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The application of cooperative group work and APOS Theory in topic of logarithms*

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Abstract. The study investigates the impact of in-class cooperative group work on high school students' mathematics performance, motivation and engagement. The research was conducted with 11th-grade students learning logarithms for the first time. The APOS Theory was applied to assess students' cognitive development with an emphasis on students' progress before and after the cooperative group work. Our findings indicate that the cooperative learning activity influenced positively the learning outcome. The post tests and the students' activities in the lessons indicate that students' motivation and engagement increased after group work and the pupils advanced to higher levels of the APOS Theory. Group level improvement was validated using Fischer's exact test, whereas individual differences were investigated through Paired Samples t-test.

Keywords: APOS Theory, teaching logarithms, Fisher's exact test, cooperative group work, Paired Samples t-test

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1. Introduction

Many researches have addressed the question of how to teach the topic of logarithms effectively to students [21, 31, 34]. This topic is complex and requires pupils to have a lot of prior knowledge (see Subsection 2.2). In our experiment, we taught the logarithms to 11th-grade students at a Hungarian high school using textbooks and exercise books [4, 17, 22]. Our students were learning the mathematics at an advanced level. The typical mistakes students make when learning about logarithms are described in [10]. Among our pupils, we detected overgeneralization and a lack of in-depth knowledge of the laws of exponents (see Figures 1, 2 and 3). Many investigations emphasize the positive effects of cooperative learning methods in a classroom environment [18, 19, 24]. In our lessons, we applied cooperative group work to examine how our students' thinking processes improved when they explained the solutions of the group work exercises to their classmates. Our experiment reinforced the experience of [24] when students with different abilities work together. We observed that the students had a deeper understanding of the topic of logarithms and were more motivated to learn after working cooperatively with each other.

The new idea in this research is the combination of the cooperative group work with the APOS (Action, Process, Object, Schema) Theory [12]. We applied this model to recognize changes in the students' performance. We classified our students into the levels of the APOS Theory before and after the cooperative group work. At the beginning of each lesson, the students completed short tests including one exercise from the previous lesson's topic. These exercises before the group work are collected into the pre-assessment, whereas those after the group work are listed in the post-assessment (see Appendix). To obtain appropriate categorizations, we utilized the results of the pre- and post-assessments, as well as our observations of the students' activity during lessons.

We split the tasks of the assessments into items. These items corresponding to the different levels of the APOS Theory and referring to the underlying mathematical object are listed in Subsection 3.1. The occurrence of the items in the tasks of the pre- and post-assessment and in the group work can be found in Table 2. In Subsection 3.2 we determine the conditions under which individuals reach the different levels of the APOS Theory. Furthermore, we demonstrate the expected solutions of two typical exercises.

After learning the topic of logarithms the pupils completed group work exercises in the lessons. Each group consisted of students from the different levels of the APOS model (see Table 1). Their task was to discuss together the solutions of the exercises on the group work worksheet (see Appendix), to share their thoughts with each other, and to demonstrate their solutions to the class. After applying the cooperative learning method in lessons, we observed a significant change in the students' attitude. Using the same evaluation methods as before the group work, the students' performance was measured after the group work resulting in a new categorization with respect to the levels of the APOS Theory. The students can reach higher levels in different ways (for instance [21]). We examine the effect of our experiment in Section 4. Using diagram (see Figure 7) and figures (see Figures 8, 9, 10, 11), we demonstrate which levels of the APOS Theory the students reached before and after the cooperative group work, and how their performance improved.

In Section 5 we prove with the help of the Fisher's exact test [16], that the completion of three items (O1, O2, N2) increased significantly after the cooperative group work (see Table 3). Furthermore, the individual development of some students was presented by the Paired Samples t-test [27] (see Table 4).

Our questions and hypothesis were as follows:

- Q1 Which levels of the APOS Theory can students reach before and after the cooperative group work?
- Q2 How can the students benefit from the experiment?
- H1 Some students can progress to a higher level of the APOS Theory during the experiment.

The organization of our paper is the following. In Section 2 we present the history of the logarithm [6, 9, 11] and the Hungarian high school curriculum, particularly at an advanced level [23]. Moreover, we explain the APOS Theory and the mental processes that are important for understanding students' mathematical thinking when learning about logarithms [12, 29, 33]. We investigated the logarithm from three aspects: as a number, an operation, and a function [5, 35]. Our experiment is described in Section 3. In Section 4 we present our results. Section 5 is devoted to the statistical analysis of our data. Section 6 summarizes our experiences, answers our questions and accepts our hypothesis.

Based on our experiment, we think that it is useful to find an appropriate categorization to measure the mathematical skills of our students, because with this method we can teach more effectively. Furthermore, it can be helpful for pupils at all levels if we apply in the lessons the forms of cooperative learning, in which students of different abilities work together. This can increase their motivation and allow them to deepen their knowledge during the explanation of the solutions of the exercises.

2. Theoretical background

2.1. The history and the curriculum of logarithm

The history of logarithm is closely linked to the theory of exponentiation. Even the Babylonians were aware of the concept of powers (2000 to 600 B.C.). They presented the first ten successive powers of numbers in base sixty in a table [6]. In the 15th century, Nicolas Chuquet (1445–1500) identified zero and negative numbers as exponents [11]. In the 16th century, John Napier (1550–1617) developed the concept of logarithm as the inverse of exponentiation. He was motivated by the desire to make calculations involving large numbers easier for astronomers [6].

In the 17th century, William Oughtred (1574–1660) created the laws of logarithms: $\log a + \log b = \log ab$, $\log a - \log b = \log \frac{a}{b}$ and $\log a^m = m \log a$. The logarithmic function was also invented in this century by James Gregory (1638–1675) [9]. Some years later, Leonhard Euler (1707–1783) was one of the first mathematicians, who gave a definition of the logarithm in connection with the exponentiation. "Resuming the equation $a^b = c$, [...] we take the exponent b such, that the power a^b becomes equal to a given number c; in which case this exponent b is said to be the logarithm of the number c" [15, p. 63]. Nowadays we use a similar definition. The logarithm of a number c to the base a is the exponent to which a must be raised to produce c, it means that $\log_a c = b \Leftrightarrow a^b = c$, where $a > 0, a \neq 1$ and $c > 0$ [22, p. 93].

In summary, scientists used the logarithms before the invention of the calculators to shorten length calculations. They had logarithm tables in which they could find the logarithm of concrete numbers. The logarithm was useful, because one could apply addition or subtraction instead of multiplication or division and multiplication or division instead of power or root. Nowadays we apply the logarithm when working with large measurements.

But what do pupils learn about logarithms in high school? Before 2020, students in Hungary learnt the definition and all laws of logarithms, the formula for changing the base, and the graph of the logarithmic function in secondary education. They also solved logarithmic equations and word problems [1]. Nowadays the pupils in the secondary education learn the definition of the logarithm, the change of base formula for computation of the value of logarithm with calculators and solve word problems [2]. The laws of logarithms, solving equations, inequalities and systems of equations, as well as the logarithmic function and its inverse being an exponential function, are topic of the mathematical education only at an advanced level [23]. The strength of the application of logarithms in practical problems could be the reason for the reduction of the curricula at intermediate level. However, we believe that learning about logarithmic rules and solving equations can improve pupils' logical thinking and help them to develop a deeper understanding of exponential expressions.

2.2. Students' understanding of logarithm

In this section, we discuss some theories which describe the mathematical thinking of the students to understand logarithm and play an important role in our experiment.

In [5] the authors divide the concept of the logarithm into three categories: logarithmic expressions as numbers, operational meaning of logarithms and logarithms as functions. The first two meanings of the logarithm are strongly connected to students' algebraic knowledge. Now, we consider the prior algebraic cognition what is useful for understanding the logarithmic equations and expressions.

Algebra as a generalized arithmetic [33] means that we apply symbols instead of concrete numbers. For example the multiplication law of logarithms can be written with symbols as follows $\log a + \log b = \log ab$ [10].

We can analyze the *algebra as a study of procedures for solving certain kinds of problems* [33]. To find the solution of the logarithmic equation $\log_2(x+5) = 3$ the pupils either can take the third power of 2 and subtract 5 from this, or using the exponentiation they write the linear equation: $x + 5 = 2^3$ [10]. In Hungary, a different approach is taken to solving logarithmic equations [22, p. 107]. The students transform the right side of the original equation into logarithmic form to obtain the equation $\log_2(x+5) = \log_2 8$. Referring to the bijectivity of the function $\log_2 x$ it reduces to the linear equation $x + 5 = 8$.

Algebra can be discussed *as the study of relationships among quantities*: "Under this conception, a variable is an argument (i.e., stands for a domain value of a function) or a parameter (i.e., stands for a number on which other numbers depend)" [33, p. 14]. In the logarithmic expression $y = \log_2 x$, the argument is x and y is the parameter [10].

Algebra as the study of structures [33] occurs in teaching of logarithms for example when we ask our students to simplify $\log_a 8 - 2\log_a 4$ or to evaluate $\log_b \sqrt{b}$. In these cases, there is no generalized pattern or equation that we can solve. Here a and b are arbitrary objects to operational rules [10].

