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On a combinatorial identity associated with Pascal's triangle

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Abstract. Let $f(x) = \cos x$, and consider the sum $\tau_n^{(f)} = \sum_k \binom{n-k}{k} f(kt)$. Using a general method due to Ahmia and Szalay on weighted sums in generalized Pascal triangle an explicit formula is developed for $\tau_n^{(f)}$. An analogous result is provided if $f(x) = \sin x$, and a strong connection to Fibonacci polynomials is also discovered in both cases.

Keywords: combinatorial identity, Pascal's triangle, weighted sum, Fibonacci polynomial

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1. Introduction

Pascal's triangle is one of the most studied objects in combinatorics, in particular there exists a huge number of identities on binomial coefficients. We refer here the two closed forms

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} \cos kx &= 2^n \cos \frac{nx}{2} \left(\cos \frac{x}{2}\right)^n \quad \text{and} \\ \sum_{k=0}^{n} \binom{n}{k} \sin kx &= 2^n \sin \frac{nx}{2} \left(\cos \frac{x}{2}\right)^n,\end{aligned} \tag{1.1}$$

see, for example (1.26) and (1.27) in [6]. On the other hand, theory of linear recurrences often plays an important role in solving combinatorial and number

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theoretical problems. The present paper describes a method to investigate the explicit form of the sum

$$\tau_n^{(f)} = \sum_k \binom{n-k}{k} f(kt), \tag{1.2}$$

where the function $f(x)$ is equal to either $\cos x$ or $\sin x$. The precise result (i.e. explicit formula) for $\tau_n^{(\cos)}$ and $\tau_n^{(\sin)}$ is presented in Theorem 2 of Section Results. Closely related identities appear in [1] in particular, identity (2.3), furthermore on pages 75–76 of the excellent book of Riordan [7].

This work is motivated essentially by the two identities (1.1). We suggest two approaches, the starting point of the first one is the paper of Ahmia and Szalay [3]. This provides a recurrence relation for the sums of binomial coefficients weighted by the terms of a given linear recurrence such that the binomial coefficients lay along an arbitrary final direction. Paper [3] is an extension of [4] where the principal theorem is able to handle arbitrary diagonal sums without weights in a generalized Pascal triangle. We believe that, using our approach analogous questions like $\sum_k \binom{n-k}{c_1+ck} \cos kt$ and $\sum_k \binom{n-k}{c_1+ck} \sin kt$ can be solved for small values of c such as $c = 2, 3$ with $0 \le c_1 < c$, and the nature of the result has similar flavor to Theorem 2. There are other possible ways to handle such questions, for example Egorychev method, i.e. application of complex integral representation of the binomial coefficients (see Part 7 of the compendium [2]). The advantage of our method is the flexibility in binomial coefficients in the above sums. On the other hand, our solution is not very efficient, it requires probably more calculations than other ways do.

Let $C_n = C_n(t) = \cos nt$ and $S_n = S_n(t) = \sin nt$ be considered as two sequences of functions. It is known that

$$C_n = (2\cos t)C_{n-1} - C_{n-2} \quad \text{and} \quad S_n = (2\cos t)S_{n-1} - S_{n-2}, \tag{1.3}$$

in other words, both sequences (C_n) and (S_n) satisfy the same binary recurrence rule with the initial values $C_0 = 1$, $C_1 = \cos t$, and $S_n = 0$, $S_1 = \sin t$, respectively. This observation ensures that we can apply the main result of [3] to the problem above. It turned out that the explicit formula we gained has a strong connection to Fibonacci polynomials, which are the basement of the second approach. The sequence of Fibonacci polynomials is defined by the initial polynomials $F_0(t) = 0$, $F_1(t) = 1$, and by the recurrence $F_n(t) = tF_{n-1}(t) + F_{n-2}(t)$. Comparing the results (of Theorem 2.1 and 2.2) provided by the two approaches we gain identities related to (1.2). We note that our method is applicable even, at least in theory, for functions $\mathcal{F}_k = f(kt)$ $(k = 0, 1, \dots)$ if the sequence $(\mathcal{F}_k)$ satisfies a homogeneous linear recurrence relation. In particular, using our approach we could prove (1.1) although this proof is not the simplest one.

