

On the group of units of the semisimple group algebras of dimension 200

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Abstract. In this study, we examine the non-metabelian groups of order 200 and analyze the structure of the unit group within their corresponding group algebra. Among the fifty-two non-isomorphic groups of order 200, only two are non-metabelian. The paper focuses on characterizing the unit group of the semi-simple group algebras associated with these two groups over any finite field.

Keywords: finite field, group algebra, unit group, non-metabelian groups

AMS Subject Classification: Primary: 16U60, Secondary: 20C05

1. Introduction

The linear combination of elements from G with coefficients from the finite field \mathcal{F}_q , where G is a finite group, is the group algebra, denoted by $\mathcal{F}_q G$ over the field \mathcal{F}_q with $q = p^k$ elements for a prime p and $k \in \mathbb{Z}^+$. Maschke's theorem [19] implies that the group algebra $\mathcal{F}_q G$ is semisimple if and only if $\text{char}(\mathcal{F}_q) \nmid n$. Thus, via Wedderburn decomposition theorem [19], $\mathcal{F}_q G$ is isomorphic to the direct sum of matrix algebras over division rings, that is,

$$\mathcal{F}_q G \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_l}(D_l), \quad n_i, l \in \mathbb{Z}^+.$$

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One may easily determine the structure of the unit group of $\mathcal{F}_q G$ from the aforementioned isomorphism. Recall that the unit group, $\mathbb{U}(\mathcal{F}_q G)$, is set of all of the invertible elements in $\mathcal{F}_q G$. See [1, 6, 22, 24, 26, 30–32] for a few noteworthy recent studies that have been conducted to determine the structure of the unit group of the semi-simple group algebra. The applications of units in number theory [14], coding theory [16], cryptography [20], etc. make the research in this area crucial. In addition to this, the recent counterexample to the renowned Kaplansky’s unit conjecture further emphasizes the need of research in this area (see [15]). Furthermore, there have been significant developments in the exploration of the unit group of modular group algebras, in addition to integral and semisimple group algebras (see [9–11, 28] and the references therein for a comprehensive and recent literature in this direction).

In view of studying the unit group of all group algebras, the groups can be categorized into two divisions: metabelian and non-metabelian. Bakshi et al. conducted a thorough analysis of the first instance in [8]. As a result, we just need to consider the non-metabelian groups. In [27], Pazderski determined possible non-metabelian group orders. The smallest non-metabelian groups have order 24, and Khan et al. [17] and Maheshwari et al. [18] have examined the unit group of the corresponding group algebras. These findings may be readily ascertained with the use of [27]. Similarly, in [7, 21, 23, 24, 26, 29], Mittal et al. and Arvind et al. categorised the unit group of group algebras of non-metabelian groups up to order 120. Mittal et al. [25] and Abhilash et al. [2–5] recently finished the work for all groups up to order 180.

It is known that there exist non-metabelian groups of order 200 as a result of [27]. There exist 52 non-isomorphic groups of order 200, two of which are non-metabelian. The objective of this paper is to take these 2 groups into consideration and use the Wedderburn decomposition to derive the unit groups of their group algebras (see [19]).

The flow of this paper is as follows. The important definitions, results and the 2 non-metablian groups to be studied in this paper are introduced in Section 2 and Section 3, respectively. Moreover, Section 3 has the main results on the unit groups of the semisimple group algebras. The final section concludes the paper in nature.

2. Preliminaries

The results and prerequisites definitions needed to support the main result are provided in this section. The following notations apply to this entire paper.

