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A sum of negative degrees of the gaps values in 2 and 3-generated numerical semigroups

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Abstract

We show explicit expressions for an inverse power series over the gaps values of numerical semigroups generated by two and three integers. As an application, a set of identities of the Hurwitz zeta functions is derived.

Keywords: numerical semigroups, gaps and non-gaps, the Hurwitz zeta function

MSC: Primary 20M14; Secondary 11P81

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1. Introduction

A sum of integer powers of gaps values in numerical semigroups $S_m = \langle d_1, \ldots, d_m\rangle$ with $\gcd(d_1, \ldots, d_m) = 1$, is referred often as the semigroup series

$$g_n(S_m) = \sum_{s\in\mathbb{N}\setminus S_m} s^n, \quad n \in \mathbb{Z},$$

where $\mathbb{N}\setminus S_m$ is known as the set of gaps of S_m and $g_0(S_m)$ is called the genus of S_m. The semigroup series $g_n(S_m)$ has been attractive by many researchers for $n \geq 0$. In particular, an explicit expression of $g_n(S_2)$ and implicit expression of $g_n(S_3)$ were given in [6] and [4], respectively. However, the series $g_n(S_m)$ for negative integers n has not seemingly treated so often. In this paper we derive a formula for semigroup series $g_{-n}(S_2) = \sum_{s\in\mathbb{N}\setminus S_2} s^{-n}$ and $g_{-n}(S_3) = \sum_{s\in\mathbb{N}\setminus S_3} s^{-n}$ $(n \geq 1)$. In fact, it will be known that such series are related with zeta functions in Number theory.

Consider a numerical semigroup $S_2 = \langle d_1, d_2\rangle$, generated by two integers $d_1, d_2 \geq 2$ with $\gcd(d_1, d_2) = 1$. Here, the Hilbert series $H(z; S_2)$ and the gaps generating function $\Phi(z; S_2)$ are given as

$$H(z; S_2) = \sum_{s\in S_2} z^s \quad \text{and} \quad \Phi(z; S_2) = \sum_{s\in\mathbb{N}\setminus S_2} z^s,$$

respectively, satisfying

$$H(z; S_2) + \Phi(z; S_2) = \frac{1}{1-z} \quad (z < 1), \tag{1.1}$$

where $\min\{\mathbb{N} \setminus S_2\} = 1$, and $\max\{\mathbb{N} \setminus S_2\} = d_1d_2 - d_1 - d_2$ is called the Frobenius number and is denoted by F_2. A rational representation (Rep) of $H(z; S_2)$ is given by

$$H(z; S_2) = \frac{1 - z^{d_1d_2}}{(1 - z^{d_1})(1 - z^{d_2})}. \tag{1.2}$$

We introduce a new generating function $\Psi_1(z; S_2)$, defined by

$$\Psi_1(z; S_2) = \int_0^z \frac{\Phi(t; S_2)}{t}\, \mathrm{d}t = \sum_{s\in\mathbb{N}\setminus S_2} \frac{z^s}{s} \quad \text{with} \quad \Psi_1(1; S_2) = g_{-1}(S_2). \tag{1.3}$$

Substituting (1.1) into (1.3), we obtain

$$\Psi_1(z; S_2) = \int_0^z \left(\frac{1}{1-t} - H(t; S_2)\right) \frac{\mathrm{d}t}{t}. \tag{1.4}$$

Since $(1 - t^{d_i})^{-1} = \sum_{k_i=0}^{\infty} t^{k_id_i}$, by substituting (1.4) into (1.2), we obtain

$$H(t; S_2) = \sum_{k_1,k_2=0}^{\infty} t^{k_1d_1+k_2d_2} - \sum_{k_1,k_2=0}^{\infty} t^{k_1d_1+k_2d_2+d_1d_2}. \tag{1.5}$$

Indeed, an expression (1.5) is an infinite series with degrees $s = k_1d_1 + k_2d_2$ running over all nodes in the following sublattice $\mathbb{K}$ of the integer lattice $\mathbb{Z}_2$.