In this paragraph, we introduce certain necessary notation and recall the aforementioned result from [3], which plays a crucial role in the investigation. Let x and y denote two non-zero real numbers. (This criteria appears in [4], but the statements remain true even if we allow for x and y to be complex numbers.) Assume that the element of the generalized Pascal triangle located in the kth position of row n is $\binom{n}{k}x^{n-k}y^k$ ($n \in \mathbb{N}$, $0 \le k \le n$), and consider the sum

$$T_n = T_n^{(r,q,p)} = \sum_{k=0}^{\omega} \binom{n-qk}{p+rk} x^{n-p-(r+q)k} y^{p+rk},$$

where the parameters r, q and p satisfy the conditions $r \in \mathbb{N}^+$, $q \in \mathbb{Z}$, $r+q > 0$ and $0 \le p < r$, further $\omega = \lfloor (n-p)/(q+r) \rfloor$. Recall the worldwide known example, when $r = q = 1$, $p = 0$, $x = y = 1$. This choice admits $T_n^{(1,1,0)} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} = F_{n+1}$, where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$ is the Fibonacci sequence. Clearly, the vector (r, q) and the non-negative integer p determine uniquely a finite ray crossing the generalized Pascal triangle if n is given. The paper of Belbachir, Komatsu and Szalay [4] gives a precise description on the sum T_n by showing that it satisfies the linear recurrence relation

$$T_n = \binom{r}{1} x T_{n-1} - \binom{r}{2} x^2 T_{n-2} + \cdots + (-1)^{r+1} \binom{r}{r} x^r T_{n-r} + y^r T_{n-r-q}. \quad (1.4)$$

Obviously, the order of the recurrence is $r+q$ if q is non-negative, and r otherwise. Observe, that (1.4) does not depend on p. Now we slightly modify the problem by including a sequence (G_n) to have the weighted sum

$$T_n = T_n^{(r,q,p),(G)} = \sum_{k=0}^{\omega} \binom{n-qk}{p+rk} x^{n-p-(r+q)k} y^{p+rk} G_k.$$

In general, the problem to describe the behavior of sequence $(T_n^{(r,q,p),(G)})$ is rather difficult, but if (G_n) is a homogeneous linear recursive sequence, then we can conclude Theorem 1.1.

Suppose that (G_n) is a real (or complex) linear homogeneous recurrence of order $s \in \mathbb{N}^+$ with given initial values $G_0, \ldots, G_{s-1}$ and with the defining identity $G_n = \sum_{j=1}^{s} A_j G_{n-j}$, where we assume $A_s \neq 0$.

Theorem 1.1 (Theorem 3.1 in [3])**.** *The terms*

$$T_n = \sum_{k=0}^{\omega} \binom{n-qk}{p+rk} x^{n-p-(r+q)k} y^{p+rk} G_k$$

satisfy the recurrence relation

$$T_n = xT_{n-1} - \sum_{j=1}^{rs-1} (-1)^j \binom{rs-1}{j} x^j (T_{n-j} - xT_{n-j-1}) + \sum_{t=1}^{s} A_t y^{rt} \sum_{j=0}^{r(s-t)} (-1)^j \binom{r(s-t)}{j} x^j T_{n-(r+q)t-j}$$

for all $n \ge \max\{rs, (r+q)s\}$.

Since our application is restricted to the case of binary recurrences $G_n = A_1G_{n-1} + A_2G_{n-2}$, further when $r = q = 1$, $p = 0$ hold we have the following consequence of Theorem 1.1.