\mathcal{F}_q	finite field of order $q = p^k$ with characteristic p and $k \geq 1$
G	finite group of order n with $p \nmid n$
e	exponent of the group G i.e., the l.c.m of the order of elements in G
ω	primitive e -th root of unity over \mathcal{F}_q
\mathbb{G}	Galois group of $\mathcal{F}_q(\omega)$ over \mathcal{F}_q , where $\mathcal{F}_q(\omega)$ is the splitting field of \mathcal{F}_q

$\mathcal{T}_{G, \mathcal{F}_q}$ collection of all s such that $\sigma(\omega) = \omega^s$, where $\sigma \in \mathbb{G}$
 C_x conjugacy class of x
 $[x, y]$ denote the commutator $x^{-1}y^{-1}xy$ of $x, y \in G$
 1 identity element of G

Definition 2.1 ([13]). (i) For any prime p , an element $x \in G$ is said to be p' -element if order of x is not divisible by p .
(ii) For any p' -element $x \in G$, the cyclotomic \mathcal{F}_q -class of $\gamma_x = \sum_{h \in C_x} h$ is the set $S_{\mathcal{F}_q}(\gamma_x) = \{\gamma_{x^s} \mid s \in \mathcal{T}_{G, \mathcal{F}_q}\}$.

Lemma 2.3 deals with the number of elements in a specific cyclotomic class, while the proposition following concerns the total count of cyclotomic \mathcal{F}_q -classes. Ferraz provides these two results in [13].

Proposition 2.2. The set of simple components of $\mathcal{F}_q G / J(\mathcal{F}_q G)$ and the set of cyclotomic \mathcal{F}_q -classes in G , where $J(\mathcal{F}_q G)$ is the Jacobson radical of $\mathcal{F}_q G$, are in 1-1 correspondence.

Lemma 2.3. Let l be the number of cyclotomic \mathcal{F}_q -classes in G . If $\mathcal{F}_{q^{m_1}}, \mathcal{F}_{q^{m_2}}, \dots, \mathcal{F}_{q^{m_l}}$ are the simple components of $Z(\mathcal{F}_q G / J(\mathcal{F}_q G))$ and S_1, S_2, \dots, S_l are the cyclotomic \mathcal{F}_q -classes of G , then $|S_i| = [\mathcal{F}_{q^{m_i}} : \mathcal{F}_q]$ with a suitable ordering of the indices, assuming that \mathbb{G} is cyclic.

In order to uniquely characterize the Wedderburn decompositions of the semi-simple group algebras, we need the following important result. See [19, Chapter 3] for its proof.

Lemma 2.4. (i) Let $\mathcal{F}_q G$ be a semi-simple group algebra and let $N \trianglelefteq G$. Then

$$\mathcal{F}_q G \cong \mathcal{F}_q(G/N) \oplus \Delta(G, N),$$

where $\Delta(G, N)$ is an ideal of $\mathcal{F}_q G$ generated by the set $\{n - 1 : n \in N\}$.

(ii) If $N = G'$ in part (i), then $\mathcal{F}_q(G/G')$ is the sum of all commutative simple components of $\mathcal{F}_q G$ and $\Delta(G, G')$ is the sum of all others.

Further, we discuss a necessary condition for the dimension of the matrix algebra in the Wedderburn decomposition (see [12]).

Lemma 2.5. Suppose $\oplus_{i=1}^t M_{n_i}(\mathcal{F}_{q^{m_i}})$ is the summand of the semisimple group algebra $\mathcal{F}_q G$, where p is the characteristics of \mathcal{F}_q . Then p does not divide any of the n_i .

Again, we discuss two very important results which helps us to find the unique Wedderburn decomposition. For proof of Lemma 2.6, we refer to [34].

