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A note on the Bricard property of projective planes

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Abstract. We show that the Bricard property does not hold in every Moufang plane.

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1. Preliminaries

Concerning preliminaries we refer to [1, 4, 5], however, for convenience of the reader we recall some basic definitions and theorems.

An incidence geometry $(\mathcal{P}, \mathcal{L}, \mathcal{I} \subset \mathcal{P} \times \mathcal{L})$ is a *projective plane* if

(P1) for every pair of distinct points A and B there is a unique line incident with A and B (we denote this line by $\overleftrightarrow{AB}$);

(P2) for every pair of distinct lines m and n there is a unique point incident with m and m (we denote this point by $m \cap n$);

(P3) there are four points no three of which are collinear.

In a projective plane an ordered triple of noncollinear points is a *triangle*. Then the points are called the *vertices*, and the lines joining the three possible distinct pairs of vertices are called *sides*.

We say that two triangles ABC and $A'B'C'$ are *centrally perspective* from a point O if the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ are incident with O. The triangles are

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called *axially perspective* from a line l if the points $\overleftrightarrow{AB} \cap \overleftrightarrow{A'B'}$, $\overleftrightarrow{AC} \cap \overleftrightarrow{A'C'}$ and $\overleftrightarrow{BC} \cap \overleftrightarrow{B'C'}$ are incident with l. A projective plane is *Dersarguesian*, if *any two triangles that are perspective from a point are perspective from a line.* This holds of and only if it can be coordinatized by a skewfield.

In this paper we focus on the *Bricard property* of projective planes:

Let ABC and $A'B'C'$ be two triangles, and let $P := \overleftrightarrow{BC} \cap \overleftrightarrow{B'C'}$, $Q := \overleftrightarrow{AC} \cap \overleftrightarrow{A'C'}$ and $R := \overleftrightarrow{AB} \cap \overleftrightarrow{A'B'}$. If $\overleftrightarrow{A'P}$, $\overleftrightarrow{B'Q}$ and $\overleftrightarrow{C'R}$ are concurrent, then $D := \overleftrightarrow{BC} \cap \overleftrightarrow{AA'}$, $E := \overleftrightarrow{AC} \cap \overleftrightarrow{BB'}$ and $F := \overleftrightarrow{AB} \cap \overleftrightarrow{CC'}$ are collinear.

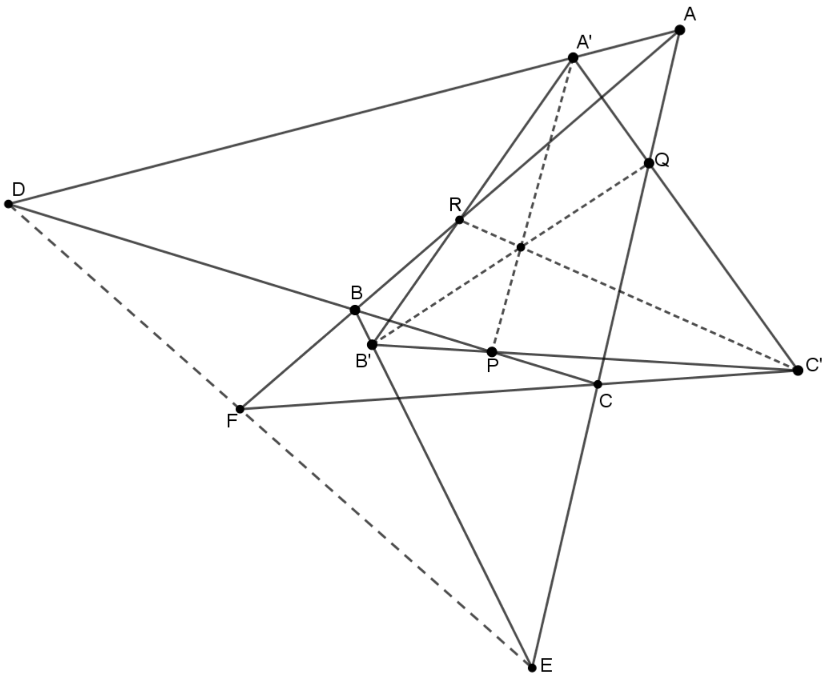

Figure 1. The Bricard property.

