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There are no k-Pell numbers expressible as products of two Fermat numbers

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Abstract. Let $k \geq 2$. A generalization of the well-known Pell sequence is the k-Pell sequence. The first k terms of the sequence are $0, \ldots, 0, 1$ and each term afterwards is given by the linear recurrence

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)}.$$

In this paper, we use Baker's method to determine all the solutions of the Diophnatine equation

$$P_n^{(k)} = (2^a + 1)(2^b + 1),$$

where a and b are positive integers. Then, we deduce that there are no k-Pell numbers expressible as products of two Fermat numbers.

Keywords: k-Pell numbers, linear form in logarithms, reduction method.

1. Introduction

The sequence of Pell numbers, denoted by $(P_n)_{n\geq 0}$, is defined by the recursive sequence given by

$$P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 2,$$

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with initials terms $P_0 = 0$ and $P_1 = 1$. For an integer $k \geq 2$, the *k-generalized Pell sequence* or, for simplicity, the *k-Pell sequence* $(P_n^{(k)})_{n \geq 2-k}$ is given by the recurrence

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)} \quad \text{for all} \quad n \geq 2,$$

with the initial conditions

$$P_{-(k-2)}^{(k)} = P_{-(k-3)}^{(k)} = \cdots = P_0^{(k)} = 0 \quad \text{and} \quad P_1^{(k)} = 1.$$

We note that $P_n^{(k)}$ is the *nth k-Pell number*. This sequence generalizes the usual Pell sequence i.e., when $k = 2$.

Below we present the values of these numbers for the first few values of k and $n \geq 1$.

k	Name	First non-zero terms
2	Pell	1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, ...
3	3-Pell	1, 2, 5, 13, 33, 84, 214, 545, 1388, 3535, 9003, 22929, 58396,...
4	4-Pell	1, 2, 5, 13, 34, 88, 228, 591, 1532, 3971, 10293, 26680, 69156, ...
5	5-Pell	1, 2, 5, 13, 34, 89, 232, 605, 1578, 4116, 10736, 28003, 73041, ...
6	6-Pell	1, 2, 5, 13, 34, 89, 233, 609, 1592, 4162, 10881, 28447, 74371,...
7	7-Pell	1, 2, 5, 13, 34, 89, 233, 610, 1596, 4176, 10927, 28592, 74815, ...
8	8-Pell	1, 2, 5, 13, 34, 89, 233, 610, 1597, 4180, 10941, 28638, 74960, ...
9	9-Pell	1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181, 10945, 28652, 75006, ...
10	10-Pell	1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181, 10946, 28656, 75020, ...

Finding the k-Pell numbers of special forms attracts the attention of many researchers. In [9], Kiliç gave some relations involving Fibonacci and k-Pell numbers showing that the k-Pell numbers can be expressed as the summation of the Fibonacci numbers. For instance, in 2018 Normenyo, Luca and Togbé [11] found all repdigits expressible as sums of three Pell numbers. In 2015, Faye and Luca [2] looked for repdigits in the usual Pell sequence and using some elementary methods to conclude that there are no Pell numbers larger than 10 which are repdigits. All the Padovan and Perrin numbers that are also in the sequence of Fermat numbers are found by Rihane, Adegbindin and Togbé in [12]. Recently, in [1] we showed that $F_6^{(4)} := 15$ is the only k-Fibonacci number, which is product of two Fermat numbers.

In this paper, we investigate the problem of finding the k-Pell numbers which are of the form $(2^a + 1)(2^b + 1)$, where a and b are nonnegative integers. This means that we determine all the k-Pell which are products of two Fermat numbers. Therefore, we will show the following result.

Theorem 1.1. *The Diophantine equation*

$$P_n^{(k)} = (2^a + 1)(2^b + 1) \tag{1.1}$$

has no solutions in nonnegative integers n, k, a, and b with $k \geq 2$.

Corollary 1.2. *There are no k-Pell numbers expressible as product of two Fermat numbers.*

We organize this paper as follows. In Section 2, we recall some results useful for the proof of Theorem 1.1. The proof of Theorem 1.1 is done in the last section.

2. Preliminary results

This section is devoted to collect a few definitions, notations, proprieties, and results, which will be used in the remaining of this work.