There is an other approach the understanding of logarithm [29, p. 1]. The author's opinion is that a mathematical concept has operational and structural sides. The structural approach means that the individual treats the mathematical concepts as abstract objects. It is static, instantaneous and integrative. The operational side of a notion has a processual perspective, what is dynamic, sequential and detailed. For example the logarithmic function can be defined not only as a set of ordered pairs [29, p. 4] but also as a rule or method which assigns to each element of a set exactly one element of the image set [30, p. 349]. The former definition corresponds to the structural approach while the latter notion reflects the operational conception. Both perspectives are necessary in the teaching and learning processes: "the structural approach generates insight; the operational approach generates result" [29, p. 28].

During our experiment, we observed clearly both aspects. First, the pupils acquired the logarithm as a dynamic transformation (operational approach), that is, they learnt the function of logarithm and its transformations and calculated the missing base, value or antilogarithm. Later, they considered the logarithm as a static notion (structural approach) and treated it as a mathematical tool for solving equations and word problems. If the pupils cannot realize these two sides of the logarithm, they cannot understand it as a whole.

There are some typical difficulties that can hinder the understanding of the concept of the logarithm. In our lessons, we mainly observed overgeneralization and a lack of prior knowledge [10, 20].

When the students make the mistake of overgeneralization, they try to use their existing knowledge to solve new tasks [10]. The mistakes belonging to this cause are that either the pupils consider the notation of logarithm ("log") as an object instead of an operation or they treat the logarithm as a common factor [20]. Therefore, they sometimes cannot use the laws of logarithms correctly (see

Figure 1).

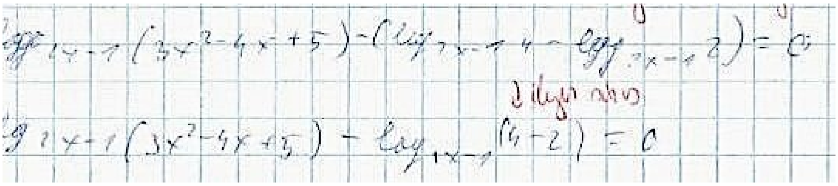

Figure 1. Example for overgeneralization. Our student S3 used the notation "log" as a variable. Probably he wanted to factorize the expression $\log_{2x-1} 4 - \log_{2x-1} 2$ similarly as in the case: $xy + xz = x(y + z)$. So he wrote $\log_{2x-1}(4 - 2)$.

Students have deficient prior knowledge when they lack an understanding of powers or exponents [34, p. 19] (see Figures 2 and 3).

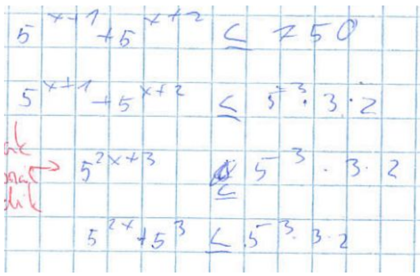

Figure 2. Incorrect application of the power identities. Our student S6 did not know the identities of exponentiation, and she had difficulties with the logarithm as well.

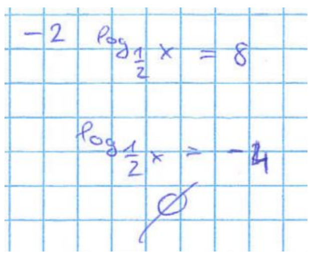

Figure 3. Lack of knowledge of negative exponents. Our student S13 did not know the negative exponents. This prevented him from solving logarithmic equations because he could not interpret some logarithmic expressions as numbers.

2.3. APOS Theory

We chose the APOS Theory as our theoretical framework, because it provides an appropriate model for describing the cognitive development of students' mathematical thinking and understanding the problem-solving process [26]. Pupils can reflect on problems using their mathematical knowledge and organize these problems into schemas by constructing or reconstructing actions, processes and objects [3]. Each individual has many schemas in different areas of mathematics, such as arithmetic, functions, and proofs. Students develop schemas to organize their individual mathematical knowledge [13]. In order to study these schemas, we must understand the structure of the APOS System and apply it to our topic.

Actions interiorize to form processes, and these processes encapsulate to form objects (see Figure 4). However, objects can also de-encapsulate into processes. Finally, these three levels can be organized into schema [14].

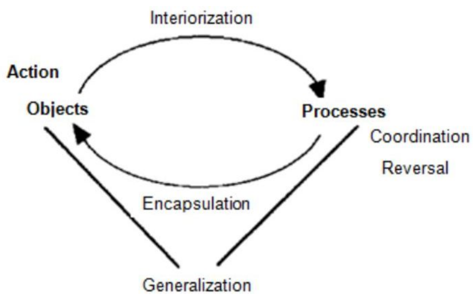

Figure 4. Schemas and their construction [13, p. 106]

The *action* is the cognitive structure of students which helps them to complete operations with the aid of the learnt concepts in an algorithmic way [8]. This is "a repeatable mental or physical manipulation of object" [25, p. 49]. At this stage, students can solve only routine procedures [28]. For example an individual can draft the graph of a function by substituting values into its expression and calculating it. If they repeat this activity, then it interiorizes into a mental process [12].

The *process* is the cognitive structure of students who can aware perform the same operations as on the stage of action, but these operations already exist in their mind. Hence they can implement these operations without external help, such as guidance from a teacher. It means that these are parts of the internal constructions of the pupils [8]. Students can reverse and coordinate their thoughts. At this stage, they can manipulate multiple processes simultaneously [28]. For example, students can correctly apply the laws of logarithms.

The *object* is the cognitive structure of students who treat the action and the process as one unit [8]. At this stage, students can understand the process as a whole and can perform conversions on it. "Encapsulation is the construction of a cognitive object through awareness of totality of a process by imagination and manipulation of it as a whole without performing subsequent action" [28, p. 3]. When an individual develops many actions, processes and objects, they can form the schema of the topic [12].

The *schema* involves the actions, processes and objects to provide a framework for problem solving [8]. At this stage, students are able to move back and forth between the levels [28], to generalize the knowledge and to use it by solving other problems. To sum up the APOS Theory describes how the students can establish mathematical concepts and mental constructs [8].

In our research, we analyzed the schemas of the logarithmic functions and of solving logarithmic equations, inequalities and systems of equations. We categorized the students according to their knowledge at the appropriate stage of this model, both before and after the group work (see Section 3).

2.4. Framework of logarithm

In [35] the concept of the logarithm given by [5] is reinterpreted and four categories are distinguished: logarithms as objects, logarithms as processes, logarithms as functions and logarithms in contextual problems. In both papers the first three stages are analogous. We did not deal with the logarithm in contextual problems, because the tasks, what our students solved during the experiment, belonged to the first three frameworks. These categories constitute a system rather than a linear progression. Now we describe them.

1. *Logarithms as numbers/ objects:* in [5] the authors state that the logarithm as number is the exponent to which the base is raised to obtain a given number. In [35] this concept is extended. The opinion of the researcher is that the students must understand that the logarithmic expressions replace numbers, and that they do not need to express them in decimal form. In our research, students must understand the definition of a logarithm and be able to express a number in logarithmic form.

2. *Logarithms as operations/ processes* mean the isomorphism between multiplicative and additive structures in both directions [5]. In [35] it is described as the transformation of the logarithmic expression. This means that the students are able to create exponential expression with the same base and recognize that the value of the logarithm is equal to the exponent. In our research, the students have to know the laws of logarithms and to solve logarithmic equations.

3. *Logarithms as functions:* in [5] it is examined how the students can connect the definition of the logarithms to their functions, construct graphs, and solve problems. According to [35] the students are able to calculate the value of the logarithmic function by substituting the x value, know the general function and identify the exponential function as its inverse. These abilities were also very important in our study. Furthermore, the pupils have to know the steps of the transformation of function.

According to the levels of the APOS Theory we examined the functional knowledge of our pupils by splitting category 3. into two components: the knowledge of the definition of the logarithmic function (F1, F2, F3, F4) and the knowledge of the transformation of function (T1, T2, T3, T4). We checked also the logarithms as operations in terms of the identities of logarithms (O1, O2, O3, O4). The logarithms as numbers were split into the levels N1, N2, N3, N4 of the APOS Theory. Moreover, we investigated the algebraic knowledge of the students about logarithms in levels A1, A2, A3, A4. We describe these levels in Subsection 3.1.

3. Method

3.1. Introduction of the group and the research method

We conducted our research at a secondary school in Hungary. We taught a group of 15 pupils the topic of logarithms. In Hungary the pupils learn this topic in the 11th class. Our students learnt mathematics at an advanced level. It means that they had 5 mathematics lessons per week. Mathematics was an important subject for them, as several of them planned to study technical or natural sciences at university.

The topic of logarithm was discussed in 35 lessons using the textbooks and exercise books [4, 17, 22]. The curriculum of the topic logarithm is described in Subsection 2.1. The pupils learnt about logarithmic functions, used the laws of logarithms by solving logarithmic equations, inequalities and systems of equations and applied the technique, how these can be transformed to linear and quadratic ones.

In the lessons the students worked either at the board or at their desks, either alone or in pairs. At the end of every lesson they were given homework to complete. They took 10-minute tests regularly at the beginning of lessons, based on the topic of the previous lessons. The tasks of these short tests formed the pre-assessment. With the help of the tasks of the pre-assessment and with the observation of the students' activities in the lessons we could do an appropriate categorization which levels the students are achieved in the APOS Theory. We divided the tasks of the tests into small parts and assigned an item to each part.