In [3] it is shown that the Bricard property follows from the Desargues property.

It is an open question if the Desargues property is necessary in a projective plane to satisfy the Bricard property. The author of [3] conjectures that the Bricard property follows from the following weaker version of the Desargues property:

(D9): *If the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are perspective from a point O, and the triplets (A_1, B_2, C_1) and (A_2, B_1, C_2) are collinear, then the two triangles are perspective from a line.*

In [5] we proved that the converse of the Bricard property does not necessarily hold even under the following, somewhat stronger condition, which is valid in *Moufang planes*:

(D10): *If two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are perspective from a point O, and O is incident to the line of $\overleftrightarrow{A_1B_1} \cap \overleftrightarrow{A_2B_2}$ and $\overleftrightarrow{A_1C_1} \cap \overleftrightarrow{A_2C_2}$, then the triangles are perspective from a line.*

However, it is unknown whether the Bricard property and its converse are equivalent, therefore it is still unknown if the Bricard property hold in every Moufang plane, or every projective plane satisfying (D10). In this paper we prove that neither (D9), nor (D10) implies the Bricard property, as we provide a counterexample for the Bricard property in a Moufang plane.

We recall that a projective plane is a Moufang plane if and only if it can be coordinatized by an alternative division ring, i.e., it is isomorphic to a projective plane over an alternative division ring. We recall that a triplet $(\mathcal{R}, +, \cdot)$ (briefly $\mathcal{R}$) is called an alternative division ring if

Let $\mathcal{R}$ be a set and $+$, $\cdot$ be binary operations on $\mathcal{R}$ such that

- $(\mathcal{R}, +)$ is a commutative group with zero element 0;
- $a \cdot 0 = 0 \cdot a = 0$ for all $a \in \mathcal{R}$;
- $(\mathcal{R}\backslash\{0\}, \cdot)$ is a loop (for a definition, see, e.g., [2]);
- $a \cdot (b + c) = a \cdot b + a \cdot c$,
- $(a + b) \cdot c = a \cdot c + b \cdot c$,
- $a \cdot (a \cdot b) = (a \cdot a) \cdot b$,
- $a \cdot (b \cdot b) = (a \cdot b) \cdot b$; $a, b, c \in \mathcal{R}$.

In the following we will write simply ab instead of $a \cdot b$. We denote the unit of $(\mathcal{R}\backslash\{0\}, \cdot)$ by 1. In an alternative division ring for all $a \in \mathcal{R}\backslash\{0\}$ there exists a unique element a^{-1} such that $aa^{-1} = a^{-1}a = 1$, canned the inverse of a. By a difficult theorem of Bruck-Kleinfield and Skornyakov, an alternative division ring either is associative or is a Cayley-Dickson algebra over some field. From this it follows that in every alternative division ring we have the *inverse property*

$$a(a^{-1}b) = (ba^{-1})a = b \quad \text{for all} \quad a \in \mathcal{R} \setminus \{0\}, b \in \mathcal{R},$$

since this holds in every Cayley-Dickson algebra.

Let $\mathcal{R}$ be an alternative division ring. The incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$, where

- $\mathcal{P} := \{[x, y, 1], [1, x, 0], [0, 1, 0] \mid x, y \in \mathcal{R}\}$;
- $\mathcal{L} := \{\langle a, 1, b\rangle, \langle 1, 0, a\rangle, \langle 0, 0, 1\rangle \mid a, b \in \mathcal{R}\}$;
- $([x, y, z], \langle a, b, c\rangle) \in \mathcal{I}$ if and only if $xa + yb + zc = 0$

is a projective plane called *the projective plane over the alternative division ring* $\mathcal{R}$.