Corollary 1.2 (Corollary 3.3 in [3]). *The terms*

$$T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^{n-2k} y^k G_k$$

satisfy the recurrence relation

$$T_n = 2xT_{n-1} + (A_1y - x^2)T_{n-2} - A_1xyT_{n-3} + A_2y^2T_{n-4}. \tag{1.5}$$

Subsequently, equation (1.5) implies

$$T_n = 2T_{n-1} + (A_1 - 1)T_{n-2} - A_1T_{n-3} + A_2T_{n-4} \tag{1.6}$$

if $x = y = 1$.

2. Results

Let $z = \cos t + i \sin t \in \mathbb{C}$, where i is the imaginary unit. We also introduce the notation $\zeta = \sqrt{1 + 4z}$ with the condition $0 \le \arg(\zeta) \le \pi$. Clearly, the complex conjugate of ζ is

$$\overline{\zeta} = \overline{\sqrt{1 + 4z}} = -\sqrt{1 + 4\overline{z}}.$$

We study together the two cases $(G_n) = (C_n)$ and $(G_n) = (S_n)$ as far as it is possible. According to (1.3), the coefficients are $A_1 = 2\cos t$ and $A_2 = -1$ in the common binary recurrence rule. The characteristic polynomial of the recurrence (1.6) in this particular case is

$$p(X) = X^4 - 2X^3 - (2\cos t - 1)X^2 + (2\cos t)X + 1 = \prod_{j=1}^{4}(X - z_j), \tag{2.1}$$

where $z_1 = (1 + \zeta)/2$, $z_2 = (1 - \zeta)/2$, $z_3 = \overline{z_2}$, $z_4 = \overline{z_1}$. Note that all zeros are simple.

Using the notation above we can formalize the first result.

Theorem 2.1. *The identities*

$$\tau_n^{(\cos)} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \cos kt = \frac{1}{2\zeta}(z_1^{n+1} - z_2^{n+1}) - \frac{1}{2\overline{\zeta}}(z_3^{n+1} - z_4^{n+1}),$$

$$\tau_n^{(\sin)} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \sin kt = \frac{i}{2\zeta}(-z_1^{n+1} + z_2^{n+1}) - \frac{i}{2\overline{\zeta}}(z_3^{n+1} - z_4^{n+1})$$

hold.

Using Fibonacci polynomials and applying another method we can prove the following theorem.

Theorem 2.2. *We have the identities*

$$\tau_n^{(\cos)} = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n-k}{k} \cos kt = \frac{1}{2} z^{n/2} F_{n+1}\left(\frac{1}{\sqrt{z}}\right) + \frac{1}{2}\overline{z}^{n/2} F_{n+1}\left(\frac{1}{\sqrt{\overline{z}}}\right),$$

$$\tau_n^{(\sin)} = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n-k}{k} \sin kt = -\frac{i}{2} z^{n/2} F_{n+1}\left(\frac{1}{\sqrt{z}}\right) + \frac{i}{2}\overline{z}^{n/2} F_{n+1}\left(\frac{1}{\sqrt{\overline{z}}}\right).$$

Corollary 2.3. *The corresponding right-hand sides of the identities in Theorems 2.1 and 2.2 are equivalent.*

Remark 2.4. One may show directly, for instance, that

$$\frac{1}{2\zeta}(z_1^{n+1} - z_2^{n+1}) - \frac{1}{2\overline{\zeta}}(z_3^{n+1} - z_4^{n+1}) = \frac{1}{2} z^{n/2} F_{n+1}\left(\frac{1}{\sqrt{z}}\right) + \frac{1}{2}\overline{z}^{n/2} F_{n+1}\left(\frac{1}{\sqrt{\overline{z}}}\right),$$

but it would require some more calculation.

3. Proof of the theorems

Theorem 2.1. The definition of z_i $(i = 1, 2, 3, 4)$ implies $z_1 + z_2 = z_3 + z_4 = 1$. We also have $z_1 - z_2 = \zeta$ and $z_3 - z_4 = -\overline{\zeta}$, further $z_1 z_2 = -z$ and $z_3 z_4 = -\overline{z}$. Observe that $z + \overline{z} = 2\cos t$ and $z - \overline{z} = 2i \sin t$.

