Annales Mathematicae et Informaticae
59 (2023) pp. 54–70
DOI: 10.33039/ami.2023.07.001
URL: https://ami.uni-eszterhazy.hu

Computation of the Wedderburn decomposition of semisimple group algebras of groups up to order 120

Gaurav Mittal[a], Rajendra Kumar Sharma[b]

[a]Defence Research and Development Organization,
New Delhi, 110054, India
gaurav.mittaltwins@gmail.com

[b]Department of Mathematics, Indian Institute of Technology Delhi,
New Delhi, 110016, India
rksharmaiitd@gmail.com

Abstract. In this paper, we discuss the Wedderburn decompositions of the semisimple group algebras of all groups up to order 120. More precisely, we explicitly compute the Wedderburn decompositions of the semisimple group algebras of 26 non-metabelian groups.

Keywords: Unit group, Finite field, Wedderburn decomposition

AMS Subject Classification: 16U60, 20C05

1. Introduction

Let G be a finite group and $\mathbb{F}_p$ be a finite field for a prime p having characteristics p. Let p be such that $p \nmid |G|$. This means that the group algebra $\mathbb{F}_pG$ is semisimple (see [13]). Due to various applications of units of group algebras (for example, in cryptography [6, 14], in coding theory [7], in isomorphism problems and exploration of Lie properties of group algebras [2] etc.), the problem of computing the Wedderburn decompositions (or unit groups) of finite semisimple group algebras is an extensively studied problem (see [1, 3, 5, 9, 11, 12, 15, 19, 21] and the references therein).

One of the major steps in the direction of computation of Wedderburn decompositions (WDs) of finite semisimple group algebras was taken in [1]. The paper [1] gave an algorithm to compute the WDs of the semisimple group algebras of all

Submitted: September 18, 2022
Accepted: July 4, 2023
Published online: July 17, 2023

metabelian groups. We recall that a finite group G is metabelian if its derived subgroup is abelian. Consequently, the entire research in this direction is shifted on to the computation of WDs of semisimple group algebras of non-metabelian groups. Mittal et al. [17] computed the WDs of semisimple group algebras of all non-metabelian groups up to order 72. Furthermore, Mittal et al. [16, 18, 22] also computed the WDs of all semisimple group algebras of all non-metabelian groups of order 108 and some non-metabelian groups of order 120. Since, the WDs of semisimple group algebras of the symmetric groups S_n can be easily computed by employing the representation theory (see [8]), the papers [18, 22] completed the task of computation of WDs of group algebras of non-metabelian groups of order 120.

Using [20] we note that the only non-metabelian groups of order less than 120 that are not yet studied in the literature are those of order 96. Hence, the main objective of this paper is to complete the task of computation of WDs of group algebras of 26 non-metabelian groups of order 96. Consequently, with this paper, the computation of the WDs of semisimple group algebras of all groups up to order 120 will be complete. From the WD, the unit group can be computed straightforwardly.

Organization of the paper. Section 2 contains certain preliminaries that play an important role in the computation of WDs. Our main results related to WDs of semisimple group algebras are discussed in Section 3. We give the complete details of computation of WDs only for a few groups among the 26 groups. This is because for the remaining groups, the details can be generated analogously. We conclude the paper in the last section.

2. Preliminaries

Let the exponent of the group G be denoted by e and let the primitive e^{th} root of unity be denoted by ε. In our work, we use the notations of [4]. Let $\mathbb{F}$ denote a finite field. Let us define

$$I_{\mathbb{F}} = \{\omega \mid \varepsilon \mapsto \varepsilon^{\omega} \text{ is an automorphism of } \mathbb{F}(\varepsilon) \text{ over } \mathbb{F}\}.$$

It can be noted that the Galois group $\mathrm{Gal}\big(\mathbb{F}(\varepsilon), \mathbb{F}\big)$ is a cyclic group. This guarantees the existence of an $s \in \mathbb{Z}_e^*$ fulfilling $\lambda(\varepsilon) = \varepsilon^s$ for any $\lambda \in \mathrm{Gal}\big(\mathbb{F}(\varepsilon), \mathbb{F}\big)$. More specifically, $I_{\mathbb{F}}$ is a subgroup of the group $\mathbb{Z}_e^*$ (multiplicative). Let g be a p-regular element of the group G. Let us define

$$\gamma_g = \sum_{h \in C(g)} h,$$

where $C(g)$ denotes the set of all those elements of G that are conjugate to the p-regular element g. For γ_g, let the cyclotomic $\mathbb{F}$-class of be represented by

$$S(\gamma_g) = \{\gamma_{g^{\omega}} \mid \omega \in I_{\mathbb{F}}\}.$$

Let $J(\mathbb{F}G)$ represent the Jacobson radical of the group algebra $\mathbb{F}G$.

Next, we discuss two important results of [4].

Theorem 2.1. *The number of cyclotomic $\mathbb{F}$-classes in G is equal to the number of simple components of $\mathbb{F}G/J(\mathbb{F}G)$.*

Theorem 2.2. *Let the number of cyclotomic $\mathbb{F}$-classes in G be π and let ε be primitive e^{th} root of unity, where e is the exponent of G. Let $S_1, \dots, S_\pi$ be the simple components of the center of $\mathbb{F}G/J(\mathbb{F}G)$ and let $Y_1, \dots, Y_\pi$ be the cyclotomic $\mathbb{F}$-classes in G. Then, $|Y_i| = [S_i : \mathbb{F}]$ for each $1 \le i \le \pi$, after suitable ordering of the indices.*

We remark that both the Theorems 2.1 and 2.2 will be very crucial for our main results. Next, we discuss a significant result that shows that in the WD of a finite group algebra $\mathbb{F}G/J(\mathbb{F}G)$, $\mathbb{F}$ is always a Wedderburn component (see [17]).

Lemma 2.3. *Let Σ_1 and Σ_2 be two algebras over $\mathbb{F}$ having finite dimension. Let Σ_2 be semisimple and let $\varphi : \Sigma_1 \to \Sigma_2$ be a homomorphism that is also surjective. Then, there holds*

$$\Sigma_1/J(\Sigma_1) \cong \Sigma_2 + \Sigma_3,$$

where Σ_3 is an another semisimple $\mathbb{F}$-algebra.

Suppose that $J(\mathbb{F}G) = 0$. Then Lemma 2.3 confirms that $\mathbb{F}$ is always a simple component of $\mathbb{F}G$. Next, we recall a result from [10] that explicitly characterizes the set $I_{\mathbb{F}}$.

Theorem 2.4. *Let $q = p^r$ for a positive integer r and a prime p and let $\mathbb{F}_q$ be a finite field. Let e be such that $\gcd(e, q) = 1$ and let ε be the primitive e^{th} root of unity. Let $o(q)$ be the order of q modulo e. Then we have*

$$I_{\mathbb{F}_q} = \{1, q, \dots, q^{o(q)-1}\} \mod e.$$

Further, we recall two important theorems from [13].

Theorem 2.5. *Let R be a commutative ring and let RG be a semisimple group algebra. Then we have*

$$RG \cong R\big(G/G'\big) \oplus \Delta(G, G').$$

Here G' is the derived subgroup of G, $R\big(G/G'\big)$ is the sum of all commutative simple components and $\Delta(G, G')$ is the sum of all non-commutative simple components of RG.