The most simple example of an alternative division ring that is not a skewfield is the alternative division ring of *octonions*. They can be constructed by the Cayley-Dickson procedure from the ring of quaternions. An octonion can be written in form

$$x = x_0 + x_1 i + x_2 j + x_3 k + x_4 l + x_5 I + x_6 J + x_7 K,$$

where x_i ($i \in \{0, 1, 2, 3, 4, 5, 6, 7\}$) are real numbers, and the rule of multiplication of the basic elements i, j, k, l, I, J, K is given by the the following table:

	i	j	k	l	I	J	K
i	-1	l	K	$-j$	J	$-I$	$-k$
j	$-l$	-1	I	i	$-k$	K	$-J$
k	$-K$	$-I$	-1	J	j	$-l$	i
l	j	$-i$	$-J$	-1	K	k	$-I$
I	$-J$	k	$-j$	$-K$	-1	i	l
J	I	$-K$	l	$-k$	$-i$	-1	j
K	k	J	$-i$	I	$-l$	$-j$	-1

The conjugate of $x = x_0 + x_1 i + x_2 j + x_3 k + x_4 l + x_5 I + x_6 J + x_7 K$ is

$$\overline{x} := x_0 - x_1 i - x_2 j - x_3 k - x_4 l - x_5 I - x_6 J - x_7 K,$$

and the norm of x is

$$\|x\| := \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2}.$$

Then the inverse of x is

$$x^{-1} = \frac{\overline{x}}{\|x\|^2}.$$

The projective plane over the octonions is called the *octonion plane*.

2. A counterexample for the Bricard property in the octonion plane

Theorem 2.1. *The Bricard property does not hold in every Moufang plane.*

Proof. Consider the following triangles ABC and $A'B'C'$ in the octonion plane:

$$A'[1,0,0],\ B'[0,1,0],\ C'[0,0,1];$$
$$A\left[-\frac{1}{2} + \frac{1}{2}i - \frac{1}{2}k + \frac{1}{2}K, -\frac{1}{2} - \frac{1}{2}i - \frac{1}{2}k + \frac{1}{2}K, 1\right],$$
$$B\left[\frac{1}{2} + \frac{1}{2}i - j - l, \frac{1}{2} - \frac{1}{2}i - j - l, 1\right],$$

$$C\left[\frac{1}{2}-\frac{1}{2}j-\frac{1}{2}k+\frac{1}{2}I,\frac{1}{2}-\frac{1}{2}j-\frac{1}{2}k-\frac{1}{2}I,1\right].$$

It is easy to see that A is incident to $\langle -1,1,i\rangle$ and $\langle -k,1,k\rangle$; B is incident to $\langle -1,1,i\rangle$ and $\langle j,1,-1\rangle$; C is incident to $\langle j,1,-1\rangle$ and $\langle -k,1,k\rangle$, therefore

$$\overleftrightarrow{AB}=\langle -1,1,i\rangle,\ \overleftrightarrow{BC}=\langle j,1,-1\rangle,\ \overleftrightarrow{AC}=\langle -k,1,k\rangle.$$

Since

$$\overleftrightarrow{A'B'}=\langle 0,0,1\rangle,\ \overleftrightarrow{B'C'}=\langle 1,0,0\rangle,\ \overleftrightarrow{A'C'}=\langle 0,1,0\rangle,$$

we get

$$P=[0,1,1],\ Q=[1,0,1],\ R=[1,1,0].$$

Therefore $\overleftrightarrow{A'P}$, $\overleftrightarrow{B'Q}$ and $\overleftrightarrow{C'R}$ are concurrent at the point $O[1,1,1]$.

We are going to show that the points $D:=\overleftrightarrow{BC}\cap\overleftrightarrow{AA'}$, $E:=\overleftrightarrow{AC}\cap\overleftrightarrow{BB'}$ and $F:=\overleftrightarrow{AB}\cap\overleftrightarrow{CC'}$ are not collinear.