$$\mathbb{K} = \{0,0\} \cup \mathbb{K}_1 \cup \mathbb{K}_2, \qquad \left\{ \begin{array}{l} \mathbb{K}_1 = \{1 \le k_1 \le d_2 - 1, \;\; k_2 = 0\}, \\ \mathbb{K}_2 = \{0 \le k_1 \le d_2 - 1, \;\; 1 \le k_2 \le \infty\}. \end{array} \right. \tag{1.6}$$

In Figure 1, as an example, we present a part of the integer lattice $\mathbb{K}$ for the numerical semigroup

$$\langle 5, 8\rangle = \{0, 5, 8, 10, 13, 15, 16, 18, 20, 21, 23, 24, 25, 26, 28, \longmapsto\},$$

where the symbol $\longmapsto$ denotes an infinite set of positive integers exceeding 28.

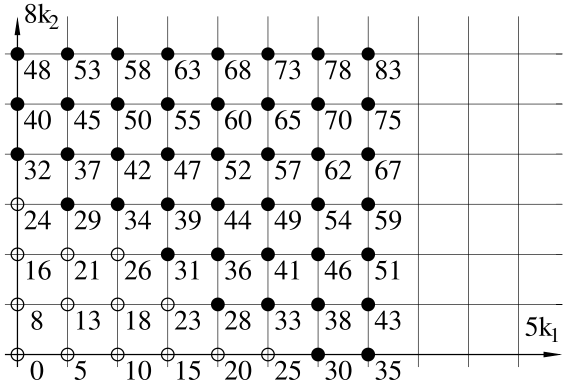

Figure 1: A part of the integer lattice $\mathbb{K} \subset \mathbb{Z}_2$ for the numerical semigroup $\langle 5, 8\rangle$. The nodes mark the non-gaps of semigroup: the values, assigned to the black and white nodes, exceed and precede $F_2 = 27$, respectively.

Proposition 1.1. *There exists a bijection between the infinite set of nodes in the integer lattice* $\mathbb{K}$ *and an infinite set of non-gaps of the semigroup* $\langle d_1, d_2\rangle$.

Proof. We have to prove two statements of existence and uniqueness:

1) Every $s \in \langle d_1, d_2\rangle$ has its Rep node in $\mathbb{K}$,

2) All $s \in \langle d_1, d_2\rangle$ have their Rep nodes in $\mathbb{K}$ only once.

1) Let $s \in \langle d_1, d_2\rangle$ be given. Then by definition of $\langle d_1, d_2\rangle$ an integer s has Rep,

$$s = k_1d_1 + k_2d_2, \quad k_1, k_2 \in \mathbb{Z}, \quad k_1, k_2 \ge 0. \tag{1.7}$$

Choose s such that $k_1 = pd_2 + q$, where $p = \lfloor k_1/d_2 \rfloor$, namely, $0 \le q \le d_2 - 1$, and $\lfloor x \rfloor$ denotes the integer part of a real number x. Then Rep (1.7) is expressed as

$$s = qd_1 + (k_2 + pd_1)d_2,$$

and s has its Rep node in $\mathbb{K}$.

2) By way of contradiction, assume that there exist two nodes $\{k_1, k_2\} \in \mathbb{K}$ and $\{l_1, l_2\} \in \mathbb{K}$ such that

$$\begin{aligned} &k_1 d_1 + k_2 d_2 = l_1 d_1 + l_2 d_2, \\ &0 \le k_1, l_1 \le d_2 - 1, \quad 0 \le k_2, l_2 \le \infty, \quad k_1 > l_1, \ \ k_2 < l_2, \end{aligned} \tag{1.8}$$

namely, that there exists such $s \in \langle d_1, d_2 \rangle$ which has two different Rep nodes in $\mathbb{K}$. Rewrite equality (1.8) as follows.