According to the underlying mathematical object the levels of the APOS model are the following.

Logarithms as functions

Definition of logarithmic function:

- *Action: F1* Students reach this level if they have the ability to calculate the value of the function at certain points by substituting the x value.
- *Process: F2* At this level the students can check the domain and the range of a logarithmic function or expression, and apply it in algebraic tasks. For example they can make assumptions for the solutions of logarithmic equations.

- *Object: F3* Students can check the zero and the monotonicity of a logarithmic function, and apply it in algebraic tasks. For example they can change the sign of inequalities if necessary.
- *Schema: F4* Students explore that the exponential function is the inverse of the logarithmic function. They can determine and graph the inverse function in the same coordinate system.

Transformation of logarithmic function:

- *Action: T1* At this level students are familiar with the graph of the general logarithmic function, but cannot manipulate it.
- *Process: T2* Students know some steps in function transformation, but they cannot solve a task about it completely or analyze the properties of functions.
- *Object: T3* Students know all steps in transformation of function and they are able to draft the graph of the transformed function and write down their properties.
- *Schema: T4* Students explore that the exponential function is the inverse of the logarithmic function. They can give correctly the inverse function and graph it in the same coordinate system.

Logarithms as operations

Logarithmic identities:

- *Action: O1* Students reach this level when they understand the concept of a logarithm. It means that they can convert a logarithmic expression into an exponential one and conversely.
- *Process: O2* At this level the students can correctly apply the laws of logarithms. They can find common new base of logarithmic expressions with more different bases.
- *Object: O3* Students can combine the logarithmic identities and with the help of this procedure they can completely solve tasks and check the obtained solutions.
- *Schema: O4* Students have a comprehensive knowledge of logarithms and they can explore rules that were not covered in the previous lessons.

Logarithms as numbers

- *Action: N1* Students are able to calculate the values of basic logarithmic expressions and to solve basic equations such as $log_a b = c$, provided two of the variables in the set of $\{a, b, c\}$ are given.

- *Process: N2* At this stage the students can calculate complex expressions involving multiple logarithmic terms using the notion of the logarithms.
- *Object: N3* Students can apply logarithms to calculate large numbers that calculators cannot define.
- *Schema: N4* Students can combine their knowledge of exponentiation and logarithms to solve complex exercises.

Algebraic knowledge about logarithms:

- *Action: A1* Students can identify a task leading either to a system of equations or to a quadratic equation.
- *Process: A2* Students reach this stage if either they can correctly express one of the two variables in a system of equations and put it back into the original system to obtain the other variable or they know the quadratic formula.
- *Object: A3* Students can transform a logarithmic equation into a quadratic one and can find the correct solution.
- *Schema: A4* Students can explore either systems of equations or the usage of the quadratic formula in tasks, where their applicability is not obvious. If the students can identify the solution of the problem at a glance without using any learnt method to solve it, then they reach this level, too.

After the students learnt the planned topic they practiced it in cooperative group work. We formed 4 groups consisting of pupils with different abilities [24, p. 449], and pointed out that social conflicts do not prevent work [19, p. 71]. Table 1 shows the composition of the groups, where action, process, object, schema refer to the group members' levels in the APOS Theory.

Table 1. The composition of the groups during the group work.

Group 1	Group 2	Group 3	Group 4
Action	Action	Action	Action
Process	Process	Process	Object
Object	Object	Object	Schema
Schema	Schema	Schema	–

Every group got the same worksheet. The task of the students was to work together and explain the solutions of the exercises to each other. The mixed composition of the groups allows us to observe the following. When children work together cooperatively, they learn to give and accept help, share their ideas and listen to other students' perspectives, seek new ways to resolve differences, solve problems and gain new knowledge and understanding [18, p. 35]. After the students worked on the worksheet, a group member presented the steps of the solution of a task on the board. The presenters were pupils, who had reached a lower level of the APOS Theory. In this way we could check, that they understood the exercises as well. All tasks were solved by at least one group. Therefore, we could discuss all issues. All students had to listen, when the exercises were presented, either because their group had not solved a particular problem, or because the presenter showed an alternative solution method.

After the group work students returned to the same classroom activities as before. They got homework and took 10-minute tests at the beginning of each lesson. The tasks of these short tests formed the post-assessment. By evaluating the post-assessment tasks and observing the students' activities in the lessons we were able to monitor their abilities and reclassified them according to the APOS Theory (see Section 4).

3.2. Categorization of the students

Our categorization of the students into the levels of APOS Theory before and after the group work is found in Figure 7. This categorization is based on the exercises of the pre- and post-assessments. Table 2 shows the codes of the items of the necessary knowledge corresponding to each level of the APOS Theory as well as the tasks, in which these items appeared before and after the group work and on the worksheet of the group work. Five items correspond to the levels action, process, object and schema according to the underlying mathematical object (see Section 3.1). An individual reached a level when they possessed at least three items of that level. The item O4 cannot be observed in the tasks of the tests. We assessed students' knowledge of this item through their activity in lessons.

Students could own an item, if they completed it in at least half of the tasks in which the item occurred. For example, an individual could reach item F2, which appeared in five tasks, if they completed this item in at least three tasks. This limit was our choice. If a student failed to solve an item less than or equal to the half of the tasks in the tests, then we could ensure in the lessons after the tests that he or she owned the knowledge of this item, only he or she was careless in the test. Our limit is legitimated by the fact, that during the learning process all students could make typical mistakes because the APOS Theory is not a linear progression [26]. Moreover, when a student needed the guide of the teacher by solutions of problems in the lessons, he or she will be placed at the action level. When someone could solve a problem at first sight or could discover new connections, the individual will be placed at the schema level. For example one student (S1) explored the rule: $\log_{a^n} b = \frac{1}{n} \log_a b$, which was not discussed in the previous lessons. He could prove and appropriately use it.

The authors created the classification into levels of the APOS Theory. We personally discussed the case of each student to place them in the appropriate level of the APOS Theory. In cases of disagreement we reviewed the documents again and found a common solution.

The topic of logarithms in connection to the APOS Theory was also discussed

Table 2. The items belonging to the levels of the APOS Theory and the tasks, in which these items occurred. The *Pr-, Po- and G-* prefixes indicate that the task occurred in the pre-assessment, in the post-assessment and in the group work, respectively. The numbers mean the ordinal number of the tasks.

Levels	Code	Tasks before	Tasks after	Group work
Action	F1	Pr1.	Po1.	G1.
	T1	Pr1.	Po1.	G1.
	O1	Pr1., Pr3., Pr4. Pr5., Pr6.	Po1., Po3., Po4. Po5., Po6.	G1., G3., G4.
	N1	Pr1., Pr3., Pr4. Pr5., Pr6., Pr7. Pr8.	Po2., Po3., Po4. Po5., Po6., Po7. Po8.	G2., G4., G5. G6., G8.
	A1	Pr3., Pr4., Pr5. Pr8.	Po3., Po4., Po5. Po8.	G5., G6.
Process	F2	Pr1., Pr3., Pr4. Pr5., Pr6.	Po1., Po3., Po4. Po5., Po6.	G1., G3., G4.
		Pr1.	Po1.	G1.
	O2	Pr2., Pr3., Pr4. Pr5., Pr7., Pr8.	Po2., Po3., Po4. Po5., Po7., Po8.	G3., G4., G5. G6., G7., G8.
	N2	Pr2., Pr5., Pr7. Pr8.	Po3., Po5., Po7. Po8.	G2., G3., G4. G5., G6., G8.
	A2	Pr3., Pr4., Pr5. Pr8.	Po3., Po4., Po5. Po8.	G5., G6.
Object	F3	Pr1., Pr6.	Po1., Po6.	G1.
	T3	Pr1.	Po1.	–
	O3	Pr2., Pr3., Pr4. Pr5., Pr7., Pr8.	Po2., Po3., Po4. Po5., Po7., Po8.	G3., G4., G5. G6., G7., G8.
	N3	Pr7.	Po7.	G4.
	A3	Pr3., Pr4.	Po3., Po4.	G6.
Schema	F4	Pr1.	Po1.	–
	T4	Pr1.	Po1.	–
	O4	–	–	–
	N4	Pr7.	Po7.	G2., G4., G6.
	A4	Pr8.	Po8.	G5.

in [21, 31]. In these articles the authors establish a categorization into the stages of the APOS Theory corresponding to the specific tasks what they discussed with their students. The fact that the logarithm is the inverse of exponentiation can be used to solve logarithmic equations [34, 35]. The adaptation of the levels of APOS Theory in [31] roots on this approach. In [21] the thinking of the students about logarithms is examined using card games and the stages of the APOS Theory are adapted to the issues of the game. We applied our categorization to the problems our students were confronted with.

The Figures 5 and 6 give the correct answer of students S5 and S7 for the transformation of the function (8.1) and the solution of the equation (8.3), respectively.

