

Differential geometry properties by using the perturbation methods

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Abstract. In this paper, we present a new method to evaluate the moving Frenet frame along the intersection curves of two parametric surfaces in the tangential intersection situations. To resolve such situation, we have combined a perturbation method with a classical method that works in transversal intersection situation. Unlike the existent methods, our method works even if the order contact of the point is more than one.

Keywords: intersection of surfaces, parametric curve, perturbation methods, differential geometry properties, tangential intersection.

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1. Introduction

Tracing the intersection curves of two parametric surfaces is a very important subject in computer-aided geometric design applications, manufacturing as well as in many other industrial fields [4, 5, 7, 8]. The intersection between two parametric surfaces can be either an empty set, several isolated points, a set of curves, a set of patches or a mixture of the aforementioned items. We focused on the case when the two surfaces meet at a collection of curves, and we shall assume that the two parametric surfaces are both so smooth enough that all the derivatives and partial derivatives are given in this paper existent and continuous. There are different ways to resolve this problem. Nevertheless, the most widely used one is the marching method because its algorithm is relatively simple and yet robust, efficient and

the required accuracy can be realized by choosing an appropriate step size. The marching methods consist generally of two basic steps:

1. Finding a starting point for each intersection curve.
2. Tracing the intersection curves from these points by using the differential geometry properties (tangent vector, normal, and binormal vectors).

In this manuscript, we will care about the computation of the tangent vector, normal vector and the binormal vectors along an intersection curve of two parametric surfaces. Many papers study this problem in the transversal intersection situations like [1, 12], but only few among them tackled the tangential intersection situations. An intersection point is considered a tangential intersection point if normal vectors of the surfaces are parallel at this point. Among the papers that suggest a solution for the tangential intersection problem is the one proposed by B. U. Düldül and M. Düldül in [2]. They combined the Rodrigues' rotation formula that used in [11] with a new proposed operator so as to determine the differential geometry properties. Their algorithm does not work if the order contact of the point is more than one. Unlike the classical marching method, another approach to trace the intersection curves without using the differential geometry properties was proposed by Grandine and Klein in [3]. This method consists of two steps. Firstly, we have to determine the topology of each intersection curve (starting and ending point). Secondly, we use the first step to trace the intersection curves (boundary value problem). However, it doesn't work in many transversal and tangential situations. This method was improved later by Oh et al. in [6]. They used the perturbation methods to determine the topology of the curves in the tangential intersection situations.

We will suggest in this paper a new method to find the moving Frenet frame at a tangential intersection point. Our method combines the perturbations methods used in [6, 9, 10] with the result proposed by X. Ye and T. Maekawa in [12]. Our solution works even if order contact of the point is at least two.

The paper is organized as follows: in the second section, we will present the definitions and the existent results we need in our work. The proposed method will be presented in section 3. The fourth section is the implementation of our method. The conclusion with future works contained in the fifth section.

2. Review of differential geometry

This section will describe the differential geometry properties along a parametric curve and a parametric surface. Then, we will briefly depict a classical method suggested to evaluate the moving Frenet frame along the intersection curves of two parametric surfaces in transversal intersection situations.

2.1. Differential geometry properties along parametric curves and parametric surfaces

This subsection will mention the differential geometry properties along a parametric curve and also for a parametric surface. For more details, we refer to [2, 12].

Let us consider $\mathbf{f}(u, v)$ and $\mathbf{g}(s, t)$, where $s, t, u, v \in \mathbb{R}$ two regular parametric surfaces in \mathbb{R}^3 . A surface is regular if its first partial derivatives are not parallel at each point on this surface. Let us consider also $\mathbf{c}(\tau)$, where $\tau \in \mathbb{R}$ an intersection curve between \mathbf{f} and \mathbf{g} with arc-length parametrization. From the elementary differential geometry, $(\mathbf{T}, \mathbf{n}, \mathbf{b})$ is the moving Frenet frame along the intersection curve \mathbf{c} , then, we have

$$\begin{aligned}\mathbf{T} &= \mathbf{c}'(\tau). \\ \mathbf{K} &= \mathbf{c}''(\tau) = \kappa \cdot \mathbf{n}.\end{aligned}\tag{2.1}$$

The binormal vector of \mathbf{c} defined as:

$$\mathbf{b} = \mathbf{T} \times \mathbf{n},\tag{2.2}$$

where τ is the curve parameter; \mathbf{T} is the unit tangent vector, \mathbf{n} is the unit normal vector and \mathbf{b} is the unit binormal vector along the curve \mathbf{c} ; κ is the curvature of the curve \mathbf{c} and \mathbf{K} is the curvature vector.