2.1. Linear forms in logarithms

For any non-zero algebraic number η of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a\prod_{j=1}^{d}\left(X-\eta^{(j)}\right)$, we denote by

$$h(\eta)=\frac{1}{d}\left(\log|a|+\sum_{j=1}^{d}\log\max\left(1,|\eta^{(j)}|\right)\right)$$

the usual absolute logarithmic height of η. In particular, if $\eta=p/q$ is a rational number with $\gcd(p,q)=1$ and $q>0$, then $h(\eta)=\log\max\{|p|,q\}$. The following properties of the logarithmic height function $h()$, which will be used in the next section without special reference, are known:

$$h(\eta\pm\gamma)\leq h(\eta)+h(\gamma)+\log 2, \tag{2.1}$$

$$h(\eta\gamma^{\pm 1})\leq h(\eta)+h(\gamma), \tag{2.2}$$

$$h(\eta^{s})=|s|h(\eta)\quad (s\in\mathbb{Z}). \tag{2.3}$$

Matveev [10] proved the following theorem. But, the version that we will use is due to Bugeaud, Mignotte and Siksek. See Theorem 9.4 in [15].

Theorem 2.1. *Let $d_{\mathbb{K}}$ be a number field of degree $d_{\mathbb{K}}$ over $\mathbb{Q}$, $\eta_1,\ldots,\eta_s$ be positive real numbers of $d_{\mathbb{K}}$, and $b_1,\ldots,b_s$ integers*

$$\Lambda:=\gamma_1^{b_1}\ldots\gamma_s^{b_s}-1\quad\textit{and}\quad B\geq\max\{|b_1|,\ldots,|b_s|\}.$$

Let

$$A_j\geq\max\{d_{\mathbb{K}}h(\gamma_j),|\log\gamma_j|,0.16\},\quad\textit{for}\quad j=1,\ldots,s.$$

be real numbers, for $i=1,\ldots,s$. Assume that $\Lambda\neq 0$, we have

$$|\Lambda|\geq\exp(-1.4\cdot 30^{s+3}s^{4.5}d_{\mathbb{K}}^{2}(1+\log d_{\mathbb{K}})(1+\log B)A_1\cdots A_s).$$

2.2. On k-generalized Pell sequence

In this subsection, we recall some facts and properties of the k-Pell sequence which will be used later. The characteristic polynomial of this sequence is

$$\Psi_k(x) = x^k - 2x^{k-1} - \cdots - x - 1.$$

In [8], Bravo et al. showed that $\Psi_k(x)$ is irreducible over $\mathbb{Q}[x]$ and has just one root $\alpha(k)$ outside the unit circle. It is real and positive so it satisfies $\alpha(k) > 2$. The other roots are strictly inside the unit circle. Furthermore, in the same paper, they showed that

$$\varphi^2(1-\varphi^{-k}) < \alpha(k) < \varphi^2, \quad \text{for all} \quad k \geq 2,$$

where $\varphi = \frac{1+\sqrt{5}}{2}$. To simplify the notation, in general, we omit the dependence on k of $\alpha(k)$ and use α. For $s \geq 2$, let

$$g_k(x) := \frac{x-1}{(k+1)x^2 - 3kx + k - 4} = \frac{x-1}{k(x^2-3x+1)+x^2-1}. \tag{2.4}$$

In [5], Bravo and Luca proved the following inequalities

$$0.276 < g_k(\alpha) < 0.5 \quad \text{and} \quad \left| g_k(\alpha^{(i)}) \right| < 1, \; 2 \leq i \leq k,$$

where $\alpha := \alpha^{(1)}, \ldots, \alpha^{(k)}$ are all the zeros of $\Psi_k(x)$. So, the number $g_k(\alpha)$ is not an algebraic integer. In addition, in 2021, Bravo, Herrera, and Luca [7] proved that the logarithmic height of $g_k(\alpha)$ satisfies

$$h(g_k(\alpha)) < 4\log(\varphi) + \log(k+1), \quad \text{for all} \quad k \geq 2. \tag{2.5}$$

With the above notation, Bravo and Herrera showed in [8] that

$$P_n^{(k)} = \sum_{i=1}^{k} g_k(\alpha^{(i)}){\alpha^{(i)}}^{n-1} \quad \text{and} \quad \left| P_n^{(k)} - g_k(\alpha)\alpha^{n-1} \right| < \frac{1}{2}, \tag{2.6}$$

for all $n \geq 1$ and $k \geq 2$. Furthermore, for $n \geq 1$ and $k \geq 2$, it was shown in [8] that

$$\alpha^{n-2} \leq P_n^{(k)} \leq \alpha^{n-1}. \tag{2.7}$$

We will finish this subsection by recalling the following lemmas.