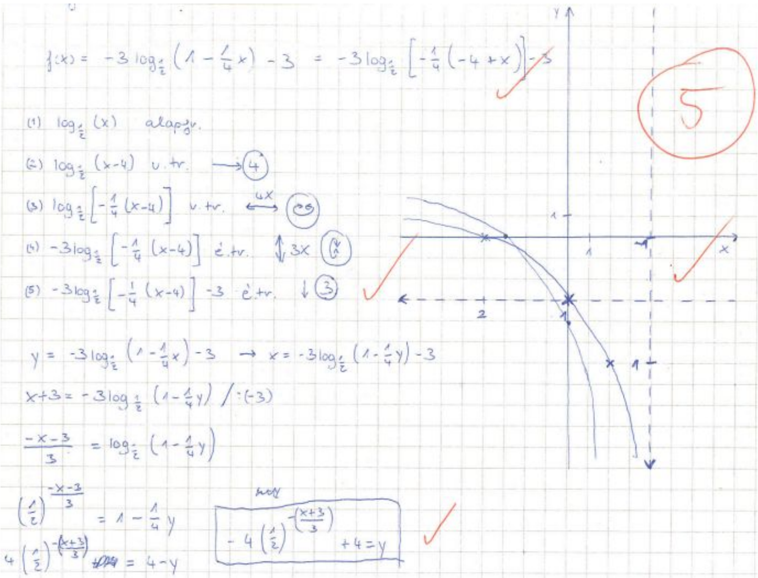

Figure 5. The correct solution of the student S5 for the function transformation and the determination of the inverse function of (8.1).

4. Results

Figure 7 shows the levels of the APOS Theory reached by the students before and after the group work, based on Table 2.

All students (S3, S6, S9, S13) who were placed at the action level before the group work developed them and moved to the process level. One student (S14) remained at the process level and two students (S2, S4) progressed to the object level from the process level at the end of the experiment.

The following example illustrates the development of student S2. Figure 8 shows his solution to the task Pr1. before the group work. Although he knew the basic steps of the function transformation, he did not apply them correctly in the task

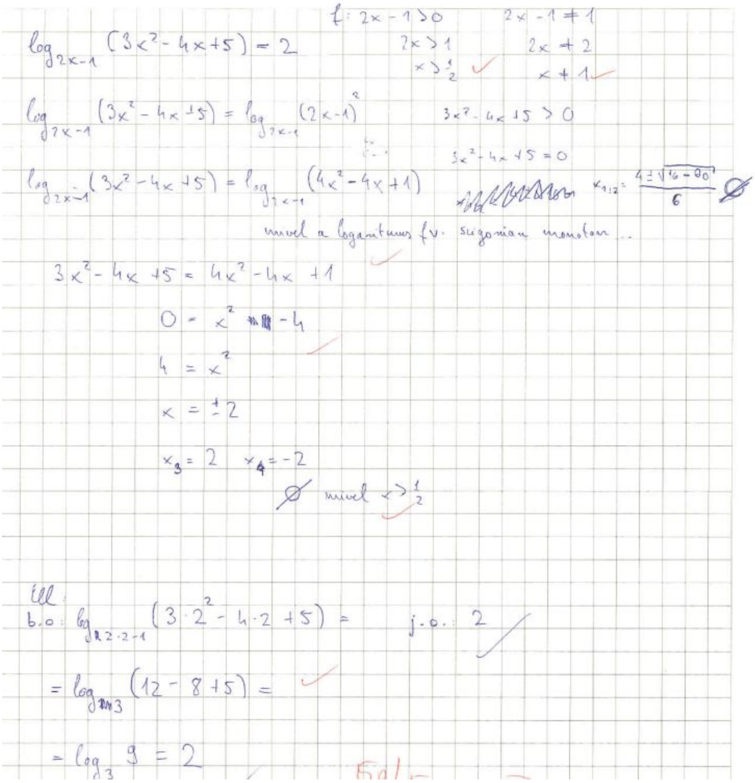

Figure 6. The correct solution of the student S7 for the logarithmic equation (8.3).

because he did not rewrite the function into the form (8.2). Consequently, he was unable to draft the graph of the function and determine its inverse. After the group work he carried out almost correctly the function transformation for the function (8.5). However, he did not find the inverse function (see Figure 9).

The Figures 10 and 11 illustrate the development of student S13, who progressed from the action level to the process level. Before the group work and after the group work the task was to find the solutions of the logarithmic equations (8.4) and (8.6), respectively. Before the group work (see Figure 10) he found the condition $x > 0$ on the variable x by testing the domain of the logarithmic expressions and using the definition of the logarithm he knew $1 = \lg 10$. However, he could not solve the task, because he did not apply the appropriate solution method. After the group work (see Figure 11) he could solve the exercise Po4. almost correctly. He could use the laws of logarithms, change the base of the logarithmic expression, introduce a new variable, transform the logarithmic equation into a quadratic one, and solve it correctly. However, he incorrectly determined the domain of the logarithmic

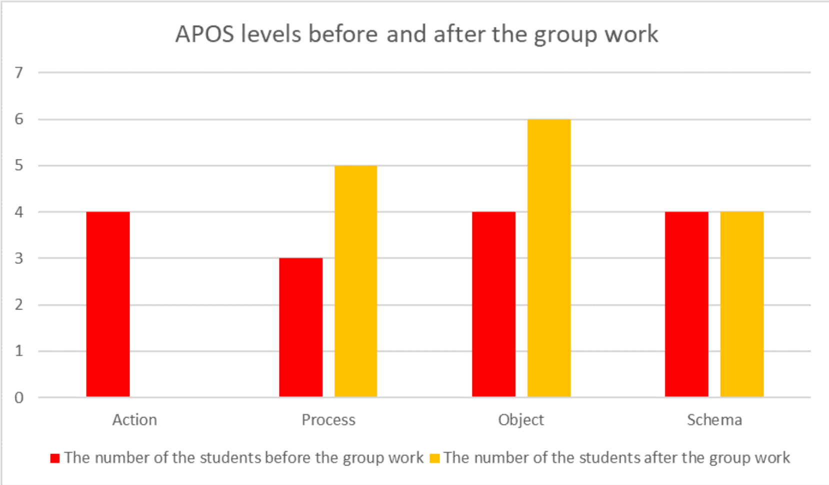

Figure 7. The development of the students belonging to the levels of the APOS Theory. Before the group work the action, the process, the object and the schema levels were reached by the students (S3, S6, S9, S13), (S2, S4, S14), (S7, S8, S12, S15) and (S1, S5, S10, S11), respectively. After the group work the process, the object and the schema levels were reached by the students (S3, S6, S9, S13, S14), (S2, S4, S7, S8, S12, S15) and (S1, S5, S10, S11), respectively.

expression. In this task he could reach the items belonging to the object level. However, he made mistakes in other tasks where these items occurred. Hence he was categorized at the process level instead of the object level.

5. Discussion

The Cronbach's Alpha of the pre-assessment is 0.912, and that of the post-assessment is 0.897.

We applied the Fisher's exact test to examine whether the cooperative learning method significantly influenced the performance of the students [16]. Our null hypothesis is that there does not exist significant connection between the improvement of the students' performance and the cooperative learning method. The alternative hypothesis is that the cooperative group work considerably affects the achievement of the pupils. We examined our hypotheses at a 5% significance level. For our statistical calculations we used the SPSS Statistics program. We summarized the results in Table 3. The first column includes the items belonging to the levels of the APOS Theory. The second and third columns contain the number of students who possessed the items before and after the cooperative group work, respectively. The fourth column includes the p-value of the Fisher's exact test.

We observed a significant difference in the numbers of the correct solutions

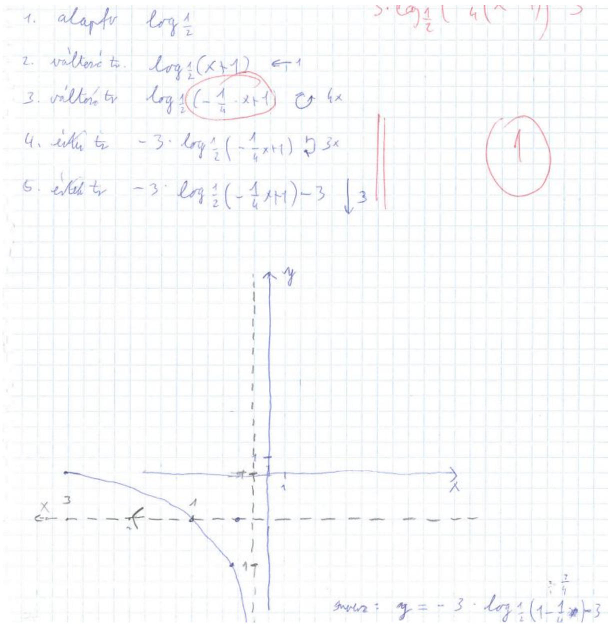

Figure 8. The wrong solution of the task Pr1. by the student S2 before the group work.