Lemma 2.6. Let p_1 and p_2 be two primes. Let \mathcal{F}_{q_1} be a field with $q_1 = p_1^{k_1}$ elements and let \mathcal{F}_{q_2} be a field with $q_2 = p_2^{k_2}$ elements, where $k_1, k_2 \geq 1$. Let both the group algebras $\mathcal{F}_{q_1} G, \mathcal{F}_{q_2} G$ be semisimple. Suppose that

$$\mathcal{F}_{q_1} G \cong \oplus_{i=1}^t M(n_i, \mathcal{F}_{q_1}), \quad n_i \geq 1$$

and $M(n, \mathcal{F}_{q^r})$ is a Wedderburn component of the group algebra $\mathcal{F}_{q^2}G$ for some $r \geq 2$ and any positive integer n , i.e.,

$$\mathcal{F}_{q^2}G \cong \bigoplus_{i=1}^{s-1} M(m_i, \mathcal{F}_{q^{2,i}}) \oplus M(n, \mathcal{F}_{q^r}), \quad m_i \geq 1.$$

Here $\mathcal{F}_{q^{2,i}}$ is a field extension of \mathcal{F}_{q^2} . Then $M(n, \mathcal{F}_{q^r})$ must be a Wedderburn component of the group algebra $\mathcal{F}_{q^1}G$ and it appears atleast r times in the Wedderburn decomposition of $\mathcal{F}_{q^1}G$.

Lemma 2.7 ([7, corollary 3.8]). *Let \mathcal{F}_qG be a finite semisimple group algebra. Then if there exists an irreducible representations of degree n over \mathcal{F}_q , then one of the Wedderburn components of \mathcal{F}_qG is $M_n(\mathcal{F}_q)$. Also, if there exists k irreducible representations of degree n over \mathcal{F}_q , then $M_n(\mathcal{F}_q)^k$ is a summand of the group algebra \mathcal{F}_qG .*

Throughout this paper, if G has j conjugacy classes, then the representatives of the conjugacy classes are denoted by $g_1 (= 1), g_2, g_3, \dots, g_j$.

3. Unit groups

The structure of the unit group of the group algebras of non-metabelian groups of order 200 is covered in this section. Two of the 52 non-isomorphic groups of order 200 are not metabelian. They are as follows:

1. $G_1 := (C_5 \times C_5) \rtimes D_8$
2. $G_2 := (C_5 \times C_5) \rtimes Q_8$

The conjugacy classes and the commutator subgroups of both the groups are taken from the GAP software (see [33]).

3.1. $G_1 := (C_5 \times C_5) \rtimes D_8$

The group G_1 has the following presentation:

$$G_1 = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^2, [x_2, x_1]x_3^{-1}, [x_3, x_1], [x_4, x_1]x_5^{-3}x_4^{-4}, [x_5, x_1]x_5^{-4}x_4^{-2}, x_2^2, [x_3, x_2], [x_4, x_2]x_5^{-1}x_4^{-4}, [x_5, x_2]x_5^{-4}x_4^{-1}, x_3^2, [x_4, x_3]x_4^{-3}, [x_5, x_3]x_5^{-3}, x_4^5, [x_5, x_4], x_5^5 \rangle.$$

The sizes (S), orders (O) and the representatives (R) of the 14 conjugacy classes of G_1 are:

R	1	x_1	x_2	x_3	x_4	x_1x_2	x_1x_4	x_1x_5	x_2x_4	x_4x_5	$x_2x_3x_4$	$x_4^2x_5$	$x_4^3x_5$	$x_4^2x_5^2$
S	1	10	10	25	8	50	20	20	20	4	20	4	4	4
O	1	2	2	2	5	4	10	10	10	5	10	5	5	5

It is clear that the exponent of G_1 is 20.

Theorem 3.1. *The unit group of the group algebra $\mathcal{F}_q G_1$ is as follows:*

- 1) for $q \equiv \{1, 9, 11, 19\} \pmod{20}$, $\mathbb{U}(\mathcal{F}_q G_1) \cong \mathcal{F}_q^{*4} \oplus GL_2(\mathcal{F}_q) \oplus GL_4(\mathcal{F}_q)^8 \oplus GL_8(\mathcal{F}_q)$.
- 2) for $q \equiv \{3, 7, 13, 17\} \pmod{20}$, $\mathbb{U}(\mathcal{F}_q G_1) \cong \mathcal{F}_q^{*4} \oplus GL_2(\mathcal{F}_q) \oplus GL_8(\mathcal{F}_q) \oplus GL_4(\mathcal{F}_q)^4$.

