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Catalan numbers which are factoriangular numbers

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Abstract. In this paper, we prove that the only Catalan numbers or middle binomial coefficients which are factoriangular numbers are 1, 2 and 5.

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1. Introduction

The Catalan numbers $\{C_n\}_{n\geq 0}$ are given by

$$C_n = \frac{(2n)!}{(n+1)!n!} = \frac{1}{n+1}\binom{2n}{n} \quad \text{for integer} \quad n \geq 0.$$

These numbers are named after the Belgian–French mathematician Eugène Charles Catalan (1814–1894). The Catalan numbers appear naturally when counting various structures. For more information on them, we refer interested readers to the books of Thomas Koshy [3] and Richard Stanley [8]. The first few Catalan numbers are:

$$1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430,\ 4862,\ 16796,\ 58786,\ 208012,\ 742900, \ldots$$

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The middle binomial coefficients $\{B_n\}_{n\geq 0}$ are given by $B_n = (n+1)C_n = \binom{2n}{n}$. The first few middle binomial coefficients are

$$1,\ 2,\ 6,\ 20,\ 70,\ 252,\ 924,\ 3432,\ 12870,\ 48620,\ 184756,\ 705432,\ 2704156,\ldots$$

A factoriangular number is the number of the form:

$$Ft_m = m! + \frac{m(m+1)}{2} \quad \text{for integer} \quad m \geq 0.$$

The factoriangular numbers have been studied first by Castillo [1]. The first few factoriangular numbers are:

$$1,\ 2,\ 5,\ 12,\ 34,\ 135,\ 741,\ 5068,\ 40356,\ 362925,\ 3628855,\ 39916866,\ 479001678,\ldots$$

Diophantine equations with factoriangular numbers were studied before. For examples, all Fibonacci numbers, all Pell numbers, all Lucas numbers, all balancing and Lucas-balancing numbers, which are factoriangulars, were found in [2, 4, 5, 7]. In this manuscript, we prove the following result:

Theorem 1.1. *The only Catalan numbers or middle binomial coefficients which are factoriangular numbers are* $1,\ 2$ *and* 5.

2. The proof of the Theorem 1.1

We consider the Diophantine equation

$$Ft_m = B_n, C_n. \tag{2.1}$$

We generated $\{Ft_m : 0 \leq m < 10^5\}$ and $\{B_n, C_n : 0 \leq n < 2.5\cdot 10^5\}$ and intersected these two sets obtaining that their intersection is $\{1, 2, 5\}$. Assume now that there are other solutions to equation (2.1). Since $C_{2.5\cdot 10^5} < B_{2.5\cdot 10^5} < Ft_{10^5}$, it follows that $n \geq 2.5\cdot 10^5$. Since $B_{2.5\cdot 10^5} > C_{2.5\cdot 10^5} > Ft_{3\cdot 10^4}$, it follows that $m \geq 3\cdot 10^4$. Now

$$Ft_m = m! + m(m+1)/2 < m^m\left(\frac{2}{m^2} + \frac{1}{m^{m-2}}\right) < m^m,$$

where the above inequality is in fact true for $m \geq 3$, and also

$$Ft_m = m! + m(m+1)/2 > m! > (m/e)^m > m^{0.9m},$$

where the right-most inequality holds for all $m > e^{10} = 22026,46\ldots$. Further,

$$2^{2n} > B_n > C_n > \frac{2^{2n}}{(n+1)(2n+1)} > 2^{1.9n}, \tag{2.2}$$

where the last inequality is equivalent to $2^{0.1n} > (2n+1)(n+1)$ which holds for all $n > 200$. Thus, we have

$$m^{0.9m} < Ft_m = B_n, C_n < 2^{2n}, \quad \text{therefore} \quad \frac{0.9m\log m}{2\log 2} < n,$$

and

$$2^{1.9n} < B_n, C_n = Ft_m < m^m, \quad \text{therefore} \quad n < \left(\frac{2}{1.9}\right)\frac{m\log m}{2\log 2} < \frac{1.1m\log m}{2\log 2}.$$

We record this as a lemma.