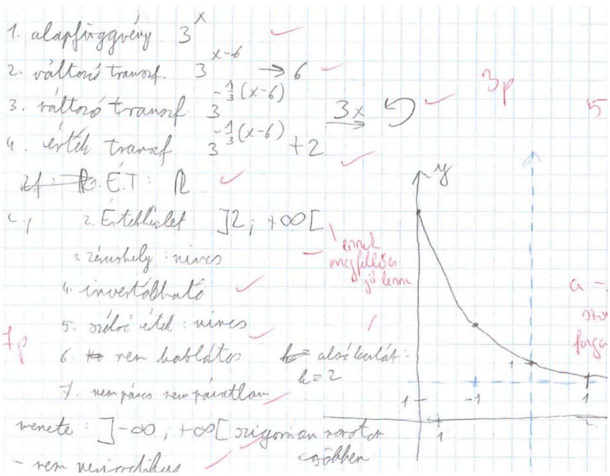

Figure 9. The solution of the task Po1. by the student S2 after the group work.

before and after the cooperative group work with respect to the items O1, O2 and N2. The reason, why the number of the students who gave correct solutions of these items increased significantly after the cooperative group work can be that these

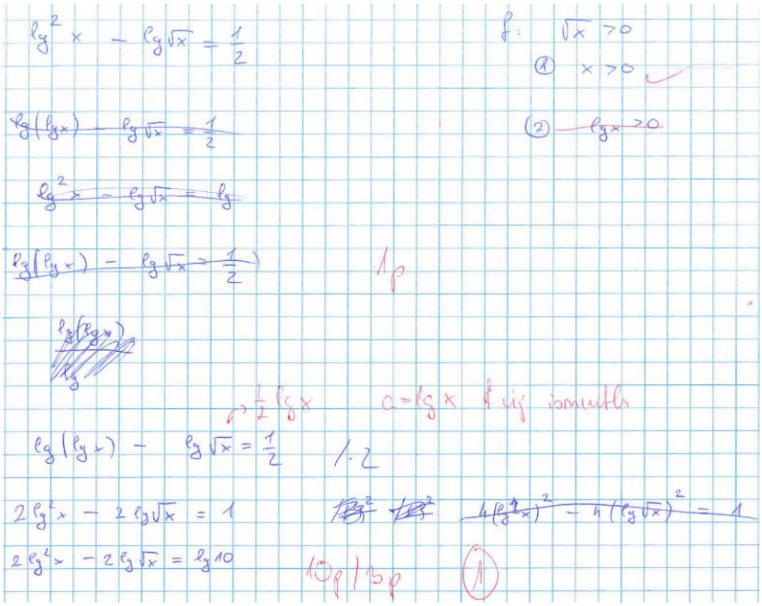

Figure 10. The wrong solution of the task Pr4. by the student S13 at the action level.

Table 3. The results of the Fisher's exact test. The * sign denotes, where the p-value is less than 0.05. In these cases the alternative hypothesis is valid.

Item	Before	After	p-Value	Item	Before	After	p-Value
F1	14	15	1.000	F3	8	10	0.710
T1	13	15	0.483	T3	8	10	0.710
O1	10	15	0.042*	O3	6	7	1.000
N1	12	15	0.224	N3	7	10	0.462
A1	13	15	0.483	A3	8	11	0.450
F2	12	14	0.598	F4	5	6	1.000
T2	13	13	1.000	T4	5	6	1.000
O2	8	14	0.035*	O4	3	3	1.000
N2	8	15	0.006*	N4	3	3	1.000
A2	12	15	0.224	A4	5	7	0.710

items very often appeared on the worksheet of the group work (see Table 2). This suggests that the cooperative learning method considerably impacted students' knowledge of these items.

Although more students gave correct solutions to the other items (expect T2, O4

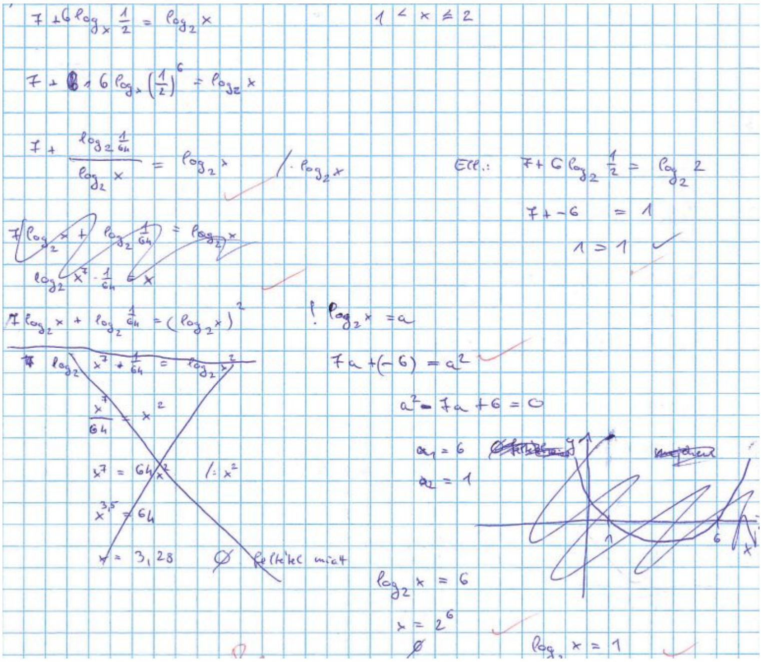

Figure 11. The almost correct solution of the task Po4. by the student S13 at the process level.

and N4), we did not obtain a significant difference before and after the cooperative group work. In the case of the items F1, T1, N1, A1 of the action level and the items F2, T2 and A2 of the process level the reason can be that the most students owned the items already before the group work. For the items F3, T3, O3, N3, A3 belonging to the object level the reason may be that these items were complex. Some students could not finish the tasks and therefore they did not complete the items T3, O3 and A3. Some pupils could not connect the functional and the algebraic meanings of the logarithm, hence only some students could answer the item F3 correctly. Some students could not apply the logarithm as a number in the tasks, so they could not complete the item N3. The same students reached the schema level before and after the group work. This could explain why the Fisher's exact test did not show a significant difference for these items.

The Fisher's exact test refers to the results of the whole group. We wish to examine the individual improvement of students, too. Six students (S2, S3, S4, S6, S9, S13) could achieve a higher level of the APOS Theory (see Figure 7). To check their individual developments we used Paired Samples t-test [27]. We compared their results in pre- and post-assessments as a percentage. The tasks with the same ordinal numbers formed a pair (for example Pr1. and Po1.). To get the paired differences we subtracted the results of the pre-assessment from that of the post-assessment. The distribution of the differences of the paired values was normal.

Our null hypothesis is that the mean of the obtained results of one student in the pre-assessment coincides with the mean of the obtained results of the same student in the post-assessment. The alternative hypothesis is that the mean of the obtained results of one student in the pre-assessment differs from the mean of the obtained results of the same student in the post-assessment. We examined our hypotheses at a 5% significance level. We used the SPSS Statistics program for our statistical calculations. We summarized the results in Table 4. The first column includes the code of the students. The second, the third and the fourth columns contain the mean, the standard deviation and the standard error mean of the paired differences, respectively. The fifth column includes the two sided p-value of the Paired Samples t-test.

Table 4. The results of the Paired Samples t-test. The * sign denotes, where the p-value is less than 0.05. In these cases the alternative hypothesis is valid.

Code	Mean	Std. Deviation	Std. error mean	p-value
S2	25.00000	37.79645	13.36306	0.104
S3	31.25000	25.87746	9.14906	0.011*
S4	34.37500	37.64852	13.31076	0.036*
S6	31.25000	25.87746	9.14906	0.011*
S9	21.87500	48.98524	17.31890	0.247
S13	28.12500	47.12730	16.66202	0.135

Because the means of the paired differences are positive for all students, we can establish that all students improved. We can reject the null hypothesis and accept the alternative hypothesis by the students S3, S4, S6. We also show the development of the students S2 and S13 in Section 4 (see Figures 8, 9, 10, 11).

The reliability of our tasks is proven by the relatively high Cronbach's Alphas. Our tests are also objective because all students wrote them under the same circumstances and received the same instructions, and we corrected them always with the help of a solution key [7, p. 6]. Moreover, our assessments are also valid, because we can assign each task to at least one item of the levels of the APOS Theory. This means that the tasks really measure students' development applying APOS Theory.

6. Conclusions

To sum up our research has allowed us to answer our questions and accept our hypothesis.

Q1 Which levels of the APOS Theory can students reach before and after the cooperative group work?

The Figure 7 shows how many students reached each level of the APOS Theory before and after the group work. The personalized classification of the students was useful in our experiment. On the one hand it was beneficial to form mixed groups for the group work, i.e. each group consisted of students at different APOS Theory levels (see Table 1). On the other hand we could objectively observe and measure the development of the students' mathematical thinking during the experiment. Therefore, we believe that in our experiment the APOS Theory helps to make the learning process more effective [21, 31].

Q2 How can the students benefit from the experiment?

The papers [18, 19, 24] demonstrate the positive effects of the group work on student performance. Our experiment reinforced this finding. The mathematical thinking and the problem solving abilities of the students developed. Six pupils who were categorized into action or process level before the group work were able to reach a higher level of APOS Theory after the group work (see Figure 7). They became more motivated and more active in the lessons after the cooperative learning. We think that these students attached great importance to the opinions of their classmates [32, p. 56]. Therefore, we detected that they made more effort to learn mathematics after the group work. As a result of their increased activity and motivation, they performed better on the post-assessment (see Table 4, and Figures 8, 9, 10, 11).