To obtain the coordinates of D, first we determine the line $\overleftrightarrow{AA'}$. Since $[1,0,0]$ is incident to it, it is of the form $\overleftrightarrow{AA'}=\langle 0,1,e\rangle$ for some octonion e. As the point A is incident to the line, we get

$$-\frac{1}{2}-\frac{1}{2}i-\frac{1}{2}k+\frac{1}{2}K+e=0,$$
$$e=\frac{1}{2}+\frac{1}{2}i+\frac{1}{2}k-\frac{1}{2}K.$$

So

$$\overleftrightarrow{AA'}=\left\langle 0,1,\frac{1}{2}+\frac{1}{2}i+\frac{1}{2}k-\frac{1}{2}K\right\rangle.$$

Next we calculate the intersection of $\overleftrightarrow{AA'}$ with the line $\overleftrightarrow{BC}=\langle j,1,-1\rangle$. If $D=[d_1,d_2,1]$, then

$$d_1j+d_2-1=0;$$
$$d_2+\frac{1}{2}+\frac{1}{2}i+\frac{1}{2}k-\frac{1}{2}K=0.$$

From the second equation we get d_2, and the first equation gives $d_1=(-d_2+1)j^{-1}=-(-d_2+1)j$; therefore

$$D=\left[-\frac{3}{2}j-\frac{1}{2}l+\frac{1}{2}I+\frac{1}{2}J,-\frac{1}{2}-\frac{1}{2}i-\frac{1}{2}k+\frac{1}{2}K,1\right].$$

We obtain the point $E=\overleftrightarrow{AC}\cap\overleftrightarrow{BB'}$ in a similar manner. Since $\overleftrightarrow{BB'}=\left\langle 1,0,-\frac{1}{2}-\frac{1}{2}i+j+l\right\rangle$ and $\overleftrightarrow{AC}=\langle -k,1,k\rangle$, the $[e_1,e_2,1]$ coordinates of E satisfy the following system of equations:

$$-e_1+e_2+k=0;$$

$$e_1 - \frac{1}{2} - \frac{1}{2}i + j + l = 0.$$

From the first equation $e_2 = (e_1 - 1)k$ and from the second equation e_1 can be expressed, therefore

$$E = \left[\frac{1}{2} + \frac{1}{2}i - j - l, -\frac{1}{2}k - I + J + \frac{1}{2}K, 1\right].$$

Finally, we determine the point $F = \overleftrightarrow{AB} \cap \overleftrightarrow{CC'}$. Since $C' = [0, 0, 1]$, the line $\overleftrightarrow{CC'}$ is of the form $\langle c, 1, 0\rangle$ for some octonion c. To obtain c we use the fact that $C \in \overleftrightarrow{CC'}$:

$$\left(\frac{1}{2} - \frac{1}{2}j - \frac{1}{2}k + \frac{1}{2}I\right)c + \left(\frac{1}{2} - \frac{1}{2}j - \frac{1}{2}k - \frac{1}{2}I\right) = 0.$$

From this equation,

$$\begin{aligned} c &= \left(\frac{1}{2} - \frac{1}{2}j - \frac{1}{2}k + \frac{1}{2}I\right)^{-1}\left(-\frac{1}{2} + \frac{1}{2}j + \frac{1}{2}k + \frac{1}{2}I\right) \\ &= \left(\frac{1}{2} + \frac{1}{2}j + \frac{1}{2}k - \frac{1}{2}I\right)\left(-\frac{1}{2} + \frac{1}{2}j + \frac{1}{2}k + \frac{1}{2}I\right) \\ &= -\frac{1}{2} + \frac{1}{2}I - \frac{1}{2}k + \frac{1}{2}j. \end{aligned}$$

Thus

$$\overleftrightarrow{CC'} = \left\langle -\frac{1}{2} + \frac{1}{2}j - \frac{1}{2}k + \frac{1}{2}I, 1, 0\right\rangle.$$

So for the coordinates $[f_1, f_2, 1]$ of F we have

$$\begin{gathered} -f_1 + f_2 + i = 0; \\ f_1\left(-\frac{1}{2} + \frac{1}{2}j - \frac{1}{2}k + \frac{1}{2}I\right) + f_2 = 0. \end{gathered}$$

From this we get

$$f_1\left(\frac{1}{2} + \frac{1}{2}j - \frac{1}{2}k + \frac{1}{2}I\right) = i,$$

whence

$$f_1 = i\left(\frac{1}{2} + \frac{1}{2}j - \frac{1}{2}k + \frac{1}{2}I\right)^{-1} = i\left(\frac{1}{2} - \frac{1}{2}j + \frac{1}{2}k - \frac{1}{2}I\right) = \frac{1}{2}i - \frac{1}{2}l + \frac{1}{2}K - \frac{1}{2}J.$$