On the other hand, since $\mathbf{c}(\tau)$ lies on both \mathbf{f} and \mathbf{g} , we may write:

$$\mathbf{c}(\tau) = \mathbf{f}(u(\tau), v(\tau)) = \mathbf{g}(s(\tau), t(\tau)).$$

Then

$$\begin{aligned}\mathbf{c}'(\tau) &= u' \cdot \mathbf{f}_u + v' \cdot \mathbf{f}_v = s' \cdot \mathbf{g}_s + t' \cdot \mathbf{g}_t, \\ \mathbf{c}''(\tau) &= u'' \cdot \mathbf{f}_u + v'' \cdot \mathbf{f}_v + (v')^2 \cdot \mathbf{f}_{vv} + (u')^2 \cdot \mathbf{f}_{uu} + 2u' \cdot v' \cdot \mathbf{f}_{uv},\end{aligned}\tag{2.3}$$

where

$$\begin{aligned}\mathbf{f}_u &= \frac{\partial \mathbf{f}(u, v)}{\partial u}, \quad \mathbf{f}_v = \frac{\partial \mathbf{f}(u, v)}{\partial v}, \quad \mathbf{g}_s = \frac{\partial \mathbf{g}(s, t)}{\partial s}, \quad \mathbf{g}_t = \frac{\partial \mathbf{g}(s, t)}{\partial t}, \\ \mathbf{f}_{uu} &= \frac{\partial^2 \mathbf{f}(u, v)}{(\partial u)^2}, \quad \mathbf{f}_{vv} = \frac{\partial^2 \mathbf{f}(u, v)}{(\partial v)^2}, \quad \mathbf{f}_{uv} = \frac{\partial^2 \mathbf{f}(u, v)}{\partial u \partial v}.\end{aligned}$$

Since \mathbf{f} and \mathbf{g} are both regular, we can define their normal vectors at a point P as:

$$\mathbf{N}_f = \frac{\mathbf{f}_u \times \mathbf{f}_v}{\|\mathbf{f}_u \times \mathbf{f}_v\|} \quad \text{and} \quad \mathbf{N}_g = \frac{\mathbf{g}_s \times \mathbf{g}_t}{\|\mathbf{g}_s \times \mathbf{g}_t\|}.$$

2.2. Differential geometry properties along an intersection curve of two parametric surfaces

In this subsection, we are going to elaborate the method suggested by X. Ye and T. Maekawa in [12] to evaluate $(\mathbf{T}, \mathbf{n}, \mathbf{b})$ along an intersection curve \mathbf{c} between

\mathbf{f} and \mathbf{g} in the transversal intersection situations. Two parametric surfaces meet transversally at a point P if $\mathbf{N}_{\mathbf{f}}$ and $\mathbf{N}_{\mathbf{g}}$ are not parallel at P .

Since $\mathbf{N}_{\mathbf{f}}$ and $\mathbf{N}_{\mathbf{g}}$ are linearly independent, we may find the tangent vector \mathbf{T} at P by using the formula:

$$\mathbf{T} = \frac{\mathbf{N}_{\mathbf{f}} \times \mathbf{N}_{\mathbf{g}}}{\|\mathbf{N}_{\mathbf{f}} \times \mathbf{N}_{\mathbf{g}}\|}. \quad (2.4)$$

In order to evaluate \mathbf{n} the normal vector at P , we need to compute u' , v' , s' and t' at P , by (2.3), we get:

$$u' = \frac{g_1 \cdot (\mathbf{T} \cdot \mathbf{f}_u) - f_1 \cdot (\mathbf{T} \cdot \mathbf{f}_v)}{e_1 \cdot g_1 - (f_1)^2}, \quad v' = \frac{e_1 \cdot (\mathbf{T} \cdot \mathbf{f}_v) - f_1 \cdot (\mathbf{T} \cdot \mathbf{f}_u)}{e_1 \cdot g_1 - (f_1)^2}, \quad (2.5)$$

$$s' = \frac{g_2 \cdot (\mathbf{T} \cdot \mathbf{g}_s) - f_2 \cdot (\mathbf{T} \cdot \mathbf{g}_t)}{e_2 \cdot g_2 - (f_2)^2}, \quad t' = \frac{e_2 \cdot (\mathbf{T} \cdot \mathbf{g}_t) - f_2 \cdot (\mathbf{T} \cdot \mathbf{g}_s)}{e_2 \cdot g_2 - (f_2)^2}, \quad (2.6)$$

where

$$e_1 = \mathbf{f}_u \cdot \mathbf{f}_u, f_1 = \mathbf{f}_v \cdot \mathbf{f}_u, g_1 = \mathbf{f}_v \cdot \mathbf{f}_v, e_2 = \mathbf{g}_s \cdot \mathbf{g}_s, f_2 = \mathbf{g}_t \cdot \mathbf{g}_s \quad \text{and} \quad g_2 = \mathbf{g}_t \cdot \mathbf{g}_t.$$