Lemma 2.2 ([4, Lemma 2.2]). *Let $k \geq 2$ and suppose that $2n - 1 \geq k/2$. If $n < \varphi^{k/2}$, then*

$$P_n^{(k)} = \frac{\varphi^{2n-1}}{\sqrt{5}}(1+\zeta), \quad \textit{where} \quad |\zeta| < \frac{32}{\varphi^{k/2}}.$$

Lemma 2.3 ([5, Lemma 2]). *Let $\alpha = \alpha(k)$ be the dominant root of the characteristic polynomial $\Psi_k(x)$ of the k-Pell sequence and consider the function $g_k(x)$ defined in (2.4). If $k \geq 30$ and $n > 1$ are integers satisfying $n < \varphi^{k/2}$, then*

$$g_k(\alpha)\alpha^n = \frac{\varphi^{2n}}{\varphi+2}(1+\zeta), \quad \textit{where} \quad |\zeta| < \frac{4}{\varphi^{k/2}}.$$

Lemma 2.4 ([13, Lemma 7]). *If $m \geq 1$, $T > (4m^2)^m$ and $T > y/(\log y)^m$. Then,*

$$y < 2^m T(\log T)^m.$$

Lemma 2.5 ([14, Lemma 2.2]). *Let $d, x \in \mathbb{R}$ and $0 < d < 1$. If $|x| < d$, then*

$$|\log(1+x)| < \frac{-\log(1-d)}{d}|x|.$$

2.3. The reduction algorithm due to Dujella and Pethő

In this subsection, we recall the following lemma, which is a slight variation of a result due to Dujella and Pethő [6] and itself is a generalization of a result of Baker and Davenport [3].

Lemma 2.6. *Let M be a positive integer, p/q be a convergent of the continued fraction of the irrational γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let*

$$\varepsilon = \|\mu q\| - M\|\gamma q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality

$$0 < m\gamma - n + \mu < AB^{-k}$$

in positive integers m, n and k with

$$m \leq M \quad \textit{and} \quad k \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

3. Proof of Theorem 1.1

In this section, we will show Theorem 1.1 in four steps corresponding to the four subsections of the section.

3.1. Setup

Clearly, we have $P_1^{(k)} = 1 = (2^1-1)(2^1-1)$, for all $k \geq 2$. Moreover, if $1 \leq n \leq k+1$, we have $P_n^{(k)} = F_{2n-1}$. Then, equation (1.1) becomes

$$F_{2n-1} = (2^a + 1)(2^b + 1).$$

From the main result of [1], we deduce that there is no solution in this case. Thus, we can assume that $n \geq k+2$, we easily see that $n \geq 4$.

Next, we give a relation between n and $a+b$. By inequalities (2.7) and (1.1), we have

$$2^{a+b} < (2^a + 1)(2^b + 1) = P_n^{(k)} \leq \alpha^{n-1}$$

and

$$\alpha^{n-2} \le P_n^{(k)} = (2^a+1)(2^b+1) \le 2^{a+b+2}.$$

Hence, we obtain

$$(a+b)\frac{\log 2}{\log \alpha} + 1 \le n \le (a+b+2)\frac{\log 2}{\log \alpha} + 2.$$

Furthermore, using the fact that $\varphi^2(1-\varphi^{-k}) < \alpha < \varphi^2$, for $k \ge 2$, where $\varphi = \frac{1}{2}(1+\sqrt{5})$, we deduce that

$$0.71(a+b) + 1 \le n < 1.45(a+b) + 4.9. \tag{3.1}$$

3.2. An inequality for n versus k

In this subsection corresponding to the second step of the proof of Theorem 1.1, we will show the following lemma, that allows us to have an upper bound of n in relation to k.

Lemma 3.1. *If (a,b,k,n) is a solution in integers of equation (1.1) with $k \ge 2$ and $n \ge k+2$, then we have the following inequality*

$$n < 1.47 \cdot 10^{29} k^8 \log^5 k. \tag{3.2}$$

Proof. We rewrite equation (1.1) into the form

$$2^{a+b} = P_n^{(k)} - 2^a - 2^b - 1.$$

Thus, from estimate (2.6), we obtain

$$\left|g_k(\alpha)\alpha^n - 2^{a+b}\right| = \left|g_k(\alpha)\alpha^n - P_n^{(k)} + 2^a + 2^b + 1\right| \le \frac{1}{2} + 2^a + 2^b + 1.$$