Since the zeros of the characteristic polynomial (2.1) are simple the fundamental theorem of linear recurrences (see [5, Theorem C.1.]) leads to the formulae

$$\tau_n^{(cos)} = \sum_{u=1}^{4} c_u z_u^n \quad \text{and} \quad \tau_n^{(sin)} = \sum_{u=1}^{4} d_u z_u^n, \tag{3.1}$$

where the coefficients c_u and d_u can be determined by verifying the above equations for $n = 0, 1, 2, 3$. This means that we must solve two systems of equations, each contains four linear equations with four unknowns.

First we find the powers of z_u in order to fix the system. Trivially, $z_1^0 = 1$, $z_1^1 = z_1$. Further

$$z_1^2 = \left(\frac{1+\zeta}{2}\right)^2 = \frac{1 + 2\zeta + \zeta^2}{4} = \frac{1 + 2\zeta + (1+4z)}{4} = \frac{1+\zeta}{2} + z = z_1 + z,$$

and $z_1^3 = (z_1 + z)z_1 = z_1^2 + z_1 z = z_1 + z + z_1 z$. Similar considerations admit the following more general result. For any $u = 1, 2$ we have

$$z_u^0 = 1, \quad z_u^1 = z_u, \quad z_u^2 = z_u + z, \quad z_u^3 = z_u + z + z_u z, \tag{3.2}$$

while

$$z_u^0 = 1, \quad z_u^1 = z_u, \quad z_u^2 = z_u + \overline{z}, \quad z_u^3 = z_u + \overline{z} + z_u\overline{z} \tag{3.3}$$

hold for $u = 3, 4$. Thus, according to (3.1) we have the system

$$\tau_0^{(\cos)} = 1 = c_1 + c_2 + c_3 + c_4,$$
$$\tau_1^{(\cos)} = 1 = c_1 z_1 + c_2 z_2 + c_3 z_3 + c_4 z_4,$$
$$\tau_2^{(\cos)} = 1 + \cos t = c_1 z_1^2 + c_2 z_2^2 + c_3 z_3^2 + c_4 z_4^2,$$
$$\tau_3^{(\cos)} = 1 + 2\cos t = c_1 z_1^3 + c_2 z_2^3 + c_3 z_3^3 + c_4 z_4^3$$

of linear equations, where we replace the corresponding powers of z_u by (3.2) and (3.3) later. The Vandermonde determinant $V(z_1, z_2, z_3, z_4)$ of the system is non-zero, therefore we have a unique solution. The solution, after a few simplification steps, appears as

$$c_1 = -\frac{(\cos t - \overline{z})i}{2\zeta \sin t} z_1, \qquad c_2 = \frac{(\cos t - \overline{z})i}{2\zeta \sin t} z_2,$$
$$c_3 = -\frac{(\cos t - z)i}{2\overline{\zeta} \sin t} z_3, \qquad c_4 = \frac{(\cos t - z)i}{2\overline{\zeta} \sin t} z_4.$$

Observe that $\cos t - \overline{z} = i \sin t$, and $\cos t - z = -i \sin t$. Hence $c_1 = z_1/(2\zeta)$, $c_2 = -z_2/(2\zeta)$, $c_3 = -z_3/(2\overline{\zeta})$, and $c_4 = z_4/(2\overline{\zeta})$. Combining these arguments with (3.1), finally we have

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \cos kt = \frac{1}{2\zeta}(z_1^{n+1} - z_2^{n+1}) - \frac{1}{2\overline{\zeta}}(z_3^{n+1} - z_4^{n+1}).$$

This proves the first part of Theorem 2.1.