The curvature \mathbf{K} lies in the normal plane of the intersection point. Hence, there are two reals a and b such that:

$$\mathbf{K} = a \cdot \mathbf{N}_{\mathbf{f}} + b \cdot \mathbf{N}_{\mathbf{g}}.$$

Then, if we denote by $\kappa_n^{\mathbf{f}} = \mathbf{K} \cdot \mathbf{N}_{\mathbf{f}}$ and $\kappa_n^{\mathbf{g}} = \mathbf{K} \cdot \mathbf{N}_{\mathbf{g}}$, we obtain:

$$\kappa_n^{\mathbf{f}} = a + b \cdot \cos(\theta). \quad (2.7)$$

$$\kappa_n^{\mathbf{g}} = a \cdot \cos(\theta) + b, \quad (2.8)$$

where $\cos(\theta) = \mathbf{N}_{\mathbf{f}} \cdot \mathbf{N}_{\mathbf{g}}$.

Now, we can evaluate a and b by (2.7) and (2.8). Hence, we may obtain:

$$\mathbf{K} = \frac{1}{\sin^2(\theta)} [(\kappa_n^{\mathbf{f}} - \kappa_n^{\mathbf{g}} \cos(\theta))\mathbf{N}_{\mathbf{f}} + (\kappa_n^{\mathbf{g}} - \kappa_n^{\mathbf{f}} \cos(\theta))\mathbf{N}_{\mathbf{g}}].$$

We note that $\kappa_n^{\mathbf{f}}$ and $\kappa_n^{\mathbf{g}}$ can be evaluated by (2.1), (2.5) and (2.6).

If $\kappa \neq 0$, we can evaluate now \mathbf{n} by the formula:

$$\mathbf{n} = \frac{\mathbf{K}}{\|\mathbf{K}\|}. \quad (2.9)$$

To evaluate the binormal vector \mathbf{b} at P , we use the formula (2.2).

2.3. Tangential intersection situations

In this subsection, we are going to display the weakness of the previous expressions in tangential intersection situations. See [2].

Let us consider P a tangential intersection point between \mathbf{f} and \mathbf{g} , then the two vectors $\mathbf{N}_{\mathbf{f}}(P)$ and $\mathbf{N}_{\mathbf{g}}(P)$ are parallel. Hence, we can not use (2.4), (2.2) and (2.9) to compute the moving Frenet frame. We have to suggest a new method to evaluate \mathbf{T} , \mathbf{n} and \mathbf{b} in the tangential intersection points.

3. The presentation of our method

In this section, we are going to present a more general method to find \mathbf{T} , \mathbf{n} and \mathbf{b} in the tangential intersection situations. We will use an analytic perturbation method that transforms tangential intersection situations into the transversal intersection situation.

Let $P = \mathbf{f}(u_0, v_0) = \mathbf{g}(s_0, t_0)$ be a tangential intersection point lying on an intersection curve \mathbf{l} between \mathbf{f} and \mathbf{g} . Then, we have two possibilities:

1. P is a singular point (\mathbf{f} and \mathbf{g} meet transversally in a small neighborhood of P).
2. P lies on a tangential intersection curve $\mathbf{l}' \subset \mathbf{l}$ (\mathbf{f} and \mathbf{g} meet tangentially along \mathbf{l}').

We aim at suggesting an idea that allows us to evaluate the moving Frenet frame $(\mathbf{T}, \mathbf{n}, \mathbf{b})$ using the ideas proposed to resolve the transversal intersection situation.

Firstly, we are going to suggest a method to evaluate $(\mathbf{T}, \mathbf{n}, \mathbf{b})$ when P is a singular point.