$$(k_1 - l_1) d_1 = (l_2 - k_2) d_2. \tag{1.9}$$

Since $\gcd(d_1, d_2) = 1$, the equality (1.9) implies that

$$k_1 - l_1 = b d_2 \quad (b \ge 1) \quad \Longrightarrow \quad k_1 = l_1 + b d_2 \quad \Longrightarrow \quad k_1 \ge d_2,$$

contradicting the assumption $\{k_1, k_2\} \in \mathbb{K}$. □

2. A sum of the inverse gaps values $g_{-1}(S_2)$

Rewrite the integral in (1.4) as follows.

$$\Psi_1(z; S_2) = \int_0^z \left(\sum_{k=0}^{\infty} t^{k-1} - \frac{H(t; S_2)}{t} \right) \mathrm{d}t, \tag{2.1}$$

where

$$\begin{aligned} \frac{H(t; S_2)}{t} &= \sum_{j=0}^{2} h_j(t; S_2), & h_0(t; S_2) &= \frac{1}{t}, \\ h_1(t; S_2) &= \sum_{k_1=1}^{d_2-1} t^{k_1 d_1 - 1}, & h_2(t; S_2) &= \sum_{k_1, k_2 \in \mathbb{K}_2} t^{k_1 d_1 + k_2 d_2 - 1}. \end{aligned}$$

By integration we obtain from (2.1),

$$\Psi_1(z; S_2) = \sum_{k=1}^{\infty} \frac{z^k}{k} - \frac{1}{d_1} \sum_{k_1=1}^{d_2-1} \frac{z^{k_1 d_1}}{k_1} - \sum_{k_1, k_2 \in \mathbb{K}_2} \frac{z^{k_1 d_1 + k_2 d_2}}{k_1 d_1 + k_2 d_2}, \tag{2.2}$$

and deduce by (1.3) and (1.6),

$$g_{-1}(S_2) = \sum_{k=1}^{\infty} \frac{1}{k} - \sum_{k_1, k_2 \in \mathbb{K}_2} \frac{1}{k_1 d_1 + k_2 d_2} - \frac{1}{d_1} \sum_{k_1=1}^{d_2-1} \frac{1}{k_1}. \tag{2.3}$$

By Proposition 1.1, after subtraction in (2.3) there is a finite number of terms left, since all terms, which exceed F_2 in the two first infinite series in (2.3), are cancelled. To emphasize that fact, we represent formula (2.3) as follows.

$$g_{-1}(S_2) = \sum_{k=1}^{c_2} \frac{1}{k} - \sum_{k_1,k_2 \in \mathbb{K}_2}^{k_1 d_1 + k_2 d_2 \le c_2} \frac{1}{k_1 d_1 + k_2 d_2} - \frac{1}{d_1} \sum_{k_1=1}^{d_2-1} \frac{1}{k_1},$$

where $c_2 = F_2 + 1$ is called the conductor of semigroup S_2.

3. A sum of the negative degrees of gaps values $g_{-n}(S_2)$

We generalize formula (2.2) and introduce a new generating function $\Psi_n(z; S_2)$ $(n \ge 2)$

$$\Psi_n(z; S_2) = \int_0^z \frac{\mathrm{d}t_1}{t_1} \int_0^{t_1} \frac{\mathrm{d}t_2}{t_2} \ldots \int_0^{t_{n-1}} \Phi(t_n; S_2) \frac{\mathrm{d}t_n}{t_n} = \sum_{s \in \mathbb{N} \setminus S_2} \frac{z^s}{s^n}, \tag{3.1}$$

where $\Psi_n(1; S_2) = g_{-n}(S_2)$ and satisfies the following recursive relation.

$$\Psi_{k+1}(t_{n-k-1}; S_2) = \int_0^{t_{n-k-1}} \frac{\mathrm{d}t_{n-k}}{t_{n-k}} \Psi_k(t_{n-k}; S_2), \quad k \ge 0,$$
$$\Psi_0(t_n; S_2) = \Phi(t_{n-1}; S_2), \quad t_0 = z.$$