Proof. The group algebra $\mathcal{F}_q G_1$ is Artinian and semisimple. We observe that the commutator subgroup $G'_1 \cong (C_5 \times C_5) \rtimes C_2$ and $\frac{G_1}{G'_1} \cong C_2 \times C_2$. Since $q = p^k$ and $p \neq 2, 5$, we split the proof in the following 2 cases.

Case 1: $q \equiv \{1, 9, 11, 19\} \pmod{20}$. In this case, the cardinality of every cyclotomic \mathcal{F}_q -class is 1. Therefore, the decomposition of the group algebra by using Proposition 2.2 and Lemma 2.3 is $\mathcal{F}_q G_1 \cong \mathcal{F}_q \oplus_{i=1}^{13} M_{n_i}(\mathcal{F}_q)$, $n_i \geq 1$. By applying (ii) of Lemma 2.4, we further deduce that

$$\mathcal{F}_q G_1 \cong \mathcal{F}_q^4 \oplus_{i=1}^{10} M_{n_i}(\mathcal{F}_q), \quad n_i \geq 2.$$

Since the dimensions of both the sides are the same, we end up with $196 = \sum_{i=1}^{10} n_i^2$. This equation has 34 different solutions. By incorporating Lemma 2.5, we conclude that p can not be 3 and 7. This means that we are remaining with 5 choices of n_i 's given as follows:

$$(2^8, 8, 10), (2^5, 4^3, 8^2), (2^4, 4, 5^4, 8), (2, 4^8, 8), (4^6, 5^3).$$

We observe that the subgroup $N := C_5 \times C_5$ is normal in G_1 and $F = G_1/N \cong D_8$. Using [8], we note that $\mathcal{F}_q F \cong \mathcal{F}_q^4 \oplus M_2(\mathcal{F}_q)$. Due to (i) of Lemma 2.4, we are remaining with $(2^8, 8, 10), (2^5, 4^3, 8^2), (2^4, 4, 5^4, 8), (2, 4^8, 8)$ choices of n_i 's. Next, we define four group homomorphism f_1, f_2, f_3, f_4 from $G_1 \rightarrow GL_4(\mathcal{F}_{11})$, where \mathcal{F}_{11} is the finite field having 11 elements, as follows:

$$\begin{aligned} x_1 \xrightarrow{f_1} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad x_2 \xrightarrow{f_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad x_3 \xrightarrow{f_1} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ x_4 \xrightarrow{f_1} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}, \quad x_5 \xrightarrow{f_1} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}, \\ x_1 \xrightarrow{f_2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad x_2 \xrightarrow{f_2} \begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 10 & 0 \end{pmatrix}, \quad x_3 \xrightarrow{f_2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ x_4 \xrightarrow{f_2} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}, \quad x_5 \xrightarrow{f_2} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned}
x_1 \xrightarrow{f_3} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & x_2 \xrightarrow{f_3} & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & x_3 \xrightarrow{f_3} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
x_4 \xrightarrow{f_3} & \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, & x_5 \xrightarrow{f_3} & \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}, \\
x_1 \xrightarrow{f_4} & \begin{pmatrix} 0 & 10 & 0 & 0 \\ 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix}, & x_2 \xrightarrow{f_4} & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & x_3 \xrightarrow{f_4} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
x_4 \xrightarrow{f_4} & \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, & x_5 \xrightarrow{f_4} & \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}.
\end{aligned}$$