Now turn our attention to the sum $\tau_n^{(sin)}$. In this case, the same machinery works, but now we have $\tau_0^{(sin)} = 0$, $\tau_1^{(sin)} = 0$, $\tau_2^{(sin)} = \sin t$, and $\tau_3^{(sin)} = 2\sin t$ on the left-hand side of the current system of four equations. Of course, on the right-hand side we replace c_u by d_u. The solution is given by

$$d_1 = -\frac{i}{2\zeta} z_1, \ d_2 = \frac{i}{2\zeta} z_2, \ d_3 = -\frac{i}{2\overline{\zeta}} z_3, \ d_4 = \frac{i}{2\overline{\zeta}} z_4,$$

and we can conclude the second statement of the theorem immediately. □

Remark 3.1. Recall that $z = \cos t + i \sin t$. The characteristic polynomial $p(X)$ has the factorization

$$p(X) = X^4 - 2X^3 - (2\cos t - 1)X^2 + (2\cos t)X + 1 = (X^2 - X - z)(X^2 - X - \overline{z}).$$

This explains why we obtained two separated parts for both $\tau_n^{(cos)}$ and $\tau_n^{(sin)}$ in Theorem 2.1:

$$X^2 - X - z = (X - z_1)(X - z_2) \quad \text{and} \quad X^2 - X - \overline{z} = (X - z_3)(X - z_4).$$

Theorem 2.2. Let the polynomial sequence $(p(t))$ be defined by $p_0(t) = 0$, $p_1(t) = 1$, and $p_n(t) = p_{n-1}(t) + tp_{n-2}(t)$ for $n \geq 2$. The connection between these polynomials and Fibonacci polynomials can be given by $p_n(t) = t^{(n-1)/2}F_n(1/\sqrt{t})$. Indeed, it is true for $n = 0, 1$ and we see by induction that

$$\begin{aligned} t^{(n-1)/2}F_n(1/\sqrt{t}) &= t^{(n-1)/2}\frac{1}{\sqrt{t}}F_{n-1}(1/\sqrt{t}) + t^{(n-1)/2}F_{n-2}(1/\sqrt{t}) \\ &= t^{(n-2)/2}F_{n-1}(1/\sqrt{t}) + t \cdot t^{(n-3)/2}F_{n-2}(1/\sqrt{t}) \\ &= p_{n-1}(t) + tp_{n-2}(t) \\ &= p_n(t). \end{aligned}$$

Recall the explicit sum formula

$$F_{n+1}(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} t^{n-2k},$$

see, for example, [8]. Hence

$$p_{n+1}(t) = t^{n/2}F_{n+1}(1/\sqrt{t}) = t^{n/2} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} t^{k-n/2} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} t^k,$$

i.e. the coefficients of polynomial $p_{n+1}(t)$ are the binomial coefficients $\binom{n-k}{k}$. Knowing that $z^k = \cos kt + i \sin kt$, we obtain immediately that

$$p_{n+1}(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} (\cos kt + i \sin kt).$$

Combining it with

$$p_{n+1}(\overline{z}) = \sum_{k=0}^{n/2} \binom{n-k}{k} (\cos kt - i \sin kt),$$

we have immediately

$$\begin{aligned} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \cos kt &= \frac{p_{n+1}(z) + p_{n+1}(\overline{z})}{2} \\ &= \frac{z^{n/2}F_{n+1}\left(\frac{1}{\sqrt{z}}\right) + \overline{z}^{n/2}F_{n+1}\left(\frac{1}{\sqrt{\overline{z}}}\right)}{2}, \end{aligned}$$

and

$$\begin{aligned} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \sin kt &= \frac{-p_{n+1}(z) + p_{n+1}(\overline{z})}{2} i \\ &= \frac{-z^{n/2}F_{n+1}\left(\frac{1}{\sqrt{z}}\right) + \overline{z}^{n/2}F_{n+1}\left(\frac{1}{\sqrt{\overline{z}}}\right)}{2} i. \end{aligned}$$

Now the proof is complete. □

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