Theorem 2.6. *Let RG be a semisimple group algebra and H be a normal subgroup of G. Then*

$$RG \cong R\big(G/H\big) \oplus \Delta(G, H).$$

Here $\Delta(G, H)$ represents the left ideal of RG and it is generated by the set $\{h - 1 : h \in H\}$.

We remark that through Theorem 2.5 one can obtain all the possible commutative simple components of the group algebra $\mathbb{F}_qG$. Further, Theorem 2.6 relates WD of the group algebra $\mathbb{F}_q(G/H)$ with that of $\mathbb{F}_qG$ for a normal subgroup H of G. Finally, we end this section by invoking an important result from [3]. This result will be very crucial in unique computation of the WD for any semisimple group algebra.

Theorem 2.7. *Let $\mathbb{F}$ be a finite field of characteristics p. Let $\Sigma = \oplus_{s=1}^{t} M_{n_s}(\mathbb{F}_s)$ be a summand of a semisimple group algebra $\mathbb{F}G$, where $\mathbb{F}_s$ denotes a finite extension of $\mathbb{F}$ for each s. Then $p \nmid n_s$ for every $1 \leq s \leq t$.*

3. WDs of non-metabelian groups of order 96

In this section, we discuss all the non-metabelian groups of order 96 along with their WDs. Up to isomorphism, we note that there are 231 groups of order 96 and 26 of them are non-metabelian. Among these 26 groups, 11 have exponent 24 and rest all have exponent 12.

3.1. Non-metabelian groups of order 96 having exponent 24

The non-metabelian groups of order 96 having exponent 24 are as follows:

1. $G_1 = A_4 \rtimes C_8$
2. $G_2 = SL(2,3) \rtimes C_4$
3. $G_3 = SL(2,3) \rtimes C_4$
4. $G_4 = C_2 \times (SL(2,3) \cdot C_2)$
5. $G_5 = C_2 \times GL(2,3)$
6. $G_6 = (C_2 \times SL(2,3)) \rtimes C_2$
7. $G_7 = (SL(2,3) \cdot C_2) \rtimes C_2$
8. $G_8 = (((C_4 \times C_2) \rtimes C_2) \rtimes C_3) \rtimes C_2$
9. $G_9 = (((C_4 \times C_2) \rtimes C_2) \rtimes C_3) \rtimes C_2$
10. $G_{10} = ((C_8 \times C_2) \rtimes C_2) \rtimes C_3$
11. $G_{11} = ((C_4 \times C_4) \rtimes C_3) \rtimes C_2)$.

3.2. Wedderburn decomposition of $\mathbb{F}_qG_1$ and some other group algebras

The presentation of $G_1 = A_4 \rtimes C_8$ is as follows:

$$\langle x, y, z, w, t, u \mid x^2y^{-1},\ [y,x],\ [z,x],\ [w,x]w^{-1},\ [t,x]u^{-1}t^{-1},$$
$$[u,x]u^{-1}t^{-1},\ y^2z^{-1},\ [z,y],[w,y],\ [t,y],\ [u,y],\ z^2,\ [w,z],$$
$$[t,z],\ [u,z],\ w^3,\ \ [t,w]u^{-1}t^{-1},\ [u,w]t^{-1},\ t^2,\ [u,t],\ u^2\,\rangle.$$

This group has 20 conjugacy classes as shown in the next table.

R	e	x	y	z	w	t	xy	xz	xt	yz	yw	yt	zw	zt	xyz
S	1	6	1	1	8	3	6	6	6	1	8	3	8	3	6
O	1	8	4	2	3	2	8	8	8	4	12	4	6	2	8

xyt	xzt	yzw	yzt	$xyzt$
6	6	8	3	6
8	8	12	4	8

where R, S and O represent representative, size and order of conjugacy classes, respectively. From the above discussion, we conclude that the exponent of G_1 is 24. Also $G_1' \cong A_4$ with $G_1/G_1' \cong C_8$. Since $p > 3$, we have $\gcd(|G_1|, p) = 1$, and so $J(\mathbb{F}_q G_1) = 0$.

Theorem 3.1. *The Wedderburn decomposition of $\mathbb{F}_q G_1$ for $q = p^k$, $p > 3$ is as follows:*

values of p and k	*Wedderburn decomposition*
k *even or* $p \equiv \{1, 17\} \mod 24$ *and* k *odd*	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^8$
$p^k \equiv \{5, 13\} \mod 24$ *and* k *odd*	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^2 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^4$ $\oplus M_3(\mathbb{F}_{q^2})^2$
$p^k \equiv \{7, 23\} \mod 24$ *and* k *odd or* $p^k \equiv \{11, 19\} \mod 24$ *and* k *odd*	$\mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^3 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^2 \oplus$ $M_2(\mathbb{F}_{q^2}) \oplus M_3(\mathbb{F}_{q^2})^3$

Proof. As $\mathbb{F}_q G_1$ is semisimple, we have $\mathbb{F}_q G_1 \cong \oplus_{r=1}^{t} M_{n_r}(\mathbb{F}_r)$, $t \in \mathbb{Z}$, where for each r, $\mathbb{F}_r$ is a finite extension of $\mathbb{F}_q$, $n_r \geq 1$. Incorporating Lemma 2.3 in above to obtain

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q \oplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r). \tag{3.1}$$

For k even and any prime $p > 3$, $p^k \equiv 1 \mod 24$. This means $|S(\gamma_g)| = 1$ for each $g \in G_1$ as $I_{\mathbb{F}} = \{1\}$ (see Theorem 2.4). Hence, (3.1) and Theorems 2.1, 2.2 imply that $\mathbb{F}_q G_1 \cong \mathbb{F}_q \oplus_{r=1}^{19} M_{n_r}(\mathbb{F}_r)$. This with $G_1/G_1' \cong C_8$ and Theorem 2.5 leads to (with suitable rearrangement of indexes) $\mathbb{F}_q G_1 \cong \mathbb{F}_q^8 \oplus_{r=1}^{12} M_{n_r}(\mathbb{F}_r)$ with $88 = \sum_{r=1}^{12} n_r^2$, $n_r \geq 2$, which gives the only possible choice $(2^4, 3^8)$ (here a^b means $(a, a, \ldots, b$ times$)$) for values of n_r's. Therefore, the required WD is

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q^8 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^8. \tag{3.2}$$

Now, we assume that k is odd. We discuss this possibility in the following 4 cases:

Case 1. $p \equiv 1 \mod 24$ or $p^k \equiv 17 \mod 24$. In this case, we have $|S(\gamma_g)| = 1$ for each $g \in G_1$ as $I_{\mathbb{F}} = \{1\}$ or $I_{\mathbb{F}} = \{1, 17\}$. Hence, WD is given by (3.2).