Clearly, the maps f_i 's are irreducible representations of G_1 of degree 4 over \mathcal{F}_{11} , i.e., f_i 's are group homomorphisms from G_1 to $GL_4(\mathcal{F}_{11})$ and f_i 's are irreducible, that is there is no matrix $U \in GL_4(\mathcal{F}_{11})$ such that

$$U^{-1}f_i(g)U = \begin{pmatrix} A(g) & B(g) \\ 0 & C(g) \end{pmatrix} \quad \text{for all } g \in G_1 \text{ and } i = 1, 2, 3, 4$$

where $A(g)$, $B(g)$, and $C(g)$ are square matrices with entries from \mathcal{F}_{11} depending on g . Therefore, Lemma 2.7 implies that $M_4(\mathcal{F}_{11})^4$ must be a summand of $\mathcal{F}_{11}G_1$. This confirms that $(2, 4^8, 8)$ is the final choice of the values of n_i 's. Therefore, we have

$$\mathcal{F}_qG_1 \cong \mathcal{F}_q^4 \oplus M_2(\mathcal{F}_q) \oplus M_4(\mathcal{F}_q)^8 \oplus M_8(\mathcal{F}_q).$$

Case 2: $q \equiv \{3, 7, 13, 17\} \pmod{20}$. In this case, the cyclotomic \mathcal{F}_q -class corresponding to g_7 includes g_8 , g_9 includes g_{11} , g_{10} includes g_{14} and g_{12} includes g_{13} , while rest of the g_i 's forms individual classes. Therefore, as per Proposition 2.2, Lemma 2.3 and (ii) of Lemma 2.4, the decomposition for this scenario is given by

$$\mathcal{F}_qG_1 \cong \mathcal{F}_q^4 \oplus_{i=1}^2 M_{n_i}(\mathcal{F}_q) \oplus_{i=3}^6 M_{n_i}(\mathcal{F}_{q^2}), \quad n_i > 1$$

which means $196 = n_1^2 + n_2^2 + 2 \sum_{i=3}^6 n_i^2$. There are 24 possibilities and by Lemma 2.6, $M_8(\mathcal{F}_q)$ must be a summand. Also, [8] implies that $\mathcal{F}_qF \cong \mathcal{F}_q^4 \oplus M_2(\mathcal{F}_q)$. Thus, (i) of Lemma 2.4 derives that

$$\mathcal{F}_qG_1 \cong \mathcal{F}_q^4 \oplus M_2(\mathcal{F}_q) \oplus M_8(\mathcal{F}_q) \oplus M_4(\mathcal{F}_{q^2})^4.$$

This completes the proof. \square

3.2. $G_2 := (C_5 \times C_5) \rtimes Q_8$

The group G_2 has the following presentation:

$$G_2 = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^2 x_3^{-1}, [x_2, x_1] x_3^{-1}, [x_3, x_1], [x_4, x_1] x_4^{-1}, [x_5, x_1] x_5^{-2}, x_2^2 x_2^{-1}, [x_3, x_2], [x_4, x_2] x_5^{-1} x_4^{-4}, [x_5, x_2] x_5^{-4} x_4^{-4}, x_3^2, [x_4, x_3] x_4^{-3}, [x_5, x_3] x_5^{-3}, x_4^5, [x_5, x_4], x_5^5 \rangle.$$

The sizes, orders and the representatives of the 8 conjugacy classes of G_2 are given below:

Representatives	1	x_1	x_2	x_3	x_4	$x_1 x_2$	$x_4 x_5$	$x_4^2 x_5$
Size	1	50	50	25	8	50	8	8
Order	1	4	4	2	5	4	5	5

It is clear that the exponent of G_2 is 20.