Dividing both sides by 2^{a+b}, we get

$$|\Lambda_1| \le \frac{1}{2^{a+b+1}} + \frac{1}{2^b} + \frac{1}{2^a} + \frac{1}{2^{a+b}} < \frac{1.5}{2^a}, \tag{3.3}$$

where

$$\Lambda_1 := g_k(\alpha)\alpha^n 2^{-(a+b)} - 1. \tag{3.4}$$

If $\Lambda_1 = 0$, then we obtain

$$g_k(\alpha) = \alpha^{-n} 2^{a+b}.$$

Thus, $g_k(\alpha)$ is an algebraic integer, which is not possible. Therefore, $\Lambda_1 \ne 0$. We can apply Theorem 2.1 to Λ_1 given by (3.4). To do this, we consider

$$(\gamma_1, b_1) := (2, -(a+b)), \quad (\gamma_2, b_2) := (\alpha, n), \quad (\gamma_3, b_3) := (g_k(\alpha), 1).$$

The algebraic numbers $\gamma_1, \gamma_2, \gamma_3$ are elements of the field $\mathbb{K} := \mathbb{Q}(\alpha)$ and $d_{\mathbb{K}} = k$. Since $h(\gamma_1) = \log 2$, $h(\gamma_2) = (\log(\alpha))/k < (2\log(\varphi))/k$, then we can take

$$A_1 := k \log 2, \quad A_2 := 2 \log \varphi.$$

The facts $h(\gamma_3) = h(g_k(\alpha)) \leq 4\log(\varphi) + \log(k+1) < 4.4 \log k$, for all $k \geq 2$. So we can take

$$A_3 := 4.4k \log k.$$

Finally, inequality (3.1) implies that we can take $B := 1.5n$. Therefore, inequality (3.3) and Theorem 2.1 tell us that

$$|\Lambda_1| > \exp\big(-2.21 \cdot 10^{12} k^4 \log^2 k \log n\big), \tag{3.5}$$

where we used the facts $1 + \log k < 2.5 \log k$ and $1 + \log(1.5n) < 2 \log n$, which are true for all $k \geq 2$ and $n \geq 4$ respectively. Combining (3.3) and (3.5), we obtain

$$a \log 2 < 2.22 \cdot 10^{12} k^4 \log^2 k \log n. \tag{3.6}$$

We go back to equation (1.1) and we rewrite it as

$$\frac{P_n^{(k)}}{2^a + 1} - 1 = 2^b \tag{3.7}$$

and consequently we have

$$\left| \frac{g_k(\alpha)\alpha^n}{2^a + 1} - 2^b \right| = \left| \frac{g_k(\alpha)\alpha^n - P_n^{(k)}}{2^a + 1} + 1 \right| \leq \frac{1}{2(2^a + 1)} + 1 < 1.2.$$

Dividing through 2^b, we obtain

$$|\Lambda_2| < \frac{1.2}{2^b}, \tag{3.8}$$

where

$$\Lambda_2 := \frac{g_k(\alpha)}{2^a + 1} \alpha^n 2^{-b} - 1.$$

If $\Lambda_2 = 0$, then we get

$$g_k(\alpha) = \alpha^{-n} 2^b (2^a + 1),$$

which is not possible since the right-hand side is an algebraic integer while the left-hand side is not. So $\Lambda_2 \neq 0$. Now, we will apply Theorem 2.1 to Λ_2 by taking

$$(\gamma_1, b_1) := (g_k(\alpha)/(2^a + 1), 1), \quad (\gamma_2, b_2) := (\alpha, n), \quad (\gamma_3, b_3) := (2, -b).$$

Clearly, $\mathbb{K} := \mathbb{Q}(\alpha)$ contains $\gamma_1, \gamma_2, \gamma_3$ and has the degree $d_{\mathbb{K}} = k$. As calculated before we take

$$A_2 := 2 \log \varphi, \quad A_3 := k \log 2, \quad \text{and} \quad B := 1.5n.$$

We need to compute A_1. The estimates (2.5) and (3.6) together with the proprieties (2.1)–(2.3) imply that the inequalities

$$\begin{aligned} h(\gamma_1) &\leq h(g_k(\alpha)) + h(2^a) + h(1) + \log 2 \\ &< 4\log(\varphi) + \log(k+1) + (a+1)\log 2 \\ &< 2.23 \cdot 10^{12} k^4 \log^2 k \log n \end{aligned}$$

hold for all $k \geq 2$. Since

$$\gamma_1 := \frac{g_k(\alpha)}{2^a+1} < \frac{1}{6} \quad \text{and} \quad \gamma_1^{-1} = \frac{2^a+1}{g_k(\alpha)} < 2^{a+3},$$

then by (3.6) we have

$$|\log \gamma_1| < (a+3)\log 2 < 2.23 \cdot 10^{12} k^4 \log^2 k \log n.$$

Thus, we conclude that

$$\max\{kh(\gamma_1), |\log \gamma_1|, 0.16\} < 2.23 \cdot 10^{12} k^5 \log^2 k \log n := A_1.$$