Case 2. $p^k \equiv 5 \mod 24$ or $p^k \equiv 13 \mod 24$. In this case, we have $S(\gamma_x) = \{\gamma_x, \gamma_{xz}\}$, $S(\gamma_{xy}) = \{\gamma_{xy}, \gamma_{xyz}\}$, $S(\gamma_{xt}) = \{\gamma_{xt}, \gamma_{xzt}\}$, $S(\gamma_{xyt}) = \{\gamma_{xyt}, \gamma_{xyzt}\}$, and $S(\gamma_g) = \{\gamma_g\}$ for the remaining representatives g of conjugacy classes. Using Theorems 2.1 and 2.2 and (3.1), we get $\mathbb{F}_q G_1 \cong \mathbb{F}_q \oplus_{r=1}^{11} M_{n_r}(\mathbb{F}_q) \oplus_{r=12}^{15} M_{n_r}(\mathbb{F}_{q^2})$. Applying Theorem 2.5 with $G_1/G_1' \cong C_8$ and $\mathbb{F}_q C_8 \cong \mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^2$ to obtain

$$\begin{aligned} &\mathbb{F}_q G_1 \cong \mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^2 \oplus_{r=1}^{8} M_{n_r}(\mathbb{F}_q) \oplus_{r=9}^{10} M_{n_r}(\mathbb{F}_{q^2}) \\ &\text{with} \ \ 88 = \sum_{r=1}^{8} n_r^2 + 2\sum_{r=9}^{10} n_r^2, \ n_r \geq 2, \end{aligned} \tag{3.3}$$

which gives 3 possibilities for values of n_r' namely $(3^8, 2^2), (2^2, 3^6, 2, 3)$ and $(2^4, 3^6)$. For uniqueness, consider a normal subgroup $H_1 = \langle t, u \rangle$ of G_1 with $K_1 = G_1/H_1 \cong C_3 \rtimes C_8$. It can be verified that K_1 has 12 conjugacy classes as shown in the table below.

R	e	x	y	z	w	xy	xz	yz	yw	zw	xyz	yzw
S	1	3	1	1	2	3	3	1	2	2	3	2
O	1	8	4	2	3	8	8	4	12	6	8	12

Also $K_1' \cong C_3$ with $K_1/K_1' \cong C_8$. For the representatives k of K_1, we have $S(\gamma_x) = \{\gamma_x, \gamma_{xz}\}$, $S(\gamma_{xy}) = \{\gamma_{xy}, \gamma_{xyz}\}$, $S(\gamma_k) = \{\gamma_k\}$ for the remaining representatives. Therefore, employ Theorems 2.1, 2.2 and 2.5 to obtain $\mathbb{F}_q K_1 \cong \mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^2 \oplus_{r=1}^4 M_{t_r}(\mathbb{F}_q)$ with $16 = \sum_{r=1}^4 t_r^2$. This gives us the only possibility (2^4) for value of t_r's. Next, incorporate Theorem 2.6 in (3.3) to deduce that $(2^4, 3^6)$ is the correct choice for n_r's and therefore, we have $\mathbb{F}_q G_1 \cong \mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^2 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_{q^2})^2$.

Case 3. $p^k \equiv 7 \mod 24$ or $p^k \equiv 23 \mod 24$. In this case, we have $S(\gamma_x) = \{\gamma_x, \gamma_{xyz}\}$, $S(\gamma_y) = \{\gamma_y, \gamma_{yz}\}$, $S(\gamma_{xy}) = \{\gamma_{xy}, \gamma_{xz}\}$, $S(\gamma_{xt}) = \{\gamma_{xt}, \gamma_{xyzt}\}$, $S(\gamma_{xyt}) = \{\gamma_{xyt}, \gamma_{xzt}\}$, $S(\gamma_{yw}) = \{\gamma_{yw}, \gamma_{yzw}\}$, $S(\gamma_{yt}) = \{\gamma_{yt}, \gamma_{yzt}\}$, $S(\gamma_g) = \{\gamma_g\}$ for the remaining representatives g of conjugacy classes. Using Theorems 2.1, 2.2 and (3.1), we get $\mathbb{F}_q G_1 \cong \mathbb{F}_q \oplus_{r=1}^5 M_{n_r}(\mathbb{F}_q) \oplus_{r=6}^{12} M_{n_r}(\mathbb{F}_{q^2})$. Applying Theorem 2.5 with $G_1/G_1' \cong C_8$ and $\mathbb{F}_q C_8 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^3$ in this to obtain

$$\begin{aligned}&\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^3 \oplus_{r=1}^4 M_{n_r}(\mathbb{F}_q) \oplus_{r=5}^8 M_{n_r}(\mathbb{F}_{q^2})\\ &\text{with } \; 88 = \sum_{r=1}^4 n_r^2 + 2\sum_{r=5}^8 n_r^2, \;\; n_r \geq 2\end{aligned} \tag{3.4}$$

which gives three possibilities for values of n_r's namely $(3^4, 2^2, 3^2), (2^2, 3^2, 2, 3^3)$ and $(2^4, 3^4)$. Further, we can verify that for the representatives k of K_1, we have $S(\gamma_x) = \{\gamma_x, \gamma_{xyz}\}$, $S(\gamma_y) = \{\gamma_y, \gamma_{yz}\}$, $S(\gamma_{xy}) = \{\gamma_{xy}, \gamma_{xz}\}$, $S(\gamma_{yw}) = \{\gamma_{yw}, \gamma_{yzw}\}$ and $S(\gamma_k) = \{\gamma_k\}$ for the remaining representatives. This with Theorems 2.1, 2.2 and 2.5 leads to $\mathbb{F}_q K_1 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^3 \oplus_{t=1}^2 M_{t_r}(\mathbb{F}_q) \oplus M_{t_3}(\mathbb{F}_{q^2})$, $t_r \geq 2$, $t_r \in \mathbb{Z}$ with $16 = \sum_{r=1}^2 t_r^2 + 2t_3^2$, which gives the only choice (2^3) for t_r's. Therefore, (3.4) and Theorem 2.6 imply that $(2^2, 3^2, 2, 3^3)$ is the correct choice for n_r's. So, we get $\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^3 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^2 \oplus M_2(\mathbb{F}_{q^2}) \oplus M_3(\mathbb{F}_{q^2})^3$.

Case 4. $p^k \equiv 11 \mod 24$ or $p^k \equiv 19 \mod 24$. In this case, we have $S(\gamma_x) = \{\gamma_x, \gamma_{xy}\}$, $S(\gamma_y) = \{\gamma_y, \gamma_{yz}\}$, $S(\gamma_{xz}) = \{\gamma_{xz}, \gamma_{xyz}\}$, $S(\gamma_{xt}) = \{\gamma_{xt}, \gamma_{xyt}\}$, $S(\gamma_{xzt}) = \{\gamma_{xzt}, \gamma_{xyzt}\}$, $S(\gamma_{yw}) = \{\gamma_{yw}, \gamma_{yzw}\}$, $S(\gamma_{yt}) = \{\gamma_{yt}, \gamma_{yzt}\}$, $S(\gamma_g) = \{\gamma_g\}$ for the remaining representatives g. Using Theorems 2.1 and 2.2 and (3.1), we get $\mathbb{F}_q G_1 \cong \mathbb{F}_q \oplus_{r=1}^5 M_{n_r}(\mathbb{F}_q) \oplus_{r=6}^{12} M_{n_r}(\mathbb{F}_{q^2})$. Further, we can easily see that rest part of this case is similar to Case 3. □

Next, we remark that for the groups G_i, where $2 \leq i \leq 8$ and $i = 10$, the Wedderburn decomposition of their group algebras can be computed by following the steps of Theorem 3.1 (see Tables 1–8). Hence, we are omitting their proofs from the paper.