Theorem 3.2. *The unit group of the group algebra $\mathcal{F}_q G_2$ for any field with characteristic not equal to 2 and 5 is given by:*

$$\mathbb{U}(\mathcal{F}_q G_2) \cong \mathcal{F}_q^{*4} \oplus GL_2(\mathcal{F}_q) \oplus GL_8(\mathcal{F}_q)^3.$$

Proof. The group algebra $\mathcal{F}_q G_2$ is Artinian and semisimple. We observe that the commutator subgroup $G_2' \cong (C_5 \times C_5) \times C_2$ and $\frac{G_2}{G_2'} \cong C_2 \times C_2$. $T_{G, \mathcal{F}_q} = \{1, 3, 7, 9, 11, 13, 17, 19\}$. So, for any $g \in G_1$, we have

$$S_{\mathcal{F}_q}(\gamma_g) = \{\gamma_{g^t} \mid t \in T_{G, \mathcal{F}_q}\} \Rightarrow S_{\mathcal{F}_q}(\gamma_g) = \{\gamma_g, \gamma_{g^3}, \gamma_{g^7}, \gamma_{g^9}, \gamma_{g^{11}}, \gamma_{g^{13}}, \gamma_{g^{17}}, \gamma_{g^{19}}\}.$$

Further, it can be verified that g^t , where $t \in T_{G, \mathcal{F}_q}$ belong to the conjugacy class of g . Consequently, $S_{\mathcal{F}_q}(\gamma_g) = \{\gamma_g\}$ for any $g \in G_1$. Therefore, we conclude that the cardinality of every cyclotomic \mathcal{F}_q -class is 1. Therefore, as per Proposition 2.2, Lemma 2.3 and (ii) of Lemma 2.4, the decomposition is

$$\mathcal{F}_q G_2 \cong \mathcal{F}_q^4 \oplus_{i=1}^4 M_{n_i}(\mathcal{F}_q),$$

and $196 = \sum_{i=1}^4 n_i^2$, where $n_i > 1$. The possible choices of n_i 's fulfilling above equation are $(2, 8, 8, 8)$, $(3, 3, 3, 13)$, $(3, 5, 9, 9)$, $(4, 4, 8, 10)$, $(5, 5, 5, 11)$, $(7, 7, 7, 7)$. We observe that the subgroup $N := \langle x_4, x_5 \rangle$ is normal in G_2 and $F = G_2/N \cong Q_8$. Using [8], we recall that $\mathcal{F}_q F \cong \mathcal{F}_q^4 \oplus M_2(\mathcal{F}_q)$. Therefore, $M_2(\mathcal{F}_q)$ must be the Wedderburn components of $\mathcal{F}_q G_2$ as per (i) of Lemma 2.4. So, $(2, 8, 8, 8)$ is the required choice, which means that $\mathcal{F}_q G_2 \cong \mathcal{F}_q^4 \oplus M_2(\mathcal{F}_q) \oplus M_8(\mathcal{F}_q)^3$. This completes the proof. \square

4. Conclusion

This work concludes the study of the unit groups of semisimple group algebras of all groups up to order 200, with the exception of the groups of order 192. We have

previously investigated the unit group of the semisimple group algebras of non-metabelian groups of order 200. It is evident that in order to characterise the unit groups in a way that is distinct from one another, different strategies are needed as the group increases in size. Researchers are encouraged by this study to develop an algorithm that can calculate the semisimple group algebra of any finite group's Wedderburn decomposition.