Therefore

$$F = \left[\frac{1}{2}i - \frac{1}{2}l - \frac{1}{2}J + \frac{1}{2}K, -\frac{1}{2} - \frac{1}{2}l + \frac{1}{2}K - \frac{1}{2}J, 1\right].$$

It is well-known that if $A[a_1, a_2, 1]$ and $B[b_1, b_2, 1]$ are points in a projective plane over an alternative division ring $\mathcal{R}$, then the points of the line $\overleftrightarrow{AB}$ are of the form

$$[t(a_1, a_2, 1) + (1 - t)(b_1, b_2, 1)],\ t \in \mathcal{R} \quad \text{or} \quad [1, x, 0].$$

Therefore, if we want to check whether D, E and F are collinear, we need to check if the coordinates of F can be combined from the coordinates of D and E in this way. Suppose that such a t octonion exists. Then, from the first coordinates of D, E and F, we get

$$t\left(-\frac{3}{2}j-\frac{1}{2}l+\frac{1}{2}I+\frac{1}{2}J\right)+(1-t)\left(\frac{1}{2}+\frac{1}{2}i-j-l\right)=\frac{1}{2}i-\frac{1}{2}l-\frac{1}{2}J+\frac{1}{2}K.$$

This equation leads to

$$t\left(-\frac{1}{2}-\frac{1}{2}i-\frac{1}{2}j+\frac{1}{2}l+\frac{1}{2}I+\frac{1}{2}J\right)=-\frac{1}{2}+j+\frac{1}{2}l-\frac{1}{2}J+\frac{1}{2}K,$$

hence

$$\begin{aligned} t&=\left(-\frac{1}{2}+j+\frac{1}{2}l-\frac{1}{2}J+\frac{1}{2}K\right)\left(-\frac{1}{2}-\frac{1}{2}i-\frac{1}{2}j+\frac{1}{2}l+\frac{1}{2}I+\frac{1}{2}J\right)^{-1}\\ &=\left(-\frac{1}{2}+j+\frac{1}{2}l-\frac{1}{2}J+\frac{1}{2}K\right)\left(-\frac{1}{3}+\frac{1}{3}i+\frac{1}{3}j-\frac{1}{3}l-\frac{1}{3}I-\frac{1}{3}J\right)\\ &=-\frac{1}{6}-\frac{5}{6}i-\frac{1}{6}j-\frac{1}{6}k+\frac{1}{6}l-\frac{1}{6}I+\frac{1}{2}J-\frac{1}{2}K. \end{aligned}$$

We check if the second coordinates can be combinated using the same coefficient. In this case the following equation would hold:

$$t\left(-\frac{1}{2}-\frac{1}{2}i-\frac{1}{2}k+\frac{1}{2}K\right)+(1-t)\left(-\frac{1}{2}k-I+J+\frac{1}{2}K\right)=-\frac{1}{2}-\frac{1}{2}l+\frac{1}{2}K-\frac{1}{2}J.$$

Here the left side is

$$\begin{aligned} &\left(-\frac{1}{6}-\frac{5}{6}i-\frac{1}{6}j-\frac{1}{6}k+\frac{1}{6}l-\frac{1}{6}I+\frac{1}{2}J-\frac{1}{2}K\right)\left(-\frac{1}{2}-\frac{1}{2}i-\frac{1}{2}k+\frac{1}{2}K\right)\\ &+\left(\frac{7}{6}+\frac{5}{6}i+\frac{1}{6}j+\frac{1}{6}k-\frac{1}{6}l+\frac{1}{6}I-\frac{1}{2}J+\frac{1}{2}K\right)\left(-\frac{1}{2}k-I+J+\frac{1}{2}K\right), \end{aligned}$$

whose real part is

$$\frac{1}{12}-\frac{5}{12}-\frac{1}{12}+\frac{1}{4}+\frac{1}{12}+\frac{1}{6}+\frac{1}{2}-\frac{1}{4}=\frac{1}{3}.$$

Otherwise, the real part of the right side is $-\frac{1}{2}$; therefore D, E and F are not collinear. □

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