Namely,

$$\Psi_1(t_{n-1}; S_2) = \int_0^{t_{n-1}} \frac{\mathrm{d}t_n}{t_n} \Psi_0(t_n; S_2),$$
$$\Psi_2(t_{n-2}; S_2) = \int_0^{t_{n-2}} \frac{\mathrm{d}t_{n-1}}{t_{n-1}} \Psi_1(t_{n-1}; S_2), \quad \ldots .$$

By integration in (3.1), we obtain

$$\Psi_n(z; S_2) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} - \frac{1}{d_1^n} \sum_{k_1=1}^{d_2-1} \frac{z^{k_1 d_1}}{k_1^n} - \sum_{k_1,k_2 \in \mathbb{K}_2} \frac{z^{k_1 d_1 + k_2 d_2}}{(k_1 d_1 + k_2 d_2)^n}.$$

Thus, for $z = 1$ we have

$$g_{-n}(S_2) = \sum_{k=1}^{\infty} \frac{1}{k^n} - \sum_{k_1=0}^{d_2-1} \sum_{k_2=1}^{\infty} \frac{1}{(k_1 d_1 + k_2 d_2)^n} - \frac{1}{d_1^n} \sum_{k_1=1}^{d_2-1} \frac{1}{k_1^n}, \quad n \ge 2. \tag{3.2}$$

Denoting the ratio d_1/d_2 by δ, we can rewrite (3.2) as

$$g_{-n}(S_2)=\sum_{k=1}^{\infty}\frac{1}{k^n}-\frac{1}{d_2^n}\sum_{k_2=1}^{\infty}\frac{1}{k_2^n}-\frac{1}{d_2^n}\sum_{k_1=1}^{d_2-1}\sum_{k_2=1}^{\infty}\frac{1}{(k_1\delta+k_2)^n}-\frac{1}{d_1^n}\sum_{k_1=1}^{d_2-1}\frac{1}{k_1^n}.$$

Making use of the Hurwitz $\zeta(n,q)=\sum_{k=0}^{\infty}(k+q)^{-n}$ and Riemann zeta functions $\zeta(n)=\zeta(n,1)$, we represent the last formula as follows.

$$g_{-n}(S_2)=\left(1-\frac{1}{d_2^n}\right)\zeta(n)-\frac{1}{d_2^n}\sum_{k_1=1}^{d_2-1}\zeta(n,k_1\delta),\quad n\geq 2. \tag{3.3}$$

On interchanging the generators d_1 and d_2 in (3.3), we obtain an alternative expression for $g_{-n}(S_2)$:

$$g_{-n}(S_2)=\left(1-\frac{1}{d_1^n}\right)\zeta(n)-\frac{1}{d_1^n}\sum_{k_2=1}^{d_1-1}\zeta\left(n,\frac{k_2}{\delta}\right). \tag{3.4}$$

4. Symmetric 3-generated numerical semigroup

We deal with symmetric numerical semigroup $S_3=\langle d_1,d_2,d_3\rangle$ generated by three integers with the Hilbert series $H(z;S_3)$, satisfying minimal relations,

$$H(z;S_3)=\frac{\left(1-z^{a_{22}d_2}\right)\left(1-z^{a_{33}d_3}\right)}{(1-z^{d_1})(1-z^{d_2})(1-z^{d_3})}\quad (a_{22},a_{33}\geq 2), \tag{4.1}$$

with $a_{11}d_1=a_{22}d_2$, $a_{33}d_3=a_{31}d_1+a_{32}d_2$ (see [3]). In this section, we prove a statement which is necessary to establish the convergence for $g_1(z,S_3)$, namely, the difference between two divergent infinite series is convergent

$$g_1(z,S_3)=\sum_{k=1}^{\infty}\frac{1}{k}-\sum_{k_2=0}^{a_{22}-1}\sum_{k_3=0}^{a_{33}-1}\sum_{k_1=0}^{\infty}\frac{1}{k_1d_1+k_2d_2+k_3d_3},\quad \sum_{j=1}^{3}k_j\geq 1. \tag{4.2}$$

The idea is to prove that after cancellation of identical terms, a finite number of terms is left in (4.2).