References

- [1] N. ABHILASH, E. NANDAKUMAR: *A brief about the units of Heisenberg group algebra of higher dimensions*, Asia Pac. J. Math. 10 (2023), p. 23, DOI: [10.28924/APJM/10-23](https://doi.org/10.28924/APJM/10-23).
- [2] N. ABHILASH, E. NANDAKUMAR, G. MITTAL, R. K. SHARMA: *Unit groups of semisimple group algebras of groups up to order 180*, Journal of Interdisciplinary Mathematics 27.6 (2024), pp. 1383–1404, DOI: [10.47974/JIM-1858](https://doi.org/10.47974/JIM-1858).
- [3] N. ABHILASH, E. NANDAKUMAR, G. MITTAL, R. K. SHARMA: *Units of the semisimple group algebras of groups of order 162*, Math Vensik Online first. August 20 (2024), DOI: [10.57016/MV-0dhsTT7h](https://doi.org/10.57016/MV-0dhsTT7h).
- [4] N. ABHILASH, E. NANDAKUMAR, R. K. SHARMA, G. MITTAL: *On the unit group of the semisimple group algebras of groups up to order 144*, Palestine Journal of Mathematics 13.2 (2024), pp. 160–171.
- [5] N. ABHILASH, E. NANDAKUMAR, R. K. SHARMA, G. MITTAL: *Structure of the unit group of the group algebras of non-metabelian groups of order 128*, Math Bohemica Online first. May 6 (2024), DOI: [10.21136/MB.2024.0017-23](https://doi.org/10.21136/MB.2024.0017-23).
- [6] S. ANSARI, M. SAHAI: *Units in $F(C_n \times Q_{12})$ and $F(C_n \times D_{12})$* , Int. Elect. J. Algebra 34 (2023), pp. 182–196.
- [7] N. ARVIND, S. PANJA: *Unit group of some finite semisimple group algebras*, Journal of Egyptian Mathematical Society 3 (2022), p. 17, URL: <https://joems.springeropen.com/article/s/10.1186/s42787-022-00151-0>.
- [8] G. K. BAKSHI, S. GUPTA, I. B. S. PASSI: *The algebraic structure of finite metabelian group algebras*, Communications in Algebra 43.1 (2015), pp. 2240–2257, DOI: [10.1080/00927872.2014.888566](https://doi.org/10.1080/00927872.2014.888566).
- [9] Z. A. BALOGH: *The structure of the unit group of some group algebras*, Miskolc Math. Notes 21.2 (2020), pp. 615–620.
- [10] Z. A. BALOGH: *Unitary units of the group algebra of modular groups*, Journal of Algebra and Its Applications 21.02, 2250027 (2022), DOI: [10.1142/S021949882250027X](https://doi.org/10.1142/S021949882250027X).
- [11] A. BOVDI: *The group of units of a group algebra of characteristic p* , Publ. Math. Debrecen 52.1-2 (1998), pp. 193–244.
- [12] C. DIETZEL, G. MITTAL: *Summands of finite group algebras*, Czechoslovak Mathematical Journal 71.4 (2021), pp. 1011–1014, DOI: [10.21136/CMJ.2020.0171-20](https://doi.org/10.21136/CMJ.2020.0171-20).
- [13] R. A. FERRAZ: *Simple components of the center of $FG/J(FG)$* , Communications in Algebra 36.9 (2008), pp. 3191–3199, DOI: [10.1080/00927870802103503](https://doi.org/10.1080/00927870802103503).
- [14] W. GAO, A. GEROLDINGER, F. H.-KOCH: *Group algebras of finite abelian groups and their applications to combinatorial problems*, Rocky Mountain J. Math. 39.3 (2009), pp. 805–823, DOI: [10.1216/RMJ-2009-39-3-805](https://doi.org/10.1216/RMJ-2009-39-3-805).
- [15] G. GARDAM: *A counterexample to the unit conjecture for group rings*, Ann. of Math. 194.3 (2021), pp. 967–979.