Table 1. Wedderburn decomposition of $\mathbb{F}_qG_2$.

values of p and k	Wedderburn decomposition
k even or $p \equiv \{1, 17\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^6 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)^2$
$p^k \equiv \{5, 13\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)^2$ $\oplus M_2(\mathbb{F}_{q^2})^2$
$p^k \equiv \{7, 23\} \mod 24$ and k odd or $p^k \equiv \{11, 19\} \mod 24$ and k odd	$\mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^2 \oplus$ $M_4(\mathbb{F}_q)^2 \oplus M_2(\mathbb{F}_{q^2}) \oplus M_3(\mathbb{F}_{q^2})$

Table 2. Wedderburn decomposition of $\mathbb{F}_qG_3$.

values of p and k	Wedderburn decomposition
k even or $p \in \{1, 5, 13, 17\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^6 \oplus M_3(\mathbb{F}_q)^4$ $\oplus M_4(\mathbb{F}_q)^2$
$p^k \equiv \{7, 11, 19, 23\} \mod 24$ and k odd	$\mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^2 \oplus$ $M_2(\mathbb{F}_{q^2})^2 \oplus M_3(\mathbb{F}_{q^2}) \oplus M_4(\mathbb{F}_{q^2})$

Table 3. Wedderburn decomposition of $\mathbb{F}_qG_4$.

values of p and k	Wedderburn decomposition
k even or $p \in \{1, 7, 17, 23\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^6 \oplus M_3(\mathbb{F}_q)^4$ $\oplus M_4(\mathbb{F}_q)^2$
$p^k \equiv \{5, 11, 13, 19\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus$ $M_4(\mathbb{F}_q)^2 \oplus M_2(\mathbb{F}_{q^2})^2$

Table 4. Wedderburn decomposition of $\mathbb{F}_qG_5$.

values of p and k	Wedderburn decomposition
k even or $p \in \{1, 11, 17, 19\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^6 \oplus M_3(\mathbb{F}_q)^4$ $\oplus M_4(\mathbb{F}_q)^2$
$p^k \equiv \{5, 7, 13, 23\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus$ $M_4(\mathbb{F}_q)^2 \oplus M_2(\mathbb{F}_{q^2})^2$

Table 5. Wedderburn decomposition of $\mathbb{F}_qG_6$.

values of p and k	Wedderburn decomposition
k even or $p \in \{1,7,13,19\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)^3$
$p^k \equiv \{5,7,13,23\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q) \oplus M_4(\mathbb{F}_{q^2})$

Table 6. Wedderburn decomposition of $\mathbb{F}_qG_7$.

values of p and k	Wedderburn decomposition
k even or $p \in \{1,11,13,23\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)^3$
$p^k \equiv \{5,7,13,23\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q) \oplus M_4(\mathbb{F}_{q^2})$

Table 7. Wedderburn decomposition of $\mathbb{F}_qG_8$.

values of p and k	Wedderburn decomposition
k even or $p \in \{1,11,13,23\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)^3$
$p^k \equiv \{5,7,17,19\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q) \oplus M_4(\mathbb{F}_{q^2})$

Table 8. Wedderburn decomposition of $\mathbb{F}_qG_{10}$.

values of p and k	Wedderburn decomposition
k even or $p \equiv 1 \mod 24$ and k odd	$\mathbb{F}_q^{12} \oplus M_2(\mathbb{F}_q)^{12} \oplus M_3(\mathbb{F}_q)^4$
$p^k \in \{7,19\} \mod 24$ and k odd	$\mathbb{F}_q^6 \oplus \mathbb{F}_{q^2}^3 \oplus M_3(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_{q^2}) \oplus M_2(\mathbb{F}_{q^2})^6$
$p^k \equiv 13 \mod 24$ and k odd	$\mathbb{F}_q^{12} \oplus M_3(\mathbb{F}_q)^4 \oplus M_2(\mathbb{F}_{q^2})^6$
$p^k \equiv 17 \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^4 \oplus M_3(\mathbb{F}_q)^4 \oplus M_2(\mathbb{F}_q)^4 \oplus M_2(\mathbb{F}_{q^2})^4$
$p^k \in \{11,23\} \mod 24$ and k odd	$\mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^5 \oplus M_3(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_{q^2}) \oplus M_2(\mathbb{F}_{q^2})^6$
$p^k \equiv 5 \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^4 \oplus M_3(\mathbb{F}_q)^4 \oplus M_2(\mathbb{F}_{q^2})^6$

3.3. Wedderburn decomposition of $\mathbb{F}_qG_{11}$

It is to be noted that for the group algebra $\mathbb{F}_qG_{11}$, WD can not be uniquely characterize only by using Theorems 2.5 and 2.6. We also need Theorem 2.7 for its unique characterization. Consequently, we separately discuss the WD of $\mathbb{F}_qG_{11}$ in the following theorem. We have $G_{11} = ((C_4 \times C_4) \rtimes C_3) \rtimes C_2)$. This group has 10 conjugacy classes.

R	e	x	y	z	t	xz	xt	zw	zt	xzt
S	1	12	32	3	3	12	12	3	6	12
O	1	2	3	4	2	8	4	4	4	8

Clearly, the exponent of G_{11} is 24 and $G'_{11} \cong (C_4 \times C_4) \rtimes C_3$ with $G_{11}/G'_{11} \cong C_2$.

Theorem 3.2. *The Wedderburn decomposition of $\mathbb{F}_qG_{11}$ for $q = p^k$, $p > 3$ is*

values of p and k	*Wedderburn decomposition*
k even or $p \in \{1, 5, 13, 17\} \mod 24$ and k odd	$\mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^6 \oplus M_6(\mathbb{F}_q)$
$p^k \equiv \{7, 11, 19, 23\} \mod 24$ *and k odd*	$\mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q)$ $\oplus M_3(\mathbb{F}_{q^2})^2$

Proof. For k even and any prime $p > 3$, $p^k \equiv 1 \mod 24$. This means $|S(\gamma_g)| = 1$ for each $g \in G_{11}$ and hence, (3.1) and Theorems 2.1, 2.2 imply that $\mathbb{F}_qG_{11} \cong \mathbb{F}_q \oplus_{r=1}^{9} M_{n_r}(\mathbb{F}_r)$. This with $G_{11}/G'_{11} \cong C_2$ and Theorem 2.5 leads to $\mathbb{F}_qG_{11} \cong \mathbb{F}_q^2 \oplus_{r=1}^{8} M_{n_r}(\mathbb{F}_r)$ with $94 = \sum_{r=1}^{8} n_r^2$, $n_r \geq 2$ which gives four possible choices for n_r's as $(2^5, 3, 4, 7), (2^3, 3, 4^3, 5), (2^2, 3^4, 5^2)$ and $(2, 3^6, 6)$. In order to seek uniqueness, consider a normal subgroup $H_{11,1} = \langle t, u \rangle$ of G_{11} with $K_{11,1} = G_{11}/H_{11,1} \cong S_4$. From [9] and Theorem 2.6, we conclude that $(2^2, 3^4, 5^2)$ and $(2, 3^6, 6)$ are the only required possibility for n_r's. Further, using Theorem 2.7, we derive that the required choice for n_r's is $(2, 3^6, 6)$. Therefore, we have the result. Next, we assume that k is odd. We discuss this possibility in the following 2 cases:

Case 1. $p^k \equiv \{1, 5, 13, 17\} \mod 24$. In this case, WD is same as in the case of k even as $|S(\gamma_g)| = 1$ for each representative g of conjugacy classes.