Lemma 2.1. *If (n,m) satisfy (2.1) and $n \geq 2.5\cdot 10^5$, then $m > 3\cdot 10^4$ and*

$$\frac{0.9m\log m}{2\log 2} < n < \frac{1.1m\log m}{2\log 2}.$$

Lemma 2.2. *If p is a prime in the interval $(\sqrt{n}, \sqrt{2n})$, then $p \mid B_n$.*

Proof. By Kummer's theorem, $p \mid B_n$ if and only if there is at least a carry when adding n to itself in base p. Since $p < \sqrt{2n} < n < p^2$, it follows that $n = ap + b$, where $a, b \in \{0, \ldots, p-1\}$ with $a \neq 0$. If both a, b are at most $(p-1)/2$, then

$$2n = (2a)p + (2b) \leq (p-1)p + (p-1) = p^2 - 1 < p^2,$$

a contradiction. Thus, one of a, b must be in $[(p+1)/2, p-1]$, therefore $p \mid B_n$. □

Let $I = (\sqrt{n}, \sqrt{2n})$. In the equation

$$B_n, C_n = m! + m(m+1)/2,$$

let us consider primes $p \in I$. Such primes divide B_n by Lemma 2.2. At most one of them divides $n+1$. Indeed, if at least two of them say $p_1 < p_2$ divide $n+1$, we would then have that $n + 1 \geq p_1 p_2 \geq \sqrt{n}(\sqrt{n}+2) = n + 2\sqrt{n}$, a contradiction. Thus, all primes $p \in I$ divide B_n and with at most one exception they divide C_n as well. If $p \in I$ divides C_n, then since

$$p < \sqrt{2n} < \left(\frac{2.2m\log m}{2\log 2}\right)^{1/2} < m,$$

it follows that p divides also $m!$; hence, $m(m+1)/2$. If there are at least four such primes say $p_1 < p_2 < p_3 < p_4$, then $m(m+1)/2$ is divisible by their product. Thus,

$$m^2 > \frac{m(m+1)}{2} \geq p_1p_2p_3p_4 > \left(\frac{0.9m\log m}{2\log 2}\right)^{4/2} > (0.6m\log m)^2 > 0.3m^2(\log m)^2,$$

which is false. Thus, there can be at most three such prime factors of C_n showing that I contains at most four primes. Hence,

$$\pi(\sqrt{2n}) - \pi(\sqrt{n}) \leq 4.$$

By [6, Corollaries 1 and 2], we have that

$$\frac{x}{\log x} < \pi(x) < \frac{5x}{4\log x} \quad \text{for} \quad x > 114.$$

Applying this with $x \in \{\sqrt{n}, \sqrt{2n}\}$, we have that

$$\pi(\sqrt{2n}) \geq \frac{\sqrt{2n}}{\log(\sqrt{2n})} \quad \text{and} \quad \pi(\sqrt{n}) < \frac{5\sqrt{n}}{4\log(\sqrt{n})}.$$

We thus get that

$$4 \geq \pi(\sqrt{2n}) - \pi(\sqrt{n}) \geq \frac{\sqrt{2n}}{\log(\sqrt{2n})} - \frac{1.25\sqrt{n}}{\log(\sqrt{n})}$$

which gives $n < 80000$, a contradiction. This finishes the proof of Theorem 1.1.

3. Concluding remarks

A similar argument shows that Diophantine equations of the form

$$B_n, C_n = m! \pm P(m), \tag{3.1}$$

where $P(X) \in \mathbb{Q}[X]$ is an integer valued polynomial have only finitely many positive integer solutions n, m. Indeed, the estimates of Lemma 2.1 apply when n is sufficiently large with respect to the degree and height (maximum absolute value of the coefficients) of the polynomial $P(X)$. Lemma 2.2 shows that all primes $p \in I$ divide B_n and all such primes also divide C_n with at most one exception. Since they are also smaller than m, it follows that they divide $P(m)$. Since such primes are in fact larger than $c_1\sqrt{m \log m}$ with some suitable positive constant c_1, it follows that for large m there cannot be more than $2k$ such primes, where k is the degree of $P(X)$. This gives that $\pi(\sqrt{2n}) - \pi(\sqrt{n}) \leq 2k + 1$, which implies that n; hence also m, is bounded. The left-hand side of equation (3.1) can be replaced by a binomial coefficient of the form $\binom{an+b}{cn+d}$ with integers $a > c \geq 1$, b, d and the resulting equations still have only finitely many solutions. We do not enter into further details.

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