The students who reached before the group work the object and schema levels of the APOS Theory, deepened their knowledge by explaining the solution of the tasks of the group work to their classmates. They were able to recall and apply what they learnt about logarithms in other topics, for example in trigonometry.

H1 Some students can progress to a higher level of the APOS Theory during the experiment.

We accepted this hypothesis using the Fisher's exact test, which shows the development of the whole group [16], and the Paired Samples t-test [27], which justifies the individual improvement of some students. Using Fisher's exact test we found a significant connection between the cooperative group work and the achievement of the students with respect to the items O1, O2, N2 (see Table 3). All students in the action level and two students in the process level before the group work could reach the process and object levels after that, respectively. Among these pupils the development of three students was significant using the Paired Samples t-test (see Table 4).

7. Limitation

The limitation of the research is, that we do not have a control group. The improvement of the performance of the students can depend on more factors. Hence we cannot be sure whether the changes was due to the effect of the group work. It can be influenced also by other variables such as the choice of tasks, the passage of time or the combination of these factors. Based on the presented observational study we can only infer the development of the individual performance of the students. Increased activity and motivation during the learning process after group work can significantly impact positive changes in student performance.

8. Appendix

Tasks of the pre-assessment

Pr1. Plot the following function and its inverse in the same coordinate system! Write the steps of the function transformation! Analyze the function!

$$f(x) = -3\log_{\frac{1}{2}}\left(1-\frac{1}{4}x\right)-3 \tag{8.1}$$

Solution:

Firstly the students have to overwrite the function (8.1) as

$$f(x) = -3\log_{\frac{1}{2}}\left[-\frac{1}{4}(-4+x)\right]-3. \tag{8.2}$$

Using form (8.2) they can graph the function. The steps of the function transformation are:

- general function: $\log_{\frac{1}{2}} x$,
- translation along the x-axis with $+4$,
- stretching the function along the x-axis to four times,
- reflection of the function on the y-axis,
- stretching the function along the y-axis to three times,
- reflection of the function on the x-axis,
- translation along the y-axis with -3.

The properties of the function are:

- domain: $]-\infty; 4[$
- range: $\mathbb{R}$
- zero: -4
- monotony: monoton decreasing
- extremum: –
- boundary: unbounded (unbounded from below and above)
- symmetry: not symmetric
- periodicity: not periodic

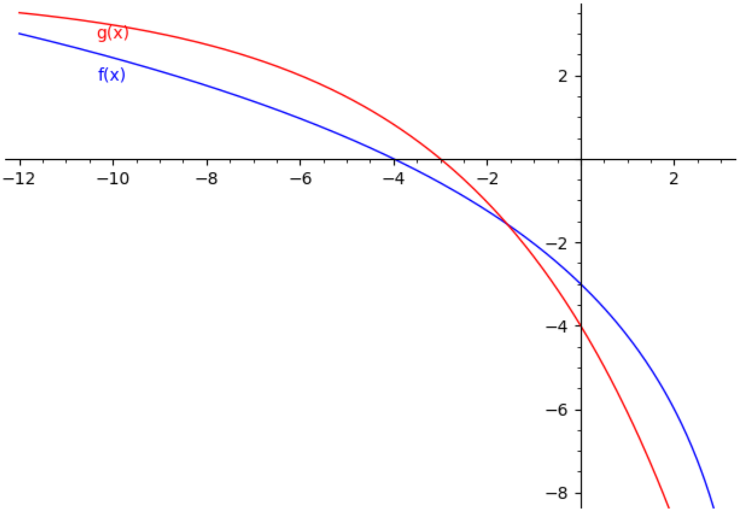

Figure 12. The graphs of the task Pr1.

The inverse of the function is:

$$f^{-1}(x) = g(x) = -4\left(\frac{1}{2}\right)^{-\frac{x+3}{3}} + 4$$

. The levels of the APOS Theory are

- Action: if the student can calculate the values of the function at certain points by substituting the x value or graph the general function.
- Process: if the student can check the domain and the range of the logarithmic function and recognize some steps of the function transformation.
- Object: if the student knows all steps in function transformation, can graph the function and describe its properties.
- Schema: if the student can determine and graph the inverse function.

Pr2. Let $x = \frac{4r^3\pi}{3}$. Determine $\log x$!

Solution:

$$\log x = \log 4 + 3\log r + \log \pi - \log 3$$

The levels of the APOS Theory are

- Action: if the student understands the concept of a logarithm with a base of 10.
- Process: if the student knows some logarithmic rules.
- Object: if the student knows all the necessary logarithmic identities.

- Schema: due to the simplicity of the task, the schema level was not identified in this case.

Pr3. Solve the logarithmic equation in the set of real numbers!

$$\log_{2x-1}(3x^2 - 4x + 5) = 2 \tag{8.3}$$

Solution:

Domain: $x > \frac{1}{2}$, $x \neq 1$, $3x^2 - 4x + 5 > 0$

$$\log_{2x-1}(3x^2 - 4x + 5) = \log_{2x-1}(2x - 1)^2$$

Since the logarithmic function is bijective

$$3x^2 - 4x + 5 = 4x^2 - 4x + 1$$

$x_1 = 2$ and $x_2 = -2$ but -2 does not satisfy the conditions.

The levels of the APOS Theory are

- Action: if the student can transform the right-hand side of (8.3) to the form $\log_{2x-1}(2x - 1)^2$.
- Process: if the student can find the domain of the expression, explore the quadratic equation and know the quadratic formula.
- Object: if the student can completely solve the task.
- Schema: due to the simplicity of the task we did not identify in this case the schema level.

Pr4. Solve the logarithmic equation in the set of real numbers!

$$\log^2 x - \log \sqrt{x} = \frac{1}{2} \tag{8.4}$$

Solution:

Domain: $x > 0$

$$\log^2 x - \frac{1}{2}\log x = \frac{1}{2}$$

$$\text{if } \log x = a$$

$$a^2 - \frac{1}{2}a = \frac{1}{2}$$

$$a_1 = 1 \text{ so } \log x = 1, \text{ i.e. } x = 10 \text{ and}$$

$$a_2 = -0.5 \text{ so } \log x = -0.5, \text{ i.e. } x = \frac{1}{\sqrt{10}}$$

The levels of the APOS Theory are

- Action: if the student can identify the task as a quadratic equation.

- Process: if the student can find the domain of the expression, and the quadratic equation and knows the quadratic formula.
- Object: if the student can completely solve the task.
- Schema: because of the simplicity of the task we did not identify in this case the schema level.

Pr5. Solve the system of equations in the set of real numbers!

$$\begin{cases} \log_{12} x + \log_{12} y = 1 + \log_{12} 5 \\ \log(2y - x) = 1 - \log 5 \end{cases}$$

Solution:

Domain: $x > 0$, $y > 0$ and $2y > x$

$$\log_{12} x + \log_{12} y = \log_{12} 12 + \log_{12} 5$$

$$\log_{12} xy = \log_{12} 60$$

Since the logarithmic function is bijective

$$xy = 60$$

$$x = \frac{60}{y}$$

$$\log(2y - x) = \log 10 - \log 5$$

$$\log(2y - x) = \log 2$$

if we substitute $x = \frac{60}{y}$, then we obtain

$$\log(2y - \frac{60}{y}) = \log 2$$

Since the logarithmic function is bijective

$$2y - \frac{60}{y} = 2$$

if $y_1 = 6$, then $x_1 = 10$

if $y_2 = -5$, then $x_2 = -12$ these do not satisfy the conditions

The levels of the APOS Theory are

- Action: if the student can identify the task as a system of equations, transform 1 to $\log_{12} 12$ and $\log 10$.
- Process: if the student finds the domain of the expression, correctly expresses one of the two variables, correctly applies the laws of logarithms, writes the quadratic equation and knows the quadratic formula.
- Object: if the student can solve the task completely.
- Schema: because of the simplicity of the task we did not identify in this case the schema level.

Pr6. Solve the logarithmic inequality in the set of real numbers!

$$\log_{0.5}(2x + 100) \geq -8$$

Solution:

Domain: $x > -50$

$$\log_{0.5}(2x + 100) \geq \log_{0.5} 256$$

Since this logarithmic function is monotone decreasing

$$2x + 100 \leq 256$$
$$-50 < x \leq 78$$

The levels of the APOS Theory are

- Action: if the student can transform -8 to $\log_{0.5} 256$.
- Process: if the student finds the domain of the expression.
- Object: if the student changes the inequality sign and solves the inequality.
- Schema: because of the simplicity of the task we did not identify in this case the schema level.

Pr7. Determine the value of A with the help of a calculator!

$$A = \frac{65^{117} \cdot 203^{107}}{18^{328} \cdot 10^{49}}$$

Solution:

Since both sides of the expression are positive

$$\log A = \log \frac{65^{117} \cdot 203^{107}}{18^{328} \cdot 10^{49}} =$$
$$= 117 \log 65 + 107 \log 203 - 328 \log 18 - 49 \log 10$$
$$\log A = -1.7164$$
$$A = 0.0192$$

The levels of the APOS Theory are

- Action: if the students can realize that they have to use the logarithm to base 10.
- Process: if the student can apply the laws of logarithms.
- Object: if the student can apply logarithms to calculate large numbers that the calculator cannot define.
- Schema: if the student can calculate the value of A.

