Annales Mathematicae et Informaticae
55 (2022) pp. 172–183
DOI: https://doi.org/10.33039/ami.2022.05.003
URL: https://ami.uni-eszterhazy.hu

On distance signless Laplacian spectral radius of power graphs of cyclic and dihedral groups*

Rezwan Ul Shaban[a], **Bilal A. Rather**[a], **S. Pirzada**[a], **A. Somasundaram**[b]

[a]Department of Mathematics, University of Kashmir, Srinagar, India
rezwanbhat21@gmail.com
bilalahmadrr@gmail.com
pirzadasd@kashmiruniversity.ac.in

[b]Department of General Sciences, Birla Institute of Technology and Science, Pilani, Dubai Campus
asomasundaram@dubai.bits-pilani.ac.in

Abstract. For a finite group $\mathcal{G}$, the power graph $\mathcal{P}(\mathcal{G})$ is a connected simple graph, whose vertex set is the set of elements of $\mathcal{G}$ and two vertices are connected by an edge if and only if one is the power of the other. In this article, we obtain sharp bounds for the distance signless Laplacian spectral radius of the power graphs of cyclic groups, dihedral and dicyclic groups. Furthermore, we characterize the extremal power graphs attaining such bounds and give some open problems.

Keywords: Distance signless Laplacian matrix, spectral radius, cyclic groups, dihedral group, power graphs

AMS Subject Classification: 05C50, 05C12, 15A18

1. Introduction

We follow the text [23] for graph theory terminology and basic definitions. A graph $G = (V(G), E(G))$ (simply written as G) consists of a vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the set of unordered pairs of elements of $V(G)$ is the edge set

*This research was supported by the SERB-DST, New Delhi under the research project number MTR/2017/000084.

Submitted: December 11, 2021
Accepted: May 11, 2022
Published online: May 18, 2022

$E(G)$. The *order* n of G is the number of elements in the set $V(G)$ and the *size* m is the number of elements in the set $E(G)$. The *neighbourhood* of $v \in V(G)$, denoted by $N(v)$, is the set of vertices incident on v. The *degree* of v, denoted by d_v, is the number of elements in $N(v)$. A graph G is said to be r regular if the degree of each vertex is r. We assume all our graphs are simple, connected and undirected. An alternating sequence of vertices and edges, beginning and ending with vertices such that no edge is traversed or covered more than once. The walk is said to be open if the initial and terminal vertices are distinct, otherwise closed. An open walk in which no vertex (and therefore no edge) is repeated is called a path and is denoted by P_n. A graph is said to be complete if it contains all possible edges and a complete graph with n vertices is denoted K_n. A graph $G(V, E)$ is said to be bipartite (or 2-partite) if its vertex set can be partitioned into two different sets V_1 and V_2 with $V = V_1 \cup V_2$ such that $uv \in E$ if and only if $u \in V_1$ and $v \in V_2$. A bipartite graph is said to be complete if $uv \in E$ for all $u \in V_1$ and $v \in V_2$. The complete bipartite graph $K_{1,n-1}$ is called a star.

The adjacency matrix of G, denoted by $A(G) = (a_{ij})$ is the matrix of order $n \times n$, defined as

$$A(G) = (a_{ij})_n = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are adjacenct,} \\ 0 & \text{otherwise.} \end{cases}$$

We denote the determinant of a matrix $M \in \mathbb{M}_n(\mathbb{C})$ by $\det(M)$. The characteristic polynomial of the matrix $A(G)$ is $\det(A(G) - xI)$, where I is the identity matrix. Since $A(G)$ is a real symmetric matrix, so the zeros of the polynomial $\det(A(G) - xI)$ are all real and can be ordered. The set of all the eigenvalues including multiplicity is known as the spectrum of $A(G)$ (or simply spectrum of G). The largest eigenvalue of $A(G)$ is called the spectral radius of G. More about the adjacency matrix can be seen in [13].