Applying Theorem 2.1 and comparing the resulting inequality with (3.8), we get

$$b < 1.62 \cdot 10^{24} k^8 \log^3 k \log^2 n,$$

where we have used the facts $1 + \log k < 2.5 \log k$ and $1 + \log n < 2.1 \log n$, which hold for $k \geq 2$ and $n \geq 4$. By inequality (3.1), we get

$$n < 1.45(a+b) + 4.9 < 2.9b + 4.9 < 4.7 \cdot 10^{24} k^8 \log^3 k \log^2 n.$$

We deduce that

$$\frac{n}{\log^2 n} < 4.7 \cdot 10^{24} k^8 \log^3 k. \tag{3.9}$$

Taking $m = 2$ and $A := 4.7 \cdot 10^{24} k^8 \log^3 k$ in Lemma 2.4 and as

$$56.81 + 8\log k + 3\log\log k < 88.4 \log k,$$

for all $k \geq 2$, we get

$$\begin{aligned} n &< 2^2(4.7 \cdot 10^{24} k^8 \log^3 k)(\log(4.7 \cdot 10^{24} k^8 \log^3 k))^2 \\ &< 1.88 \cdot 10^{25} k^8 \log^3 k (56.81 + 8\log k + 3\log\log k)^2 \\ &< 1.47 \cdot 10^{29} k^8 \log^5 k. \end{aligned}$$

This establishes (3.2) and finishes the proof of Lemma 3.1. □

3.3. The case $2 \leq k \leq 600$

This subsection is the third step of the proof of Theorem 1.1, that consists in studying the the main equation when $k \in [2, 600]$ by using Lemma 2.6. Consider

$$\Gamma_1 := \log(\Lambda_1 + 1) = n \log \alpha - (a + b) \log 2 + \log(g_k(\alpha)). \tag{3.10}$$

Since $a \geq 1$, then by (3.3), we have $|\Lambda_1| < 0.75$. Hence, applying Lemma 2.5 with $d = 0.75$, we get

$$|\Gamma_1| < \frac{-\log(1 - 0.75)}{0.75} |\Lambda_1| < 2.8 \cdot 2^{-a}. \tag{3.11}$$

Replacing (3.10) into (3.11) and dividing through by $\log 2$, we obtain

$$\left| n \left(\frac{\log \alpha}{\log 2} \right) - (a + b) + \frac{\log(g_k(\alpha))}{\log 2} \right| < 4.1 \cdot 2^{-a}. \tag{3.12}$$

To apply Lemma 2.6 to (3.12), we take

$$\gamma := \frac{\log \alpha}{\log 2}, \quad \mu := \frac{\log(g_k(\alpha))}{\log 2}, \quad A := 4.1, \quad \text{and} \quad B := 2.$$

We have $\gamma \notin \mathbb{Q}$ since if we assume the contrary, then there exist coprime integers a and b such that $\gamma = a/b$, then we get that $\alpha^b = 2^a$. Let $\sigma \in Gal(\mathbb{K}/\mathbb{Q})$ such that $\sigma(\alpha) = \alpha_i$, for some $i \in \{2, \ldots, k\}$. Applying this to the above relation and taking absolute values we get $1 < 2^a = |\alpha_i| < 1$, which is a contradiction.

For each $k \in [2, 600]$, we find a good approximation of γ and a convergent p_ℓ/q_ℓ of the continued fraction of γ such that $q_\ell > 6M_k$ and $\varepsilon = \varepsilon(k) = \|\mu q\| - M_k\|\gamma q\| > 0$, where $M_k := \lfloor 1.47 \cdot 10^{29} k^8 \log^5 k \rfloor$, which is is an upper bound of n from Lemma 3.1. After doing this, we use Lemma 2.6 on inequality (3.12). A computer program with Mathematica revealed that the maximum value of $\frac{\log(Aq/\varepsilon)}{\log B}$ over all $k \in [2, 600]$ is $196.318\ldots$, which is an upper bound of a by Lemma 2.6.