As the transversality propriety is both stable and generic on an intersection, we can perturb analytically the point P by considering a new point $P(\varsigma) \in \mathbf{l}$ defined as:

$$P(\varsigma) = \mathbf{f}(u_0 + \varsigma, v_0(\varsigma)) = \mathbf{g}(s_0(\varsigma), t_0(\varsigma)), \quad (3.1)$$

where $\varsigma \neq 0$ is appropriate and small so that the two following conditions are satisfied:

1. For every $\varsigma' \in [0, |\varsigma|]$, the point $P(\varsigma')$ is a transversal intersection point.
2. $\lim_{\varsigma' \rightarrow 0} P(\varsigma') = P$.

The former condition allows us to compute the tangent vector at $P(\varsigma')$. The latter condition allows us to compute the tangent vector at P as the limit of the tangent at $P(\varsigma')$ when $\varsigma' \rightarrow 0$. For more information about the choice of ς see [6].

The point $P(\varsigma)$ exists because P is a singular point.

As $P(\varsigma')$ is a transversal intersection point, we can evaluate the tangent vector $\mathbf{T}(\varsigma') = \mathbf{T}(P(\varsigma'))$ in term of ς' by the formula:

$$\mathbf{T}(\varsigma') = \frac{\mathbf{N}_{\mathbf{f}}(P(\varsigma')) \times \mathbf{N}_{\mathbf{g}}(P(\varsigma'))}{\|\mathbf{N}_{\mathbf{f}}(P(\varsigma')) \times \mathbf{N}_{\mathbf{g}}(P(\varsigma'))\|}.$$

As the perturbation is analytic with respect ς' , also the vector $\mathbf{T}(\varsigma')$ vary continuously in a small neighborhood of $\varsigma = 0$, then, \mathbf{T} is given by:

$$\mathbf{T} = \lim_{\varsigma' \rightarrow 0} \mathbf{T}(\varsigma'). \quad (3.2)$$

Now, we can evaluate $u'(\zeta'), v'(\zeta'), s'(\zeta')$ and $t'(\zeta')$ at $P(\zeta')$ in term of ζ' by (2.5) and (2.6). Hence we may evaluate $\mathbf{K}(\zeta') = \mathbf{K}(P(\zeta'))$ in term of ζ' . Therefore,

$$\mathbf{n} = \lim_{\zeta' \rightarrow 0} \frac{\mathbf{K}(\zeta')}{\|\mathbf{K}(\zeta')\|}, \quad (3.3)$$

because the perturbation is analytic with respect ζ' and $\mathbf{K}(\zeta')$ vary continuously in a small neighborhood of $\zeta = 0$.

Finally, we can evaluate \mathbf{b} directly by the formula (2.2).

Remark 3.1. If P is a branch point, we have to find all the points P_i which verify the conditions above.

Secondly, we are going to present a method to evaluate \mathbf{T} , \mathbf{n} and \mathbf{b} when P belongs to a tangential intersection curve.

Consider $P = \mathbf{f}(u_0, v_0) = \mathbf{g}(s_0, t_0)$ a tangential intersection point lying on a tangential intersection curve \mathbf{I}' , then, P is not a singular point. Therefore, we can not use the same idea proposed to resolve the singular point situation, because in a small neighborhood of P , all the points are tangential. Since the transversality is both stable and generic on an intersection, we can perturb \mathbf{f} analytically by a small and appropriate $\zeta \neq 0$. Then, the curve \mathbf{I}' will be replaced by two new curves $\mathbf{l}_1(\zeta)$ and $\mathbf{l}_2(\zeta)$. In order to evaluate the moving frame at P , we have to find thz points $P(\zeta) = P(u_0, v(\zeta), s(\zeta), t(\zeta))$ such that:

$$\mathbf{f}(u_0, v(\zeta), \zeta) = \mathbf{g}(s(\zeta), t(\zeta)). \quad (3.4)$$

Suppose that we get two points $P_1(\zeta) \in \mathbf{l}_1(\zeta)$ and $P_2(\zeta) \in \mathbf{l}_2(\zeta)$ which verify:

1. $P_1(\zeta')$ and $P_2(\zeta')$ are transversal intersection points for every $\zeta' \in [0, |\zeta|]$.
2. $\lim_{\zeta \rightarrow 0} P_1(\zeta) = \lim_{\zeta \rightarrow 0} P_2(\zeta) = P$.

We will take one of them ($P_1(\zeta)$ for example) and evaluate \mathbf{T} , \mathbf{n} and \mathbf{b} by (3.2), (3.3) and (2.2) like the former situation.

Remark 3.2. The perturbation that we used must conserve all the intersection properties in a small neighborhood of P .