We consider the sublattice $\widetilde{\mathbb{L}}=\mathbb{L}\cup\{0,0,0\}$ of the integer lattice $\mathbb{Z}_3$, where

$$\mathbb{L}=\bigcup_{\substack{k_1=0\\ k_1+k_2+k_3\geq 1}}^{\infty}\mathbb{L}_{k_1},\qquad \mathbb{L}_{k_1}=\bigcup_{k_2,k_3}^{\infty}\{k_1,k_2,k_3\},$$

with $0\leq k_2<a_{22}$ and $0\leq k_3<a_{33}$. In Figure 2, we present a part of the integer lattice $\widetilde{\mathbb{L}}$ for the numerical semigroup $\langle 4,7,10\rangle$.

Proposition 4.1. *There exists a bijection between the infinite set of nodes in the integer lattice $\widetilde{\mathbb{L}}$ and an infinite set of non-gaps of the semigroup $\langle d_1,d_2,d_3\rangle$.*

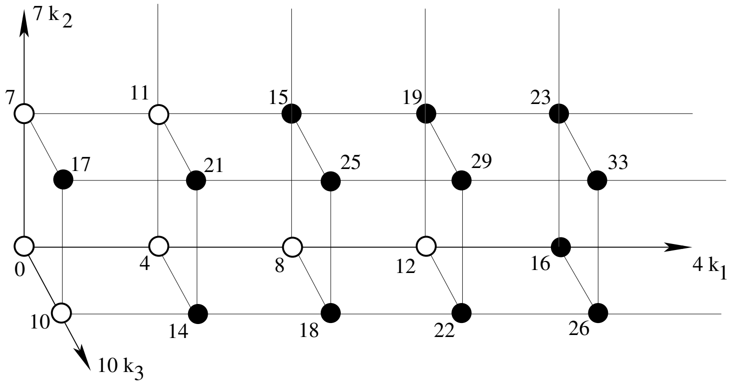

Figure 2: A part of the integer lattice $\widetilde{\mathbb{L}} \subset \mathbb{Z}_3$ for $\langle 4, 7, 10 \rangle$. The nodes mark the non-gaps of semigroup: the values, assigned to the black and white nodes, exceed and precede the Frobenius number $F_3 = 13$.

Proof. We have to prove both existence and uniqueness.

1) Every $s \in \langle d_1, d_2, d_3 \rangle$ has its representative node in $\widetilde{\mathbb{L}}$.

2) All $s \in \langle d_1, d_2, d_3 \rangle$ have their representative nodes in $\widetilde{\mathbb{L}}$ only once.

1) Let $s \in \langle d_1, d_2, d_3 \rangle$ be given. Then by definition of $\langle d_1, d_2, d_3 \rangle$ an integer s has a representation,

$$s = k_1 d_1 + k_2 d_2 + k_3 d_3, \quad 0 \le k_1, k_2, k_3 < \infty. \tag{4.3}$$

Choose s such that

$$k_2 = p_2 a_{22} + q_2, \quad k_3 = p_3 a_{33} + q_3, \quad \text{namely,} \quad p_2 = \left\lfloor \frac{k_2}{a_{22}} \right\rfloor, \ p_3 = \left\lfloor \frac{k_3}{a_{33}} \right\rfloor, \tag{4.4}$$
$$p_2, p_3, q_2, q_3 \in \mathbb{Z}, \quad p_2, p_3 \ge 0, \quad 0 \le q_2 < a_{22}, \quad 0 \le q_3 < a_{33}.$$