Now, we consider $1 \leq a \leq 196$ and

$$\Gamma_2 := \log(\Lambda_2 + 1) = n \log \alpha - b \log 2 + \log(g_k(\alpha)/(2^a + 1)). \tag{3.13}$$

Since $b \geq 1$, then by (3.8), we have $|\Lambda_2| < 0.6$. Thus, by Lemma 2.5 with $d = 0.6$ we deduce that

$$|\Gamma_2| < \frac{-\log(1 - 0.6)}{0.6} |\Lambda_2| < 1.9 \cdot 2^{-b}. \tag{3.14}$$

Replacing (3.13) into (3.14) and dividing through by $\log 2$, we obtain

$$\left| n \left(\frac{\log \alpha}{\log 2} \right) - b + \frac{\log(g_k(\alpha)/(2^a + 1))}{\log 2} \right| < 2.8 \cdot 2^{-b}. \tag{3.15}$$

To apply Lemma 2.6 to (3.15), this time for $1 \leq a \leq 196$ we take

$$\gamma := \frac{\log \alpha}{\log 2}, \quad \mu_a := \frac{\log(g_k(\alpha)/(2^a + 1))}{\log 2}, \ (1 \leq a \leq 196), \quad A := 2.8, \quad B := 2.$$

As seen before $\gamma \notin \mathbb{Q}$. Again, for each $(k,a) \in [2,600] \times [1,196]$, we find a good approximation of γ and a convergent p_ℓ/q_ℓ of the continued fraction of γ such that $q_\ell > 6M_k$ and $\varepsilon = \varepsilon(k) = \|\mu q\| - M_k\|\gamma q\| > 0$, where $M_k := \lfloor 1.47 \cdot 10^{29} k^8 \log^5 k \rfloor$, which is is an upper bound of $n-1$ from Lemma 3.1. After doing this, we use Lemma 2.6 on inequality (3.15). A computer search with Mathematica revealed that the maximum value of $\frac{\log(Aq/\varepsilon)}{\log B}$ over all $(k,a) \in [1,600] \times [1,196]$ is $211.196\ldots$, which according to Lemma 2.6, is an upper bound of b.

Hence, we deduce that the possible solutions (a,b,k,n) of equation (1.1) for which $k \in [2,600]$ satisfy $1 \le a \le b \le 212$. Therefore, we use inequalities (3.1) to obtain $n \le 616$.

Finally, we use Mathematica to compare $P_n^{(k)}$ and $(2^a+1)(2^b+1)$, for $4 \le n \le 616$ and $1 \le a \le b \le 212$, with $n < 1.45(a+b)+4.9$ and checked that equation (1.1) has no solutions.

3.4. The case $k > 600$

For the last step of the proof of Theorem 1.1, we will show that equation (1.1) has no solutions when $k > 600$.

3.4.1. An absolute upper bound on $k > 600$

Lemma 3.2. *If (n,k,a,b) is a solution of the Diophantine equation (1.1) with $k > 600$ and $n \ge k+2$, then k and n are bounded as*

$$k < 1.4 \cdot 10^{31} \quad \textit{and} \quad n < 4.2 \cdot 10^{287}.$$

Proof. For $k > 600$, we have

$$n < 1.47 \cdot 10^{29} k^8 \log^5 k < \varphi^{k/2}.$$

Since $n < \varphi^{k/2}$, Lemma 2.3 and inequality (3.3) imply

$$\begin{aligned}\left|\frac{\varphi^{2n}}{\varphi+2} - 2^{a+b}\right| &\le \left|g_k(\alpha)\alpha^n - 2^{a+b}\right| + \left|g_k(\alpha)\alpha^n - \frac{\varphi^{2n}}{\varphi+2}\right| \\ &\le \frac{3}{2} + 2^a + 2^b + \frac{\varphi^{2n}}{\varphi+2}|\zeta| \\ &\le \frac{3}{2} + \frac{\varphi^{2n-2}}{2^a} + \frac{4\varphi^{2n}}{(\varphi+2)\varphi^{k/2}},\end{aligned}$$

where we have used that $2^{a+b} < \alpha^{n-1} < \varphi^{2n-2}$. Dividing both sides by $\varphi^{2n}/\varphi+2$ and using the fact that $n \ge k+2$ we obtain

$$\left|(\varphi+2)2^{a+b}\varphi^{-2n} - 1\right| < \frac{3(\varphi+2)}{2\varphi^{2k+4}} + \frac{\varphi+2}{\varphi^2 2^a} + \frac{4}{\varphi^{k/2}} < \frac{7}{\varphi^\lambda}, \tag{3.16}$$

where $\lambda := \min\{k/2, a\}$. Let

$$\Lambda_3 := (\varphi+2)2^{a+b}\varphi^{-2n} - 1. \tag{3.17}$$

If $\Lambda_3 = 0$, then we obtain

$$(\varphi + 2)2^{a+b} = \varphi^{2n}.$$

Thus, $(\varphi + 2)2^{a+b}$ is an unit in $\mathbb{Q}(\sqrt{5})$, which is not possible. Therefore, $\Lambda_3 \neq 0$. We can apply Theorem 2.1 to Λ_3 given by (3.17). To do this, we consider

$$(\gamma_1, b_1) := (\varphi + 2, 1), \quad (\gamma_2, b_2) := (2, a + b), \quad (\gamma_3, b_3) := (\varphi, -2n).$$