Remark 3.3. If (3.4) has no solution to verify the aforementioned conditions, we have to replace the equation $u = u_0$ by $v = v_0$.

Now, we will give some illustrative examples.

4. Examples

In this section, we will implement our method on some examples.

4.1. Example 1

In the first example, we will evaluate the moving Frenet frame at a singular intersection point.

Consider the intersection between the two parametric surfaces $\mathbf{f}(u, v)$ and $\mathbf{g}(s, t)$, where $u, v, s, t \in \mathbb{R}$ that are defined as:

$$\begin{aligned}\mathbf{f}(u, v) &= [\cos(u) \cos(v), \sin(u) \cos(v), \sin(v)]. \\ \mathbf{g}(s, t) &= [0.5 \cos(s) + 0.5, 0.5 \sin(s), t].\end{aligned}$$

(See the Figure 1.)

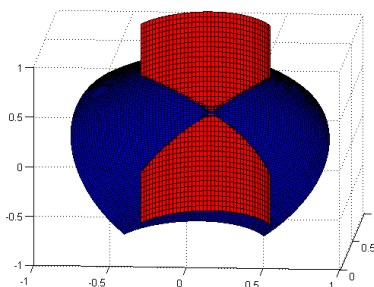


Figure 1. Branch point.

We want to determine the tangent \mathbf{T} , the normal \mathbf{n} and the binormal \mathbf{b} at the point $P = \mathbf{f}(0, 0) = \mathbf{g}(0, 0)$. P is tangential intersection point because

$$\mathbf{N}_f(P) = \mathbf{N}_g(P) = (1, 0, 0).$$

If we select a small and appropriate ς and by (3.1), we get two points $P_1(\varsigma)$ and $P_2(\varsigma)$ where:

$$\begin{aligned}P_1(\varsigma) &= \mathbf{f} \left(2 \arctan \left(\frac{\sqrt{1 - \sqrt{\cos(\varsigma)}}}{\cos(\varsigma) + 1} \right), \varsigma \right). \\ P_2(\varsigma) &= \mathbf{f} \left(-2 \arctan \left(\frac{\sqrt{1 - \sqrt{\cos(\varsigma)}}}{\cos(\varsigma) + 1} \right), \varsigma \right).\end{aligned}$$

For every ς' ($0 < |\varsigma'| \leq |\varsigma|$), the two surfaces \mathbf{f} and \mathbf{g} meet transversally at $P_1(\varsigma')$ and $P_1(\varsigma')$. We have $\lim_{\varsigma' \rightarrow 0} P_1(\varsigma') = \lim_{\varsigma' \rightarrow 0} P_2(\varsigma') = P$. Thus, by (3.2) we get:

$$\mathbf{T}_1 = (0, 0.7071, 0, 7071) \quad \text{and} \quad \mathbf{T}_2 = (0, -0.7071, 0, 7071).$$

Hence,

$$\mathbf{n} = (-1, 0, 0).$$

Thus, by (2.2) we obtain:

$$\mathbf{b}_1 = (0, -0.7071, 0, 7071) \text{ and } \mathbf{b}_2 = (0, 0.7071, 0, 7071).$$

Remark 4.1. We note that we have two tangent vectors at P , then, this point is a branch point.

4.2. Example 2

In the second example, we will implement our method on a second order contact point (cusp point).

Consider the intersection between the two parametric surfaces $\mathbf{f}(u, v)$ and $\mathbf{g}(s, t)$, where $u, v, s, t \in \mathbb{R}$ that are defined as:

$$\begin{aligned} \mathbf{f}(u, v) &= [u, v, u^3 - v^2]. \\ \mathbf{g}(s, t) &= [s, t, 0]. \end{aligned}$$

(See the Figure 2.)

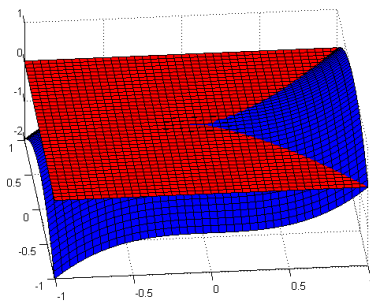


Figure 2. Cusp point.

The point $P = \mathbf{f}(0, 0) = \mathbf{g}(0, 0)$ is a cusp point, then, it is a second order contact point. Since P is a singular point, we use the point

$$P(\zeta) = \mathbf{f}\left(\sqrt[3]{\zeta^2}, \zeta\right) = \mathbf{g}\left(\sqrt[3]{\zeta^2}, \zeta\right),$$

where ζ is small and appropriate.