Pr8. Express $\log 6$ with the help of a and b if $a = \log 48$ and $b = \log 72$!

Solution:

$$\begin{cases} a = 4\log 2 + \log 3 \\ b = 3\log 2 + 2\log 3 \end{cases}$$

from the first equation

$$\log 3 = a - 4\log 2$$

if we substitute it into the second equation

$$3\log 2 + 2(a - 4\log 2) = b$$

$$\log 2 = \frac{2a - b}{5}$$

$$\log 3 = a - 4 \cdot \frac{2a - b}{5} = \frac{-3a + 4b}{5}$$

$$\log 6 = \log 2 + \log 3 = \frac{2a - b}{5} + \frac{-3a + 4b}{5} = \frac{3b - a}{5}$$

The levels of the APOS Theory are

- Action: if the student can identify the problem as a system of equations.
- Process: if the student can apply the laws of logarithms to express a and b and can correctly express one of the two variables.
- Object: if the student can completely solve the task.
- Schema: due to the simplicity of the task, the schema level was not identified in this case.

Worksheet of the group work

G1. The Figure 13 shows the graph of the function $h\colon \mathbb{R}^+ \to \mathbb{R}; x \mapsto \log_a x$.

(a) Determine the value of a!

(b) Determine the following function values: $h(9^5); h(\frac{1}{27^3}); h(3^{1.8})$!

(c) What is the value of b, if $h(b) = -4$?

G2. Simplify the following expression!

$$\log_{\frac{1}{3}} log_3 \sqrt[3^n]{\sqrt[3^{n-1}]{\ldots \sqrt[9]{\sqrt[3]{3}}}} =$$

G3. Assume $a > 0, b > 0$. Prove that if $a^2 + b^2 = 7ab$, then $\log \frac{a+b}{3} = \frac{1}{2}(\log a + \log b)$!

G4. Determine the value of the following expression with the help of a calculator!

$$21.11^{100} - 21.1^{100}$$

G5. Given $\log 12 = 1.0792$ and $\log 18 = 1.2553$. Determine the values of $\log 2$; $\log 3$; $\log 5$ in decimal!

G6. How many elements are in the solution set of the equation $\sqrt[\log x]{x} = 10^{x^4}$?

G7. Let $c = \log_a b$. Express the following logarithms in the term of c!

$$\log_a (ab); \quad \log_a \frac{a}{b}; \quad \log_a b^3; \quad \frac{\log a}{\log b}$$

G8. Solve the equation in the set of integers!

$$\frac{\log_x (35 - x^3)}{\log_x (5 - x)} = 3$$

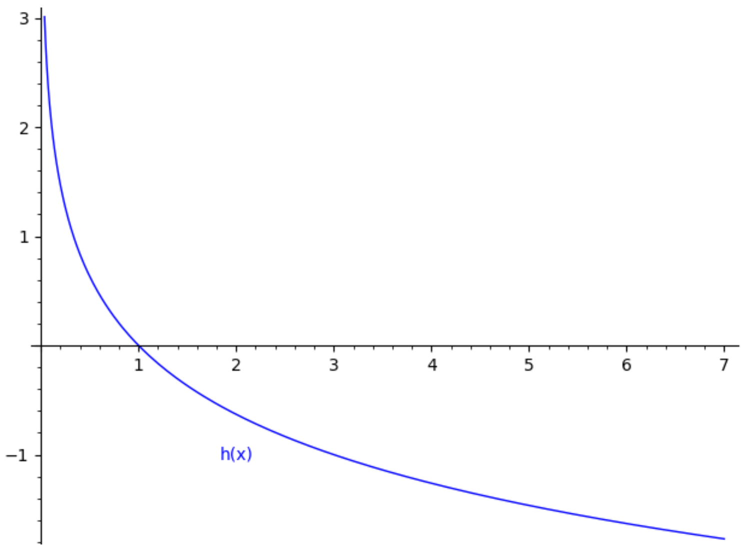

Figure 13. The function of the task G1.

Solutions of Worksheet of the group work

G1. The Figure 13 shows the graph of the function $h \colon \mathbb{R}^+ \to \mathbb{R}; x \mapsto \log_a x$.

(a) Determine the value of a!
$a = \frac{1}{3}$

(b) Determine the following function values: $h(9^5); h(\frac{1}{27^3}); h(3^{1.8})$!

$$h(9^5) = -10,$$

$$h\left(\frac{1}{27^3}\right) = 9,$$
$$h(3^{1.8}) = -1.8.$$

(c) What is the value of b, if $h(b) = -4$?
$b = 81$

G2. Simplify the following expression!

$$\log_{\frac{1}{3}} log_3 \sqrt[3^n]{\sqrt[3^{n-1}]{\dots \sqrt[9]{\sqrt[3]{3}}}} =$$
$$= \log_{\frac{1}{3}} log_3 3^{\frac{1}{3^1 \cdot 3^2 \cdots 3^n}} =$$
$$= \log_{\frac{1}{3}} \frac{1}{3^1 \cdot 3^2 \cdots 3^n} = \log_{\frac{1}{3}} \left[\left(\frac{1}{3}\right)^1 \cdot \left(\frac{1}{3}\right)^2 \cdots \left(\frac{1}{3}\right)^n\right] =$$
$$= \log_{\frac{1}{3}} \left(\frac{1}{3}\right)^{1+2+\cdots+n} = \frac{n(n+1)}{2}$$

G3. Assume $a > 0, b > 0$. Prove that if $a^2 + b^2 = 7ab$, then $\log \frac{a+b}{3} = \frac{1}{2}(\log a + \log b)$!

$$(a+b)^2 = 9ab$$
$$a + b = 3\sqrt{ab}$$

Since both sides of the equation are positive:

$$\log(a+b) = \log(3\sqrt{ab})$$
$$\log(a+b) = \log 3 + \log\sqrt{ab}$$
$$\log(a+b) - \log 3 = \log\sqrt{ab}$$
$$\log\frac{a+b}{3} = \frac{1}{2}(\log a + \log b)$$

G4. Determine the value of the following expression with the help of a calculator!

$$21.11^{100} - 21.1^{100}$$

$$\log 21.11^{100} = 100 \log 21.11 = 132.4488$$
$$21.11^{100} = 10^{132.4488} = 10^{132} \cdot 10^{0.4488} = 2.8106 \cdot 10^{132}$$

$$\log 21.1^{100} = 100 \log 21.1 = 132.4282$$
$$21.1^{100} = 10^{132.4282} = 10^{132} \cdot 10^{0.4282} = 2.6804 \cdot 10^{132}$$

$$2.8106 \cdot 10^{132} - 2.6804 \cdot 10^{132} = 1.302 \cdot 10^{131}$$

G5. Given $\log 12 = 1.0792$ and $\log 18 = 1.2553$. Determine the values of $\log 2$; $\log 3$; $\log 5$ in decimal!

$$\log 12 = \log 2 \cdot 2 \cdot 3 = \log 2 + \log 2 + \log 3 = 2\log 2 + \log 3$$
$$\log 18 = \log 2 \cdot 3 \cdot 3 = \log 2 + \log 3 + \log 3 = \log 2 + 2\log 3$$

If $\log 2 = a$ and $\log 3 = b$, then we have

$$\begin{cases} 2a + b = 1.0792 \\ a + 2b = 1.2553 \end{cases}$$

From the second equation we obtain

$$a = 1.2553 - 2b$$

if we substitute it into the first equation

$$2(1.2553 - 2b) + b = 1.0792$$
$$b = 0.4771, \text{ i.e. } \log 3 = 0.4771$$
$$a = 0.3011, \text{ i.e. } \log 2 = 0.3011$$

$$\log 5 = \log \frac{10 \cdot 3}{3 \cdot 2} = \log 10 \cdot 3 - \log 3 \cdot 2 = \log 10 + \log 3 - \log 3 - \log 2 =$$
$$= 1 - 0.3011 = 0.6989$$

G6. How many elements are in the solution set of the equation $\sqrt[\log x]{x} = 10^{x^4}$?

Domain: $x > 0$

$$x^{\frac{1}{\log x}} = 10^{x^4}$$

Since both sides of the equation are positive:

$$\log x^{\frac{1}{\log x}} = \log 10^{x^4}$$
$$\frac{1}{\log x} \log x = x^4$$
$$1 = x^4$$
$$x_1 = -1 \text{ and } x_2 = 1$$

Taking into account the condition for the domain there is only one element in the solution set. It is $x_2 = 1$.