By substituting (4.4) into (4.3), we get

$$s = k_1 d_1 + (p_2 a_{22} + q_2) d_2 + (p_3 a_{33} + q_3) d_3. \tag{4.5}$$

Combining (4.5) with minimal relations (4.1), we obtain

$$\begin{aligned} s &= (k_1 + p_2 a_{11}) d_1 + p_3 (a_{31} d_1 + a_{32} d_2) + q_2 d_2 + q_3 d_3 \\ &= (k_1 + p_2 a_{11} + p_3 a_{31}) d_1 + (p_3 a_{32} + q_2) d_2 + q_3 d_3. \end{aligned} \tag{4.6}$$

If $p_3 a_{32} + q_2 < a_{22}$, then s has its representative node in $\widetilde{\mathbb{L}}$. But, if $p_3 a_{32} + q_2 \ge a_{22}$, let us write

$$p_3 a_{32} + q_2 = p_4 a_{22} + q_4, \quad p_4 \ge 0, \quad 0 \le q_4 < a_{22}, \quad p_4 = \left\lfloor \frac{p_3 a_{32} + q_2}{a_{22}} \right\rfloor. \tag{4.7}$$

Substitute (4.7) into (4.6) and get

$$s = (k_1 + p_2 a_{11} + p_3 a_{31} + p_4 a_{11}) d_1 + q_4 d_2 + q_3 d_3,$$

and s still has its representative node in $\widetilde{\mathbb{L}}$.
2) By way of contradiction, assume that there exist two nodes $\{k_1, k_2, k_3\} \in \widetilde{\mathbb{L}}$ and $\{l_1, l_2, l_3\} \in \widetilde{\mathbb{L}}$ such that

$$k_1 d_1 + k_2 d_2 + k_3 d_3 = l_1 d_1 + l_2 d_2 + l_3 d_3, \tag{4.8}$$

$$0 \le k_1 \ne l_1 < \infty, \quad 0 \le k_2 \ne l_2 < a_{22}, \quad 0 \le k_3 \ne l_3 < a_{33}. \tag{4.9}$$

The case, when one of the differences $k_j - l_j$ vanishes, will be considered later. Suppose that $k_1 - l_1 > 0$, and $k_2 - l_2 < 0$, $k_3 - l_3 < 0$. In fact, due to (4.9) we also have to include the upper bound

$$0 < l_2 - k_2 < a_{22}, \quad 0 < l_3 - k_3 < a_{33}. \tag{4.10}$$

Rewrite (4.8) as

$$(k_1 - l_1) d_1 = (l_2 - k_2) d_2 + (l_3 - k_3) d_3,$$

where $k_1 - l_1 \ge a_{11}$, otherwise (due to minimal relations) equation (4.8) would have trivial solution $k_j = l_j$ $(j = 1, 2, 3)$. But the last contradicts (4.9), namely, $k_1 \ne l_1, k_2 \ne l_2, k_3 \ne l_3$.

If so, represent $k_1 - l_1 = u_1 a_{11} + v_1$ with $u_1 \ge 1$, $0 \le v_1 < a_{11}$, then

$$(u_1 a_{11} + v_1) d_1 = u_1 a_{22} d_2 + v_1 d_1 = (l_2 - k_2) d_2 + (l_3 - k_3) d_3. \tag{4.11}$$

Rewrite (4.11) as

$$(l_3 - k_3) d_3 = v_1 d_1 + (u_1 a_{22} - (l_2 - k_2)) d_2, \tag{4.12}$$

and note that the both terms on the right-hand side in (4.12) are positive by (4.10),

$$0 < l_2 - k_2 < a_{22} < u_1 a_{22}. \tag{4.13}$$

However, $0 < l_3 - k_3 < a_{33}$ by (4.10), and (due to minimal relations) equation (4.12) has only a trivial solution, $l_3 = k_3$, $v_1 = 0$, $l_2 = k_2 + u_1 a_{22}$. But the last contradicts an inequality (4.13).