Case 2. $p^k \equiv \{7, 11, 19, 23\} \mod 24$. In this case, we have $S(\gamma_z) = \{\gamma_z, \gamma_{zw}\}$, $S(\gamma_{xz}) = \{\gamma_{xz}, \gamma_{xzt}\}$, $S(\gamma_g) = \{\gamma_g\}$, for the remaining representatives g of conjugacy classes. Using Theorems 2.1, 2.2 and (3.1), we reach to $\mathbb{F}_qG_{11} \cong \mathbb{F}_q \oplus_{r=1}^{5} M_{n_r}(\mathbb{F}_q) \oplus_{r=6}^{7} M_{n_r}(\mathbb{F}_{q^2})$. Now incorporate Theorem 2.4 to obtain $\mathbb{F}_qG_{11} \cong \mathbb{F}_q^2 \oplus_{r=1}^{4} M_{n_r}(\mathbb{F}_q) \oplus_{r=5}^{6} M_{n_r}(\mathbb{F}_{q^2})$ with $94 = \sum_{r=1}^{4} n_r^2 + 2\sum_{r=5}^{6} n_r^2$, $n_r \geq 2$. Further, again consider the normal subgroup $H_{11,1}$. This with Theorem 2.6 yields $\mathbb{F}_qG_{11} \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^2 \oplus M_{n_1}(\mathbb{F}_q) \oplus_{r=2}^{3} M_{n_r}(\mathbb{F}_{q^2})$, $72 = n_1^2 + 2\sum_{r=2}^{3} n_r^2$, $n_r \geq 2$. The possible choices for n_r's satisfying this are $(2, 3, 5)$ and $(6, 3, 3)$. By the same logic given for the case when k is even, we derive that $(6, 3, 3)$ is the required choice. □

3.4. Wedderburn decomposition of $\mathbb{F}_qG_9$

Next, we discuss the WD of $\mathbb{F}_qG_9$ (see Table 9). We mention that, unfortunately for this particular group, our theory is not enough to uniquely characterize the WD of its corresponding group algebra when $p^k \in \{5, 13\}$ mod 24. We obtain that WD is one of the following two possibilities: $\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_{q^2}) \oplus M_2(\mathbb{F}_{q^2})^2$; $\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^4 \oplus M_2(\mathbb{F}_{q^2}) \oplus M_4(\mathbb{F}_{q^2})$. Consequently, we make use of computer package GAP in this case for uniquely determine WD.

Table 9. Wedderburn decomposition of $\mathbb{F}_q G_9$.

values of p and k	Wedderburn decomposition
k even or $p \in \{1, 17\}$ mod 24 and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^6 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)^2$
$p^k \in \{5, 13\}$ mod 24 and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_{q^2})$ $\oplus M_2(\mathbb{F}_{q^2})^2$
$p^k \in \{7, 11, 19, 23\}$ mod 24 and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_2(\mathbb{F}_{q^2})^2$ $\oplus M_4(\mathbb{F}_{q^2})$

This completes the computation of WDs of semisimple group algebras of non-metabelian groups of order 96 having exponent 24. Next, we proceed to compute the WDs of semisimple group algebras of non-metabelian groups of order 96 having exponent 12.

3.5. Non-metabelian groups of order 96 having exponent 12

1. $G_{12} = ((C_4 \times C_2) \rtimes C_4) \rtimes C_3$
2. $G_{13} = A_4 \rtimes Q_8$
3. $G_{14} = C_4 \times S_4$
4. $G_{15} = (C_4 \times A_4) \rtimes C_2$
5. $G_{16} = (C_2 \times C_2 \times A_4) \rtimes C_2$
6. $G_{17} = (C_2 \times C_2 \times Q_8) \rtimes C_3$
7. $G_{18} = ((C_2 \times C_2 \times C_2) \rtimes (C_2 \times C_2)) \rtimes C_3$
8. $G_{19} = ((C_2 \times C_2 \times C_2) \rtimes (C_2 \times C_2)) \rtimes C_3$
9. $G_{20} = ((C_2 \times Q_8) \rtimes C_2) \rtimes C_3$
10. $G_{21} = C_2 \times (A_4 \rtimes C_4)$
11. $G_{22} = C_2 \times C_2 \times S_4$
12. $G_{23} = ((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3) \rtimes C_2$
13. $G_{24} = C_4 \times SL(2, 3)$
14. $G_{25} = C_2 \times C_2 \times SL(2, 3)$
15. $G_{26} = C_2 \times (((C_4 \times C_2) \rtimes C_2) \rtimes C_3)$.

3.6. Wedderburn decomposition of $\mathbb{F}_q G_{12}$ and some other group algebras

The presentation of $G_{12} = ((C_4 \times C_2) \rtimes C_4) \rtimes C_3$ is

$$\langle x, y, z, w, t, u \mid x^3,\ [y, x]t^{-1}w^{-1}z^{-1}y^{-1},\ [z, x]u^{-1}t^{-1}y^{-1},\ [w, x]u^{-1}t^{-1}w^{-1},$$
$$[t, x]u^{-1}w^{-1},\ [u, x],\ y^2t^{-1}w^{-1},\ [z, y]u^{-1},\ [w, y],\ [t, y],\ [u, y],\ z^2w^{-1},$$

$[w,z],\ [t,z],\ [u,z],\ w^2,\ [t,w],\ [u,w],\ t^2,\ [u,t],\ u^2\,\rangle$.

This group has 12 conjugacy classes as shown in the table below.

R	e	x	y	w	t	u	x^2	xw	yz	yw	yt	x^2y
S	1	16	6	3	3	1	16	16	6	6	6	16
O	1	3	4	2	2	2	3	6	4	4	4	6

From above discussion, we see that exponent of G_{12} is 12. Also $G'_{12} \cong (C_4 \times C_2) \rtimes C_4$ with $G_{12}/G'_{12} \cong C_3$. Since $p > 3$, we have $\gcd(|G_{12}|, p) = 1$, and so $J(\mathbb{F}_q G_{12}) = 0$. We are now ready to give the WD and unit group of $\mathbb{F}_q G_{12}$ for $p > 3$.

Theorem 3.3. *The WD with unit group of $\mathbb{F}_q G_{12}$ for $q = p^k$, $p > 3$ is as follows:*

values of p and k	*Wedderburn decomposition*
k even or $p \equiv 1 \mod 12$ and k odd	$\mathbb{F}_q^3 \oplus M_2(\mathbb{F}_q)^3 \oplus M_3(\mathbb{F}_q)^5 \oplus M_6(\mathbb{F}_q)$
$p^k \equiv 5 \mod 12$ *and k odd*	$\mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^5$ $\oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^2})$
$p^k \equiv 7 \mod 12$ *and k odd*	$\mathbb{F}_q^3 \oplus M_2(\mathbb{F}_q)^3 \oplus M_3(\mathbb{F}_q)$ $\oplus M_6(\mathbb{F}_q) \oplus M_3(\mathbb{F}_{q^2})^2$
$p^k \equiv 11 \mod 12$ *and k odd*	$\mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)$ $\oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^2}) \oplus M_3(\mathbb{F}_{q^2})^2$