The algebraic numbers $\gamma_1, \gamma_2, \gamma_3$ are elements of the field $\mathbb{K} := \mathbb{Q}(\sqrt{5})$ and $d_{\mathbb{K}} = 2$. We take $B := 2n$. Since $h(\gamma_1) \leq h(\varphi) + h(2) + \log 2 = \frac{\log \varphi}{2} + 2\log 2$, $h(\gamma_2) = \log 2$, and $h(\gamma_3) = \frac{\log \varphi}{2}$ then we can consider

$$A_1 := \log \varphi + 4\log 2, \quad A_2 := 2\log 2, \quad A_3 := \log \varphi.$$

By the application of Theorem 2.1, we get

$$|\Lambda_3| > \exp\big(-2.86 \cdot 10^{12} \log n\big),$$

where we use the fact that $1 + \log(2n) < 2.3 \log n$ which hold for all $n \geq 4$. By comparing the resulting inequality with (3.16), we obtain

$$\lambda < 5.95 \cdot 10^{12} \log n. \tag{3.18}$$

Now, we distinguish two cases according to λ.

Case 1: $\boldsymbol{\lambda = k/2}$. By Lemma 3.1 and inequality (3.18), it follows that

$$k < 1.19 \cdot 10^{13} \log\big(1.47 \cdot 10^{29} k^8 \log^5 k\big).$$

Solving the above inequality and inserting the result in the inequality of Lemma 3.1, one gets

$$k < 4.5 \cdot 10^{15} \quad \text{and} \quad n < 1.6 \cdot 10^{162}.$$

Case 2: $\boldsymbol{\lambda = a}$. In this case, it comes from (3.18) that

$$a < 5.95 \cdot 10^{12} \log n. \tag{3.19}$$

We go back to equation (3.7) and we use Lemma 2.3 to obtain

$$\begin{aligned} \left| 2^b - \frac{\varphi^{2n}}{(\varphi+2)(2^a+1)} \right| &\leq \frac{1}{2^a+1} \left| \frac{\varphi^{2n}}{\varphi+2} - g_k(\alpha)\alpha^n \right| + \left| \frac{g_k(\alpha)\alpha^n}{2^a+1} - 2^b \right| \\ &\leq |\zeta| \frac{\varphi^{2n}}{(\varphi+2)(2^a+1)} + \frac{1}{2(2^a+1)}. \end{aligned}$$

If we divide the above inequality by $\frac{\varphi^{2n}}{(\varphi+2)(2^a+1)}$ and we use the fact that $n \geq k+2$, we get

$$\left|(\varphi+2)(2^a+1)\varphi^{-2n}2^b - 1\right| \leq \frac{4}{\varphi^{k/2}} + \frac{\varphi+2}{2\varphi^{2k+4}} < \frac{4.3}{\varphi^{k/2}}. \tag{3.20}$$

We apply Theorem 2.1 with the data

$$t := 3, \quad (\eta_1, b_1) := ((\varphi+2)(2^a+1), 1), \quad (\eta_2, b_2) := (\varphi, -2n), \quad (\eta_3, b_3) := (2, b),$$

and

$$\Gamma_4 := (\varphi+2)(2^a+1)\varphi^{-2n}2^b - 1.$$

We show that $\Gamma_4 \neq 0$ using the same method used to show that $\Gamma_3 \neq 0$. As calculated before, we take

$$\mathbb{K} := \mathbb{Q}(\sqrt{5}), \quad d_{\mathbb{K}} := 2, \quad A_2 := \log\varphi, \quad A_3 := 2\log 2 \quad \text{and} \quad B := 2n.$$

Moreover, using (3.19), we obtain

$$\begin{aligned} h(\eta_1) &\leq h(\varphi) + h(2) + ah(2) + h(1) + 2\log 2 \\ &\leq \log\varphi/2 + 3\log 2 + 5.95\cdot 10^{12}\log n \log 2 \\ &\leq 4.13\cdot 10^{12}\log n. \end{aligned}$$