Now, consider the case when

$$a_{33} > k_3 - l_3 > 0, \quad 0 < l_1 - k_1, \quad 0 < l_2 - k_2 < a_{22},$$

and write

$$(k_3 - l_3) d_3 = (l_1 - k_1) d_1 + (l_2 - k_2) d_2. \tag{4.14}$$

But (due to minimal relations) equation (4.14) has only trivial solution $k_j = l_j$ $(j = 1, 2, 3)$, that contradicts (4.9), namely, $k_1 \ne l_1, k_2 \ne l_2, k_3 \ne l_3$.

Next, consider the case when

$$l_1 - k_1 = 0, \quad 0 < l_2 - k_2 < a_{22}, \quad a_{33} > k_3 - l_3 > 0, \tag{4.15}$$

and write

$$(k_3 - l_3)d_3 = (l_2 - k_2)d_2. \tag{4.16}$$

But (due to minimal relations) equation (4.16) has only a trivial solution, $l_3 = k_3$, $l_2 = k_2$, that contradicts (4.15). For similar reasons the case

$$k_3 - l_3 = 0, \quad 0 < k_1 - l_1 < a_{11}, \quad 0 < l_2 - k_2 < a_{22}, \tag{4.17}$$

leads to an equality

$$(k_1 - l_1)d_1 = (l_2 - k_2)d_2,$$

which also has only a trivial solution, $l_1 = k_1$, $l_2 = k_2$, that contradicts (4.17). Thus, what is left

$$l_1 = k_1, \quad l_2 = k_2, \quad l_3 = k_3,$$

and the result is proven. □

5. Identities for the Hurwitz zeta function

As an application, our argument can be deduced to the multiplication theorem in Hurwitz zeta functions. Indeed, combining formulas (3.3) and (3.4), we get an identity

$$\delta^n \sum_{k=1}^{d_2-1} \zeta(n, k\delta) = (1 - \delta^n)\,\zeta(n) + \sum_{k=1}^{d_1-1} \zeta\left(n, \frac{k}{\delta}\right).$$

Another spinoff of formulas (3.3) and (3.4) is a set of identities for Hurwitz zeta functions. For example, consider the numerical semigroup $\langle 3, 4\rangle$ with three gaps $\mathbb{N} \setminus \langle 3, 4\rangle = \{1, 2, 5\}$. Substituting it into (3.3) and (3.4), we have

$$\zeta\left(n, \frac{3}{4}\right) + \zeta\left(n, \frac{6}{4}\right) + \zeta\left(n, \frac{9}{4}\right) = (4^n - 1)\zeta(n) - \left(4^n + 2^n + \left(\frac{4}{5}\right)^n\right)$$

and

$$\zeta\left(n, \frac{4}{3}\right) + \zeta\left(n, \frac{8}{3}\right) = (3^n - 1)\zeta(n) - \left(3^n + \left(\frac{3}{2}\right)^n + \left(\frac{3}{5}\right)^n\right),$$

respectively.

We shall show that the identity (3.3) can be deduced to the multiplication theorem in Hurwitz zeta functions (see, e.g., [1, p.249], [2, (16), p.71]). It is similar for (3.4).

Since $\gcd(d_1, d_2) = 1$, if $k_1 d_1 \equiv k_2 d_1 \pmod{d_2}$ then $k_1 \equiv k_2 \pmod{d_2}$. Therefore,

$$\zeta\left(n, \left\{\frac{d_1}{d_2}\right\}\right) + \zeta\left(n, \left\{\frac{2d_1}{d_2}\right\}\right) + \cdots + \zeta\left(n, \left\{\frac{(d_2-1)d_1}{d_2}\right\}\right)$$

$$= \zeta\left(n, \frac{1}{d_2}\right) + \zeta\left(n, \frac{2}{d_2}\right) + \cdots + \zeta\left(n, \frac{d_2-1}{d_2}\right),$$

where $\{x\}$ denotes the fractional part of a real number x. There exists a nonnegative integer a such that

$$\frac{ad_1}{d_2} < 1 < \frac{(a+1)d_1}{d_2}.$$

Then for any integer k' with $a < k' \le d_2 - 1$ there exists a positive integer l' such that $1 \le k'd_1 - l'd_2 < d_2$, and