For every ζ' verifies $0 < |\zeta'| \leq |\zeta|$ the two surfaces \mathbf{f} and \mathbf{g} meet transversally at $P(\zeta')$ and $\lim_{\zeta' \rightarrow 0} P(\zeta') = P$.

By (3.2), we get:

$$\mathbf{T} = (1, 0, 0).$$

Hence, by (3.3) we obtain:

$$\mathbf{n} = (0, 1, 0).$$

Thus, by (2.2) we get:

$$\mathbf{b} = (0, 0, 1).$$

Now, we are going to implement our method at a point lying on a tangential intersection curve.

4.3. Example 3

Consider the intersection between $\mathbf{f}(u, v)$ and $\mathbf{g}(s, t)$, where $u, v, s, t \in \mathbb{R}$ that are defined as:

$$\mathbf{f}(u, v) = \left[8u - 1, 15v - \frac{15}{2}, 0 \right].$$

$$\mathbf{g}(u, v) = \left[\cos(2\pi s)(2 + \cos(2\pi t)) - \frac{1}{5}, \sin(2\pi s)(2 + \cos(2\pi t)), 1 + \sin(2\pi t) \right].$$

(See the Figure 3.)

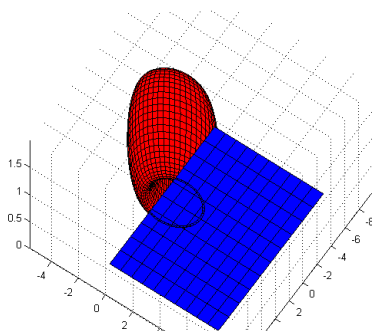


Figure 3. Tangential intersection curve situation.

The surfaces \mathbf{f} and \mathbf{g} meet at a tangential intersection curve \mathbf{l} which contain the point $P = \mathbf{f}(0, 0.3778) = \mathbf{g}(0.6845, 0.75)$.

To obtain the transversal intersection, we have to perturb \mathbf{g} analytically as

$$\mathbf{g}(\varsigma) = \left[\cos(2\pi s) \left(2 + \cos(2\pi t) - \frac{1}{5} \right), \sin(2\pi s)(2 + \cos(2\pi t)), 1 + \sin(2\pi t) - \varsigma \right],$$

where ς is small and appropriate. (See the Figure 4.)

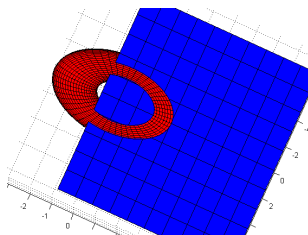


Figure 4. After the perturbation.

By (3.4), we find the point $P(\varsigma) = (u(\varsigma), v(\varsigma), s(\varsigma), t(\varsigma))$ where:

$$u(\varsigma) = 0, \quad v(\varsigma) = \frac{1}{2} - \frac{1}{15} \left[\frac{1}{2} - \sqrt{1 - \frac{16}{25\sqrt{-\varsigma(\varsigma-2)}} + 2} \right]^2 \sqrt{1 - (\varsigma-2) + 2},$$

$$s(\zeta) = \frac{1}{2} + \frac{1}{2\pi} \left[\arcsin \left(\frac{4}{5\sqrt{-\zeta(\zeta-2)}} + 10 \right) \right], \quad t(\zeta) = 1 + \frac{\arcsin(\zeta-1)}{2\pi}.$$

For every ζ' ($0 < |\zeta'| \leq |\zeta|$), the point $P(\zeta')$ is a transversal intersection point and $\lim_{\zeta' \rightarrow 0} P(\zeta') = P$. Then, we can evaluate \mathbf{T} by (3.2), such that:

$$\mathbf{T} = (-0.9169, 0.4, 0).$$

By (3.3) we get:

$$\mathbf{n} = (0.4, 0.9169, 0).$$

Thus, by (2.2) we obtain:

$$\mathbf{b} = (0, 0, 1).$$

5. Conclusion

We have developed a method to evaluate the tangent vector, normal vector and the binormal vector of a tangential intersection point between two parametric surfaces. We use the perturbation methods that allow us to use the expressions suggested by T. Maekawa and X. Ye to resolve the transversal intersection situations in the tangential intersection situations. If the curvature \mathbf{K} vanishes, we can not evaluate the normal vector and the binormal vector by using our method. This case will be a future work.

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