- [16] T. HURLEY: *Convolutional codes from units in matrix and group rings*, International Journal of Information and Coding Theory 50.3 (2009), pp. 431–463, URL: <https://www.ijpam.eu/contents/2009-50-3/9/index.html>.
- [17] M. KHAN, R. K. SHARMA, J. B. SRIVASTAVA: *The unit group of FS_4* , Acta Mathematica Hungarica 118.1-2 (2008), pp. 105–113, DOI: [10.1007/s10474-007-6169-4](https://doi.org/10.1007/s10474-007-6169-4).
- [18] S. MAHESHWARI, R. K. SHARMA: *The unit group of the group algebra $\mathcal{F}_qSL(2, \mathbb{Z}_3)$* , Journal of Algebra Combinatorics Discrete Structures and Applications 3 (2016), pp. 1–6, URL: <https://jacodesmath.com/index.php/jacodesmath/article/view/24/0>.
- [19] C. P. MILIES, S. K. SEHGAL: *An Introduction to Group Rings*, Netherlands: Springer Dordrecht, 2002, URL: <https://link.springer.com/book/9781402002380>.
- [20] G. MITTAL, S. KUMAR, S. NARAIN, S. KUMAR: *Group ring based public key cryptosystems*, J. Disc. Math. Sci. Crypt. 25.6 (2022), pp. 1683–1704.
- [21] G. MITTAL, R. K. SHARMA: *Computation of the Wedderburn decomposition of semisimple group algebras of groups up to order 120*, Annales Mathematicae et Informaticae Preprint (2023), DOI: [10.33039/ami.2023.07.001](https://doi.org/10.33039/ami.2023.07.001).
- [22] G. MITTAL, R. K. SHARMA: *Computation of Wedderburn decomposition of groups algebras from their subalgebra*, Bull. Korean Math. Soc. 59.3 (2022), pp. 781–787, DOI: [10.4134/BKMS.b210478](https://doi.org/10.4134/BKMS.b210478).
- [23] G. MITTAL, R. K. SHARMA: *On unit group of finite group algebras of Non-Metabelian groups of Order 108*, Journal of Algebra Combinatorics Discrete Structures and Applications 8.2 (2021), pp. 59–71, URL: <https://jacodesmath.com/index.php/jacodesmath/article/view/158>.
- [24] G. MITTAL, R. K. SHARMA: *On unit group of finite group algebras of non-metabelian groups upto order 72*, Mathematica Bohemica 146.4 (2021), pp. 429–455, URL: <https://articles.math.cas.cz/10.21136/MB.2021.0116-19>.
- [25] G. MITTAL, R. K. SHARMA: *The unit groups of semisimple group algebras of some non-metabelian groups of order 144*, Math. Bohemica 148.4 (2023), pp. 631–646, DOI: [10.21136/MB.2022.0067-22](https://doi.org/10.21136/MB.2022.0067-22).
- [26] G. MITTAL, R. K. SHARMA: *Unit group of semisimple group algebras of some non-metabelian groups of order 120*, Asian-European J. Math. 15.3 (2022), p. 2250059, DOI: [10.1142/S1793557122500590](https://doi.org/10.1142/S1793557122500590).
- [27] G. PAZDERSKI: *The orders to which only belong metabelian groups*, Netherlands: Math. Nachr., 1980.
- [28] M. SAHAI, S. ANSARI: *Group of Units of Finite Group Algebras for Groups of Order 24*, Ukr. Math. J. 75 (2023), pp. 244–261.
- [29] R. K. SHARMA, G. MITTAL: *Unit group of semisimple group algebra $\mathbb{F}_qSL(2, \mathbb{Z}_5)$* , Mathematica Bohemica 147.1 (2022), pp. 1–10, URL: <https://articles.math.cas.cz/10.21136/MB.2021.0104-20>.
- [30] R. K. SHARMA, J. B. SRIVASTAVA, M. KHAN: *The unit group of $\mathbb{F}A_4$* , Publ. Math. Debrecen 71.1-2 (2007), pp. 21–26.
- [31] R. K. SHARMA, J. B. SRIVASTAVA, M. KHAN: *The unit group of $\mathbb{F}S_3$* , Acta Math. Acad. Paedagog. Nyházi. 23.2 (2007), pp. 129–142.
- [32] G. TANG, Y. WEI, Y. LI: *Units group of group algebras of some small groups*, Czechoslovak Math. J. 64.1 (2014), pp. 149–157.
- [33] THE GAP SOFTWARE – GROUPS: *Algorithms and Programming*, version 4.12.2, 2022.
- [34] R. WAXMAN: *On the Wedderburn component of a group algebra* (2019), URL: <https://math.stackexchange.com/questions/4677421/on-the-wedderburn-component-of-a-group-algebra>.