Proof. As $\mathbb{F}_q G_{12}$ is semisimple, we have $\mathbb{F}_q G_{12} \cong \oplus_{r=1}^{t} M_{n_r}(\mathbb{F}_r)$, $t \in \mathbb{Z}$, where for each r, $\mathbb{F}_r$ is a finite extension of $\mathbb{F}_q$, $n_r \geq 1$. As in Theorem 3.1, we can write

$$\mathbb{F}_q G_{12} \cong \mathbb{F}_q \oplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r). \tag{3.5}$$

For k even and any prime $p > 3$, $p^k \equiv 1 \mod 12$. This means $|S(\gamma_g)| = 1$ for each $g \in G_{12}$ as $I_{\mathbb{F}} = \{1\}$. Hence, (3.5) and Theorems 2.1 and 2.2 imply that $\mathbb{F}_q G_{12} \cong \mathbb{F}_q \oplus_{r=1}^{11} M_{n_r}(\mathbb{F}_r)$. This with $G_{12}/G'_{12} \cong C_3$ and Theorem 2.5 leads to (with suitable rearrangement of indexes)

$$\mathbb{F}_q G_{12} \cong \mathbb{F}_q^3 \oplus_{r=1}^{9} M_{n_r}(\mathbb{F}_r) \text{ with } 93 = \sum_{r=1}^{9} n_r^2,\ n_r \geq 2 \tag{3.6}$$

which gives four possible choices for n_r's namely $(2,2,2,2,2,2,2,4,7)$, $(2,2,2,2,2,4,4,4,5)$, $(2,2,2,2,3,3,3,5,5)$, and $(2,2,2,3,3,3,3,3,6)$. We consider the normal subgroup $H_1 = \langle wu, t\rangle \cong C_2 \times C_2$ with $G_{12}/H_1 \cong SL(2,3)$. From [17], we know that WD of $\mathbb{F}_q G_{12}/H_1$ contains $M_2(\mathbb{F}_q)$ as well as $M_3(\mathbb{F}_q)$. So, Theorem 2.6 implies that the choices $(2,2,2,2,2,2,2,4,\ 7)$ and $(2,2,2,2,2,4,4,4,5)$ are no longer in race. For uniqueness, we consider another normal subgroup $H_2 = \langle u\rangle$ with $K_2 = G_{12}/H_2 \cong (C_4 \times C_4) \rtimes C_3$. Using [1], we note that $\mathbb{F}_q K_2 \cong \mathbb{F}_q^3 \oplus M_3(\mathbb{F}_q)^5$. This with Theorem 2.6 imply that $(2,2,2,3,3,3,3,3,6)$ is the only possibility for n_r's. Therefore, we have

$$\mathbb{F}_q G_{12} \cong \mathbb{F}_q^3 \oplus M_2(\mathbb{F}_q)^3 \oplus M_3(\mathbb{F}_q)^5 \oplus M_6(\mathbb{F}_q). \tag{3.7}$$

Next, we assume that k is odd. We discuss this possibility in the following 4 cases:

Case 1. $p \equiv 1 \mod 12$. In this case, we have $|S(\gamma_g)| = 1$ for each $g \in G_{12}$ as $I_{\mathbb{F}} = \{1\}$. Hence, Wedderburn decomposition is given by (3.7).

Case 2. $p^k \equiv 5 \mod 12$. In this case, we have $S(\gamma_x) = \{\gamma_x, \gamma_{x^2}\}$, $S(\gamma_{xw}) = \{\gamma_{xw}, \gamma_{x^2y}\}$, $S(\gamma_g) = \{\gamma_g\}$ for the remaining representatives g of conjugacy classes. Using Theorems 2.1, 2.2 and (3.5), we get $\mathbb{F}_q G_{12} \cong \mathbb{F}_q \oplus_{r=1}^{7} M_{n_r}(\mathbb{F}_q) \oplus_{r=8}^{9} M_{n_r}(\mathbb{F}_{q^2})$. Applying Theorem 2.5 with $G_{12}/G'_{12} \cong C_3$ and $\mathbb{F}_q C_3 \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2}$ to obtain that

$$\mathbb{F}_q G_{12} \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus_{r=1}^{7} M_{n_r}(\mathbb{F}_q) \oplus M_{n_8}(\mathbb{F}_{q^2}) \text{ with } 93 = \sum_{r=1}^{7} n_r^2 + 2n_8^2,\ n_r \geq 2. \tag{3.8}$$

Further, we note using [1] that $\mathbb{F}_q K_2 \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus M_3(\mathbb{F}_q)^5$. Therefore, (3.8) and Theorem 2.6 imply that $\mathbb{F}_q G_{12} \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus M_3(\mathbb{F}_q)^5 \oplus_{r=1}^{2} M_{n_r}(\mathbb{F}_q) \oplus M_3(\mathbb{F}_{q^2})$ with $48 = \sum_{r=1}^{2} n_r^2 + 2n_3^2$, $n_r \geq 2$. This gives the only possibility $(2, 6, 2)$ for n_r's which means the required WD is

$$\mathbb{F}_q G_{12} \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus M_3(\mathbb{F}_q)^5 \oplus M_2(\mathbb{F}_q) \oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^2}).$$

Case 3. $p^k \equiv 7 \mod 12$. In this case, we have $S(\gamma_y) = \{\gamma_y, \gamma_{yz}\}$, $S(\gamma_{yw}) = \{\gamma_{yw}, \gamma_{yt}\}$, $S(\gamma_g) = \{\gamma_g\}$ for the remaining representatives g of conjugacy classes. Using Theorems 2.1, 2.2 and (3.5), we get $\mathbb{F}_q G_{12} \cong \mathbb{F}_q \oplus_{r=1}^{7} M_{n_r}(\mathbb{F}_q) \oplus_{r=8}^{9} M_{n_r}(\mathbb{F}_{q^2})$. Applying Theorem 2.5 with $G_{12}/G'_{12} \cong C_3$ and $\mathbb{F}_q C_3 \cong \mathbb{F}_q^3$ in above to obtain

$$\begin{aligned} &\mathbb{F}_q G_{12} \cong \mathbb{F}_q^3 \oplus_{r=1}^{5} M_{n_r}(\mathbb{F}_q) \oplus_{r=6}^{7} M_{n_r}(\mathbb{F}_{q^2}) \\ &\text{with } \ 93 = \sum_{r=1}^{5} n_r^2 + 2\sum_{r=6}^{7} n_r^2,\ n_r \geq 2. \end{aligned} \tag{3.9}$$

Further, we observe that $\mathbb{F}_q K_2 \cong \mathbb{F}_q^3 \oplus M_3(\mathbb{F}_q) \oplus M_{t_r}(\mathbb{F}_{q^2})^2$. Therefore, (3.9) and Theorem 2.6 imply that $\mathbb{F}_q G_{12} \cong \mathbb{F}_q^3 \oplus M_3(\mathbb{F}_q) \oplus_{r=1}^{4} M_{n_r}(\mathbb{F}_q) \oplus M_3(\mathbb{F}_{q^2})^2$ with $48 = \sum_{r=1}^{4} n_r^2$, $n_r \geq 2$. This gives the only possibility $(2, 2, 2, 6)$ for n_r's which means that the WD is

$$\mathbb{F}_q G_{12} \cong \mathbb{F}_q^3 \oplus M_3(\mathbb{F}_q) \oplus M_2(\mathbb{F}_q)^3 \oplus M_6(\mathbb{F}_q) \oplus M_3(\mathbb{F}_{q^2})^2.$$