G7. Let $c = \log_a b$. Express the following logarithms in the term of c!

$$\log_a (ab); \qquad \log_a \frac{a}{b}; \qquad \log_a b^3; \qquad \frac{\log a}{\log b}$$

$$\log_a (ab) = \log_a a + \log_a b = 1 + c$$

$$\log_a \frac{a}{b} = \log_a a - \log_a b = 1 - c$$
$$\log_a b^3 = 3 \log_a b = 3c$$
$$\frac{\log_a a}{\log_a 10} \cdot \frac{\log_a 10}{\log_a b} = \frac{\log_a a}{\log_a b} = \frac{1}{c}$$

G8. Solve the equation in the set of integers!

$$\frac{\log_x (35 - x^3)}{\log_x (5 - x)} = 3$$

Domain: $\sqrt[3]{35} > x$; $5 > x$; $x > 0$ and $x \neq 1$

$$\log_x (35 - x^3) = 3 \log_x (5 - x)$$
$$\log_x (35 - x^3) = \log_x (5 - x)^3$$

Since the logarithmic function is bijective:

$$35 - x^3 = 125 - 75x + 15x^2 - x^3$$
$$15x^2 - 75x + 90 = 0$$
$$x^2 - 5x + 6 = 0$$
$$x_1 = 2 \text{ and } x_2 = 3$$

Tasks of the post-assessment

Po1. Plot the following function and its inverse in the same coordinate system! Write the steps of the function transformation! Analyze the function!

$$f(x) = 3^{(2 - \frac{1}{3}x)} + 2 \tag{8.5}$$

Solution:

Firstly the students have to overwrite the function (8.5) as

$$f(x) = 3^{-\frac{1}{3}(x-6)} + 2$$

Using this form they can graph the function. The steps of the function transformation are

- basic function: 3^x,
- translation along the x-axis with $+6$,
- stretching the function along the x-axis to three times,
- reflection of the function on the y-axis,
- translation along the y-axis with $+2$.

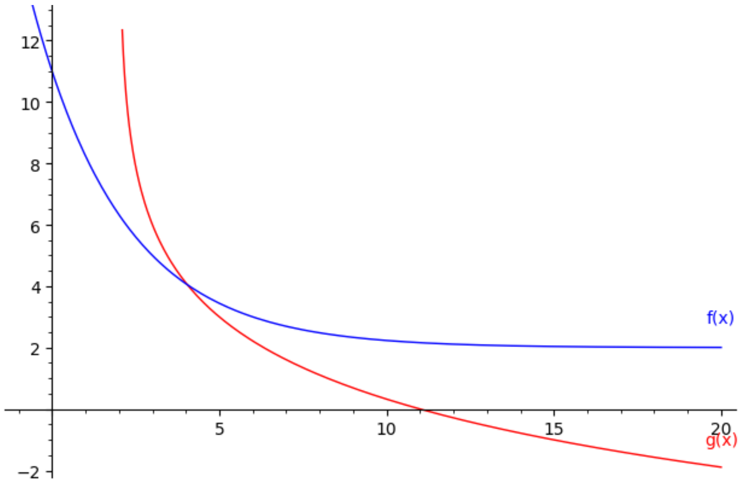

Figure 14. The graphs of the task Po1.

The properties of the function are

- domain: $\mathbb{R}$
- range: $]2;+\infty[$
- zero: –
- monotony: monoton decreasing
- extremum: –
- boundary: unbounded since bounded from below but unbounded from above
- symmetry: not symmetric
- periodicity: not periodic.

The inverse of the function is:

$$f^{-1}(x) = g(x) = -3\log_3(x-2) + 6$$

The levels of the APOS Theory are

- Action: if the student can calculate the values of the function at certain points by substituting the x value or graph the general function.
- Process: if the student can check the domain and the range of the logarithmic function and recognize some steps of the function transformation.
- Object: if the student knows all steps in function transformation, can graph the function and describe its properties.

- Schema: if the student can determine and graph the inverse function.

Po2. What is the value of the following expression $\log_a \frac{(a^2)^3 \cdot (a^{-4})^5 \cdot a^{16}}{a^2}$ ($a > 0$ and $a \neq 1$)?

Solution:

$$\log_a a^{6-20+16-2} = \log_a a^0$$
$$\log_a 1 = 0$$

The levels of the APOS Theory are

- Action: if the student understands the meaning of the logarithm to the base a.
- Process: if the student knows some rules of exponentiation.
- Object: if the student knows all of the necessary rules of exponentiation and can calculate the value of the expression.
- Schema: due to the simplicity of the task, the schema level was not identified in this case.

Po3. Solve the logarithmic equation in the set of real numbers!

$$\log(x+7) + \log(3x+1) = 2$$

Solution:

Domain: $x > -\frac{1}{3}$, $x > -7$

$$\log[(x+7)(3x+1)] = \log 100$$

Since the logarithmic function is bijective

$$3x^2 + 22x + 7 = 100$$

$$x_1 = 3 \text{ and } x_2 = -\frac{31}{3} \text{ but } -\frac{31}{3} \text{ does not satisfy the conditions.}$$

The levels of the APOS Theory are

- Action: if the student can transform 2 to $\log 100$.
- Process: if the student can correctly apply the logarithmic identities, explore the quadratic equation and knows the quadratic formula.
- Object: if the student can completely solve the task.
- Schema: due to the simplicity of the task we did not identify in this case the schema level.

Po4. Solve the logarithmic equation in the set of real numbers!

$$7 + 6\log_x \frac{1}{2} = \log_2 x \tag{8.6}$$

Solution:

Domain: $x > 0$, $x \neq 1$

$$7 + 6\frac{\log_2 \frac{1}{2}}{\log_2 x} = \log_2 x$$
$$\text{if } \log_2 x = a$$
$$7a - 6 = a^2$$
$$a_1 = 1 \text{ so } \log_2 x = 1, \text{ i.e. } x = 2 \text{ and}$$
$$a_2 = 6 \text{ so } \log_2 x = 6, \text{ i.e. } x = 64$$

The levels of the APOS Theory are

- Action: if the students figure out that they need to find a common new base to solve the task.
- Process: if the student finds the domain of the expression and the common new base, writes the quadratic equation and knows the quadratic formula.
- Object: if the student can completely solve the task.
- Schema: because of the simplicity of the task we did not identify in this case the schema level.

Po5. Solve the system of equations in the set of real numbers!

$$\begin{cases} x + y = 0.2 \\ \dfrac{\log x + \log y}{2} = \log \dfrac{x+y}{2} \end{cases}$$

Solution:

Domain: $x > 0$, $y > 0$

$$\frac{\log xy}{2} = \log \frac{0.2}{2}$$
$$\text{if we substitute } x = 0.2 - y$$
$$\frac{\log[(0.2 - y)y]}{2} = -1$$
$$\log(0.2y - y^2) = -2$$
$$\log(0.2y - y^2) = \log 0.01$$
$$\text{Since the logarithmic function is bijective}$$
$$0.2y - y^2 = 0.01$$
$$\text{hence } y = 0.1 \text{ and } x = 0.1$$

The levels of the APOS Theory are

- Action: if the student can identify the task as a system of equations and can calculate that $\log \frac{0.2}{2} = -1$.
- Process: if the student can find the domain of the expression, correctly express one of the two variables, correctly apply the laws of logarithms, write the quadratic equation and knows the quadratic formula.
- Object: if the student can completely solve the task.
- Schema: due to the simplicity of the task we did not identify in this case the schema level.

Po6. Solve the logarithmic inequality in the set of real numbers!

$$\log_{\frac{1}{3}}(2x+4) + 2 \geq 0$$

Solution:

Domain: $x > -2$

$$\log_{\frac{1}{3}}(2x+4) \geq -2$$
$$\log_{\frac{1}{3}}(2x+4) \geq \log_{\frac{1}{3}} 9$$

Since this logarithmic function is monotone decreasing

$$2x + 4 \leq 9$$
$$-2 < x \leq 2.5$$

The levels of the APOS Theory are

- Action: if the student can transform -2 to $\log_{\frac{1}{3}} 9$.
- Process: if the student finds the domain of the expression.
- Object: if the student changes the inequality sign and solves the inequality.
- Schema: because of the simplicity of the task we did not identify in this case the schema level.

Po7. Determine the value of A with the help of a calculator!

$$A = \frac{\sqrt[12]{23.8^{1100}}}{59.47^{68}}$$

Solution:

Since both sides of the expression are positive

$$\log A = \log \frac{23.8^{\frac{1100}{12}}}{59.47^{68}} =$$
$$= \frac{1100}{12} \log 23.8 - 68 \log 59.47$$

$$\log A = 5.534$$
$$A = 341948.93$$

The levels of the APOS Theory are

- Action: if the students can realize that they have to use the logarithm to base 10.
- Process: if the student can apply the laws of logarithms.
- Object: if the student can apply logarithms to calculate large numbers that the calculator cannot define.
- Schema: if the student can calculate the value of A.

Po8. Express $\log 15$ in terms of a and b if $a = \log 75$ and $b = \log 45$!

Solution:

$$\begin{cases} a = 2\log 5 + \log 3 \\ b = \log 5 + 2\log 3 \end{cases}$$

From the first equation we obtain

$$\log 3 = a - 2\log 5$$

if we substitute it into the second equation

$$\log 5 + 2(a - 2\log 5) = b$$
$$\log 5 = \frac{2a-b}{3}$$
$$\log 3 = a - 2\cdot\frac{2a-b}{3} = \frac{-a+2b}{3}$$
$$\log 15 = \log 3 + \log 5 = \frac{2b-a}{3} + \frac{2a-b}{3} = \frac{a+b}{3}$$

The levels of the APOS Theory are

- Action: if the student can identify the problem as a system of equations.
- Process: if the student can apply the laws of logarithms to express a and b, and correctly express one of the two variables.
- Object: if the student can completely solve the task.
- Schema: because of the simplicity of the task we did not identify in this case the schema level.

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