Relations for generalized Fibonacci and Tribonacci sequences*, Notes Number Theor. Disc. Math. 25.1 (2019), pp. 178–192, doi: 10.7546/nntdm.2019.25.1.178-192.

[5] OEIS Foundation Inc.: *The On-Line Encyclopedia of Integer Sequences*, Published electronically at https://oeis.org, 2024, url: http://oeis.org.