Case 4. $p^k \equiv 11 \mod 12$. In this case, we can verify that $S(\gamma_y) = \{\gamma_y, \gamma_{yz}\}$, $S(\gamma_{yw}) = \{\gamma_{yw}, \gamma_{yt}\}$, $S(\gamma_x) = \{\gamma_x, \gamma_{x^2}\}$, $S(\gamma_{xw}) = \{\gamma_{xw}, \gamma_{x^2y}\}$, and $S(\gamma_g) = \{\gamma_g\}$ for the representatives e, w, t and u. Using Theorems 2.1, 2.2 and (3.5), we get $\mathbb{F}_q G_{12} \cong \mathbb{F}_q \oplus_{r=1}^{3} M_{n_r}(\mathbb{F}_q) \oplus_{r=4}^{7} M_{n_r}(\mathbb{F}_{q^2})$. Applying Theorem 2.5 with $\mathbb{F}_q C_3 \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2}$ in above to obtain

$$\begin{aligned} &\mathbb{F}_q G_{12} \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus_{r=1}^{3} M_{n_r}(\mathbb{F}_q) \oplus_{r=4}^{6} M_{n_r}(\mathbb{F}_{q^2}) \\ &\text{with } \ 93 = \sum_{r=1}^{3} n_r^2 + 2\sum_{r=4}^{6} n_r^2,\ n_r \geq 2. \end{aligned} \tag{3.10}$$

Further, we see that $\mathbb{F}_qK_2 \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus M_3(\mathbb{F}_q) \oplus M_{t_r}(\mathbb{F}_{q^2})^2$. Therefore, (3.10) and Theorem 2.6 imply that $\mathbb{F}_qG_{12} \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus M_3(\mathbb{F}_q) \oplus M_3(\mathbb{F}_{q^2})^2 \oplus_{r=1}^{2} M_{n_r}(\mathbb{F}_q) \oplus M_{n_3}(\mathbb{F}_{q^2})$ with $48 = \sum_{r=1}^{2} n_r^2 + 2n_3^2$, which means the only possibility for n_r's is $(2, 6, 2)$. Thus, the required WD is

$$\mathbb{F}_qG_{12} \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus M_3(\mathbb{F}_q) \oplus M_3(\mathbb{F}_{q^2})^2 \oplus M_2(\mathbb{F}_q) \oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^2}). \qquad \square$$

Next, we remark that for the groups G_i, where $13 \leq i \leq 26$, the WD of their group algebras can be computed by following the steps of Theorem 3.2 and Theorem 3.3 (see Tables 10–23). Hence, we are omitting their proofs from the paper.

Table 10. Wedderburn decomposition of $\mathbb{F}_qG_{13}$.

values of p and k	Wedderburn decomposition
k even or $p^k \equiv \pm1 \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^5 \oplus M_3(\mathbb{F}_q)^4 \oplus M_6(\mathbb{F}_q)$
$p^k \equiv \pm5 \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^3 \oplus M_3(\mathbb{F}_q)^4 \oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^2})$

Table 11. Wedderburn decomposition of $\mathbb{F}_qG_{14}$.

values of p and k	Wedderburn decomposition
k even or $p^k \in \{1, 5\} \mod 12$ and k odd	$\mathbb{F}_q^8 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^8$
$p^k \in \{7, 11\} \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^2 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_2(\mathbb{F}_{q^2}) \oplus M_3(\mathbb{F}_{q^2})$

Table 12. Wedderburn decomposition of $\mathbb{F}_qG_{15}$.

values of p and k	Wedderburn decomposition
k even or $p^k \in \{1, 5\} \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^5 \oplus M_3(\mathbb{F}_q)^4 \oplus M_6(\mathbb{F}_q)$
$p^k \in \{7, 11\} \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^3 \oplus M_3(\mathbb{F}_q)^4 \oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^2})$

Table 13. Wedderburn decomposition of $\mathbb{F}_qG_{16}$.

values of p and k	Wedderburn decomposition
k even or $p^k \in \{1, 7\} \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^5 \oplus M_3(\mathbb{F}_q)^4 \oplus M_6(\mathbb{F}_q)$
$p^k \in \{5, 11\} \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^3 \oplus M_3(\mathbb{F}_q)^4 \oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^2})$

Table 14. Wedderburn decomposition of $\mathbb{F}_q G_{17}$.

values of p and k	Wedderburn decomposition
k even or $p^k \in \{1,7\} \mod 12$ and k odd	$\mathbb{F}_q^3 \oplus M_2(\mathbb{F}_q)^3 \oplus M_3(\mathbb{F}_q)^5 \oplus M_6(\mathbb{F}_q)$
$p^k \in \{5,11\} \mod 12$ and k odd	$\mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^5$ $\oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^2})$

Table 15. Wedderburn decomposition of $\mathbb{F}_q G_{18}$.

values of p and k	Wedderburn decomposition
k even or $p^k \in \{1,7\} \mod 12$ and k odd	$\mathbb{F}_q^3 \oplus M_3(\mathbb{F}_q)^5 \oplus M_4(\mathbb{F}_q)^3$
$p^k \in \{5,11\} \mod 12$ and k odd	$\mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus M_3(\mathbb{F}_q)^5 \oplus M_4(\mathbb{F}_q)$ $\oplus M_4(\mathbb{F}_{q^2})$

Table 16. Wedderburn decomposition of $\mathbb{F}_q G_{19}$.

values of p and k	Wedderburn decomposition
k even or $p^k \in \{1,7\} \mod 12$ and k odd	$\mathbb{F}_q^{12} \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)^3$
$p^k \in \{5,11\} \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^4 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)$ $\oplus M_4(\mathbb{F}_{q^2})$

Table 17. Wedderburn decomposition of $\mathbb{F}_q G_{20}$.

values of p and k	Wedderburn decomposition
k even or $p^k \in \{1,7\} \mod 12$ and k odd	$\mathbb{F}_q^{12} \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)^3$
$p^k \in \{5,11\} \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^4 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)$ $\oplus M_4(\mathbb{F}_{q^2})$

Table 18. Wedderburn decomposition of $\mathbb{F}_q G_{21}$.

values of p and k	Wedderburn decomposition
k even or $p^k \in \{1,7\} \mod 12$ and k odd	$\mathbb{F}_q^8 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^8$
$p^k \in \{5,11\} \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^2 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^4$ $\oplus M_3(\mathbb{F}_{q^2})^2$

Table 19. Wedderburn decomposition of $\mathbb{F}_q G_{22}$.

values of p and k	Wedderburn decomposition
for any k and p	$\mathbb{F}_q^8 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^8$

Table 20. Wedderburn decomposition of $\mathbb{F}_q G_{23}$.

values of p and k	Wedderburn decomposition
for any k and p	$\mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^6 \oplus M_6(\mathbb{F}_q)$

Table 21. Wedderburn decomposition of $\mathbb{F}_q G_{24}$.