$$\zeta\left(n, \frac{k'd_1}{d_2}\right) = \zeta\left(n, \frac{k'd_1 - l'd_2}{d_2}\right) - \left(\frac{d_2}{k'd_1 - l'd_2}\right)^n - \left(\frac{d_2}{k'd_1 - (l'-1)d_2}\right)^n - \cdots - \left(\frac{d_2}{k'd_1 - d_2}\right)^n, \quad (5.1)$$

where

$$\frac{k'd_1 - l'd_2}{d_2} = \left\{\frac{k'd_1}{d_2}\right\}.$$

For any positive integer r, there exist integers x and y such that $r = xd_1 + yd_2$. If $0 \le x < d_2$, then r can be expressed uniquely. Thus, if $y \ge 0$, then $r \in S_2$. If $y < 0$, then $r \notin S_2$. The largest integer is given by $(d_2 - 1)d_1 - d_2$, that is exactly the same as the Frobenius number $F(d_1, d_2)$. Thus, $k'd_1 - l''d_2 \notin S_2$ for all l'' with $1 \le l'' \le l'$ in (5.1). In addition, if $k_1 d_1 - l_1 d_2 = k_2 d_1 - l_2 d_2$, then by $\gcd(d_1, d_2) = 1$ we have $d_1 | (k_1 - k_2)$ and $d_2 | (l_1 - l_2)$. As $0 < k_1, k_2 < d_2$ and $0 < l_1, l_2 < d_1$, we get $k_1 = k_2$ and $l_1 = l_2$. Thus, all such numbers of the form $kd_1 - ld_2 \notin S_2$ are different.

In [5, (3.32)] for a real ξ and $d = \gcd(d_1, d_2)$

$$\sum_{k=0}^{d_2-1} \left\lfloor \frac{kd_1 + \xi}{d_2} \right\rfloor = d \left\lfloor \frac{\xi}{d} \right\rfloor + \frac{(d_1-1)(d_2-1)}{2} + \frac{d-1}{2}. \quad (5.2)$$

Hence, by (5.2) with $d = 1$ and $\xi = 0$, the total number of non-representable positive integers of the form $kd_1 - ld_2$ ($a < k < d_2$, $l = 1, 2, \dots, \lfloor kd_1/d_2 \rfloor - 1$) is

$$\sum_{k=1}^{d_2-1} \left\lfloor \frac{kd_1}{d_2} \right\rfloor = \frac{(d_1-1)(d_2-1)}{2},$$

which is exactly the same as the number of integers without non-negative integer representations by d_1 and d_2, that was given by Sylvester in 1882. Therefore, the right-hand side of (3.3) is

$$\left(1 - \frac{1}{d_2^n}\right)\zeta(n) - \frac{1}{d_2^n} \sum_{k_1=1}^{d_2-1} \zeta\left(n, \frac{k_1 d_1}{d_2}\right)$$

$$= \left(1 - \frac{1}{d_2^n}\right)\zeta(n) - \frac{1}{d_2^n}\left(\sum_{k_1=1}^{d_2-1}\zeta\left(n, \left\{\frac{k_1 d_1}{d_2}\right\}\right) - d_2^n \sum_{s\in\mathbb{N}\setminus S_2} s^{-n}\right)$$

$$= \left(1 - \frac{1}{d_2^n}\right)\zeta(n) - \frac{1}{d_2^n}\sum_{k_1=1}^{d_2-1}\zeta\left(n, \frac{k}{d_2}\right) + \sum_{s\in\mathbb{N}\setminus S_2} s^{-n}.$$

On the other hand, the left-hand side of (3.3) is

$$g_{-n}(S_2) = \sum_{s\in\mathbb{N}\setminus S_2} s^{-n}.$$

Therefore, we obtain that

$$\sum_{k=1}^{d_2}\zeta\left(n, \frac{k}{d_2}\right) = d_2^n\zeta(n),$$

which is the multiplication theorem in Hurwitz zeta functions.

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