Thus, we take $A_1 := 8.26\cdot 10^{12}\log n$. According to Theorem 2.1 and inequality (3.20), we get

$$\exp(-7.26\cdot 10^{24}(\log n)^2) < \frac{4.3}{\varphi^{k/2}}.$$

Consequently, one has

$$k < 3.02\cdot 10^{25}(\log n)^2.$$

From this and Lemma 3.1, it follows that

$$k < 1.4\cdot 10^{31} \quad \text{and} \quad n < 4.2\cdot 10^{287}. \tag{3.21}$$

So, in all cases inequalities (3.21) hold. Therefore, the lemma is proved. □

3.4.2. Reducing the bound on *k*

Now, we try to reduce the obtained bounds. We put

$$\Gamma_3 := \log(\Lambda_3 + 1) = (a+b)\log 2 - 2n\log\varphi + \log(\varphi+2). \tag{3.22}$$

Since $a \geq 5$, then by (3.3), we have $|\Lambda_3| < 0.64$. Hence, applying Lemma 2.5 with $d = 0.64$, we get

$$|\Gamma_3| < \frac{-\log 0.36}{0.64}|\Lambda_3| < 11.2\varphi^{-\lambda}. \tag{3.23}$$

Replacing (3.22) into (3.23) and dividing through by $\log\varphi$, we obtain

$$\left|(a+b)\left(\frac{\log 2}{\log\varphi}\right) - 2n + \frac{\log(\varphi+2)}{\log\varphi}\right| < 23.3\varphi^{-\lambda}. \tag{3.24}$$

To apply Lemma 2.6 to (3.24), we take

$$\gamma := \frac{\log 2}{\log\varphi}, \quad \mu := \frac{\log(\varphi+2)}{\log\varphi}, \quad A := 23.3 \quad \text{and} \quad B := \varphi.$$

We know that $\gamma \notin \mathbb{Q}$. We put now $M := 8.4 \cdot 10^{287}$, which is an upper bound on $(a+b)$ and we use Lemma 2.6 on (3.24) in order to obtain an upper bound on λ. A computer search with Maple shows that q_{579} satisfies the conditions of Lemma 2.6 and that $\lambda \leq 1427$.

Case 1: $\lambda = k/2$. In this case one obtains $k \leq 2854$.

Case 2: $\lambda = a$. In this case, one gets that $a \leq 1427$. Now, we define

$$\Lambda_4 := b \log 2 - 2n \log \varphi + \log((\varphi + 2)(2^a + 1)) = \log(\Gamma_4 + 1).$$

Since $k > 600$, then from (3.20) we have $|\Gamma_4| < 0.01$. Hence by 2.5, we deduce that

$$|\Gamma_4| < -\frac{\log(0.99)}{0.01}|\Lambda_4| < 4.4\varphi^{-k/2}$$

and so

$$\left| b\left(\frac{\log 2}{\log \varphi}\right) - 2n + \frac{\log(\varphi + 2)(2^a + 1)}{\log \varphi} \right| < 9.2\varphi^{-k/2}.$$

For $1 \leq a \leq 1427$, we apply Lemma 2.6 with the parameters

$$\gamma := \frac{\log 2}{\log \varphi}, \quad \mu := \frac{\log(\varphi + 2)(2^a + 1)}{\log \varphi}, \quad A := 9.2, \quad \text{and} \quad B := \varphi.$$

Furthermore, Lemma 3.2 implies that we can take $M := 8.4 \cdot 10^{287}$. Using Maple, we find that q_{579} satisfies the hypotheses of Lemma 2.6. Furthermore, according to Lemma 2.6 we obtain $k \leq 2866$. So In all cases we obtain $k \leq 2866$.

This upper bound on of k with Lemma 3.1 gives that $n < 2.2 \cdot 10^{61}$. So, we apply again Lemma 2.6 with the same above data but this time we take $M := 4.4 \cdot 10^{61}$. With the help of Maple we obtain that q_{126} satisfies the conditions of Lemma 2.6 and that $k < 648$.

With this new bound, we get $n < 5.2 \cdot 10^{55}$. So, we apply again Lemma 2.6 with the same above data but this time we take $M := 1.04 \cdot 10^{56}$. With the help of Maple, we see that q_{114} satisfies the conditions of Lemma 2.6 and that $k < 598$, which contradicts our assumption that $k > 600$. Therefore, we have no solutions (n, k, a, b) to equation (1.1) with $k > 600$. This completes the proof of Theorem 1.1.

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