values of p and k	Wedderburn decomposition
k even or $p \equiv 1 \mod 12$ and k odd	$\mathbb{F}_q^{12} \oplus M_2(\mathbb{F}_q)^{12} \oplus M_3(\mathbb{F}_q)^4$
$p^k \equiv 5 \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^4 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^4$ $\oplus M_2(\mathbb{F}_{q^2})^4$
$p^k \equiv 7 \mod 12$ and k odd	$\mathbb{F}_q^6 \oplus \mathbb{F}_{q^2}^3 \oplus M_2(\mathbb{F}_q)^6 \oplus M_3(\mathbb{F}_q)^2$ $\oplus M_2(\mathbb{F}_{q^2})^3 \oplus M_3(\mathbb{F}_{q^2})$
$p^k \equiv 11 \mod 12$ and k odd	$\mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^5 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^2$ $\oplus M_2(\mathbb{F}_{q^2})^5 \oplus M_3(\mathbb{F}_{q^2})$

Table 22. Wedderburn decomposition of $\mathbb{F}_q G_{25}$.

values of p and k	Wedderburn decomposition
k even or $p^k \in \{1, 11\} \mod 12$ and k odd	$\mathbb{F}_q^{12} \oplus M_2(\mathbb{F}_q)^{12} \oplus M_3(\mathbb{F}_q)^4$
$p^k \in \{5, 7\} \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^4 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^4$ $\oplus M_2(\mathbb{F}_{q^2})^4$

Table 23. Wedderburn decomposition of $\mathbb{F}_q G_{26}$.

values of p and k	Wedderburn decomposition
k even or $p \equiv 1 \mod 12$ and k odd	$\mathbb{F}_q^{12} \oplus M_2(\mathbb{F}_q)^{12} \oplus M_3(\mathbb{F}_q)^4$
$p^k \equiv 5 \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^4 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^4$ $\oplus M_2(\mathbb{F}_{q^2})^4$
$p^k \equiv 7 \mod 12$ and k odd	$\mathbb{F}_q^{12} \oplus M_3(\mathbb{F}_q)^4 \oplus M_2(\mathbb{F}_{q^2})^6$
$p^k \equiv 11 \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^4 \oplus M_3(\mathbb{F}_q)^4 \oplus M_2(\mathbb{F}_{q^2})^6$

4. Conclusion

We have computed the WDs of semisimple group algebras of non-metabelian groups of order 96. Hence, this study completes the computation of WDs of semisimple group algebras of all groups up to order 120. In future, this paper motivates the study of unit groups of the group algebras of non-metabelian groups having order greater than 120.

References

[1] G. K. Bakshi, S. Gupta, I. B. S. Passi: *The Algebraic Structure of Finite Metabelian Group Algebras*, Communications in Algebra 43.1 (2015), pp. 2240–2257, doi: 10.1080/00927872.2014.888566.

[2] A. Bovdi, J. Kurdics: *Lie Properties of the Group Algebra and the Nilpotency Class of the Group of Units*, Journal of Algebra 212.1 (1999), pp. 28–64, doi: 10.1006/jabr.1998.7617.

[3] C. Dietzel, G. Mittal: *Summands of Finite Group Algebras*, Czechoslovak Mathematical Journal 71.4 (2021), pp. 1011–1014, doi: 10.21136/CMJ.2020.0171-20.

[4] R. A. Ferraz: *Simple Components of the Center of $FG/J(FG)$*, Communications in Algebra 36.9 (2008), pp. 3191–3199, doi: 10.1080/00927870802103503.

[5] S. Gupta, S. Maheshwary: *Finite Semisimple Group Algebra of a Normally Monomial Group*, International Journal of Algebra and Computation 29.1 (2019), pp. 159–177, doi: 10.1142/S0218196718500674.

[6] B. Hurley, T. Hurley: *Group ring cryptography*, International Journal of Pure and Applied Mathematics 69.1 (2011), pp. 67–86, url: https://www.ijpam.eu/contents/2011-69-1/8/index.html.

[7] P. Hurley, T. Hurley: *Codes from Zero-Divisors and Units in Group Rings*, International Journal of Information and Coding Theory 1.1 (2009), pp. 57–87, doi: 10.1504/IJICOT.2009.024047.

[8] G. D. James: *The Representation Theory of the Symmetric Groups*, Berlin, Heidelberg: Springer, 1978, doi: 10.1007/BFb0067708.

[9] M. Khan, R. K. Sharma, J. B. Srivastava: *The Unit Group of FS_4*, Acta Mathematica Hungarica 118.1-2 (2008), pp. 105–113, doi: 10.1007/s10474-007-6169-4.

[10] R. Lidl: *Introduction to Finite Fields and Their Applications*, United Kingdom: Cambridge University Press, 2000, doi: 10.1017/CBO9781139172769.

[11] N. Makhijani, R. K. Sharma, J. B. Srivastava: *A Note on the Structure of $F_{p^k}A_5/J(F_{p^k}A_5)$*, Acta Scientiarum Mathematicarum 82.2-3 (2016), pp. 29–43, doi: 10.14232/actasm-014-311-2.

[12] N. Makhijani, R. K. Sharma, J. B. Srivastava: *The Unit Group of $\boldsymbol{F}_q[D_{30}]$*, Serdica Mathematical Journal 41.2-3 (2015), pp. 185–198, url: http://www.math.bas.bg/serdica/n23_15.html.

[13] C. P. Milies, S. K. Sehgal: *An Introduction to Group Rings*, Netherlands: Springer Dordrecht, 2002, url: https://link.springer.com/book/9781402002380.

[14] G. Mittal, S. Kumar, S. Kumar: *A Quantum Secure ID-Based Cryptographic Encryption Based on Group Rings*, Sadhana 47.35 (2022), doi: 10.1007/s12046-022-01806-5.

[15] G. Mittal, R. K. Sharma: *Computation of Wedderburn Decomposition of Groups Algebras from their Subalgebra*, Bulletin of the Korean Mathematical Society 59.3 (2022), pp. 781–787, doi: 10.4134/BKMS.b210478.

[16] G. Mittal, R. K. Sharma: *On Unit Group of Finite Group Algebras of Non-Metabelian Groups of Order 108*, Journal of Algebra Combinatorics Discrete Structures and Applications 8.2 (2021), pp. 59–71, URL: https://jacodesmath.com/index.php/jacodesmath/article/view/158.

[17] G. Mittal, R. K. Sharma: *On Unit Group of Finite Group Algebras of Non-Metabelian Groups Upto Order 72*, Mathematica Bohemica 146.4 (2021), pp. 429–455, DOI: 10.21136/MB.2021.0116-19.

[18] G. Mittal, R. K. Sharma: *Unit Group of Semisimple Group Algebras of Some Non-Metabelian Groups of Order 120*, Asian-European Journal of Mathematics 15.3 (2022), p. 2250059, DOI: 10.1142/S1793557122500590.

[19] G. Mittal, R. K. Sharma: *Wedderburn Decomposition of a Semisimple Group Algebra FqG from a Subalgebra of Factor Group of G*, International Electronic Journal of Algebra 32 (2022), pp. 91–100, DOI: 10.24330/ieja.1077582.

[20] G. Pazderski: *The orders to which only belong metabelian groups*, Mathematische Nachrichten 95.1 (1980), pp. 7–16, DOI: 10.1002/mana.19800950102.

[21] S. Perlis, G. L. Walker: *Abelian Group Algebras of Finite Order*, Transactions of the American Mathematical Society 68.3 (1950), pp. 420–426, DOI: 10.2307/1990406.

[22] R. K. Sharma, G. Mittal: *Unit Group of Semisimple Group Algebra* $F_qSL(2,5)$, Mathematica Bohemica 147.1 (2022), pp. 1–10, DOI: 10.21136/MB.2021.0104-20.