In a graph G, the *distance* between the two vertices $u, v \in V(G)$, denoted by $d(u, v)$, is defined as the length of the smallest path between them. The *distance matrix* indexed by the vertices of a connected graph G, denoted by $\mathcal{D}(G)$, is defined as

$$\mathcal{D}(G) = (d_{uv})_n = \begin{cases} 0 & \text{if } u = v, \\ d(u, v) & \text{otherwise.} \end{cases}$$

A complete survey of the matrix $\mathcal{D}(G)$ is given in [8]. The *transmission* of the vertex v (or transmission degree), denoted by $\mathrm{Tr}(v)$ (or Tr_v), is defined to be the sum of the distances from v to all other vertices in G, that is, $\mathrm{Tr}(v) = \sum_{u \in V(\mathcal{G})} d(u, v)$. We observe that the transmission of v_i is same as the i^{th} row sum of the matrix $\mathcal{D}(G)$.

Let $\mathrm{Tr}(G) = \mathrm{diag}(\mathrm{Tr}_1, \mathrm{Tr}_2, \dots, \mathrm{Tr}_n)$ be the diagonal matrix of vertex transmissions of G. The authors in [10] introduced the distance Laplacian

$$\mathcal{L}(G) = \mathrm{Tr}(G) - \mathcal{D}(G)$$

and the distance signless Laplacian

$$\mathcal{Q}(G) = \mathrm{Tr}(G) + \mathcal{D}(G)$$

for the distance matrix of a connected graph G. These matrices are real symmetric and positive semi-definite (definite), so the spectrum is real and non negative. In this article, we focus on the matrix $\mathcal{Q}(G)$, and we denote its eigenvalues by ρ_i's. We order them as $\rho_n \leq \rho_{n-1} \leq \cdots \leq \rho_1$, where ρ_1 is known as the distance signless Laplacian spectral radius of G. Since $\mathcal{Q}(G)$ is irreducible, so by *Perron–Frobenius theorem*, ρ_1 is a simple eigenvalue and the entries of its corresponding eigenvector are positive. Further information about the matrix $\mathcal{Q}(G)$ can be seen in [2–7, 9–11, 24–27].

Kelarev and Quinn [19] defined the directed power graph of a semigroup S as a directed graph with vertex set S in which two distinct vertices $x, y \in S$ are joined by an arc from x to y if and only if $x \neq y$ and $y^i = x$, for some positive integer i. Chakrabarty et al. [15] defined the undirected power graph $\mathcal{P}(G)$ of a group G as an undirected graph with vertex set as G and two vertices $x, y \in G$ are adjacent if and only if $x^i = y$ or $y^j = x$, for some $2 \leq i, j \leq n$. Such graphs have valuable applications and are related to the automata theory [20], besides being useful in characterizing the finite groups. More on power graphs can be seen in [1, 14, 15]. Laplacian spectrum of power graphs of finite cyclic and dihedral groups have been investigated in [16], where it is shown that the Laplacian spectral radius of the power graph of any finite group coincides with the order of group $\mathcal{G}$. Panda [22] studied the Laplacian spectral properties including vertex connectivity, Laplacian integrability and others. Spectral properties of the adjacency matrix of $\mathcal{P}(\mathcal{G})$ were investigated in [21]. Other spectral results of the power graphs can be seen in [12, 17].

The identity of the group G is denoted by e. The proper power graph of $\mathcal{P}(G)$, denoted by $\mathcal{P}(G^*) = \mathcal{P}(G \setminus \{e\})$, is obtained by removing the vertex e. Let $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{n-1}\}$ be the cyclic group of integers modulo n. Then by U_n, we denote the set

$$\{\overline{a} \in \mathbb{Z}_n \mid 1 \leq \overline{a} < n,\ \gcd(\overline{a}, n) = 1\}$$

and $U_n^* = U_n \cup \{\overline{0}\}$. $\mathbb{M}_n(\mathbb{F})$ denotes the set of $n \times n$ matrices with entries from the field $\mathbb{F}$. For other undefined notations and terminology, the readers are referred to [13, 18, 23].

The rest of the paper is organized as follows. In Section 2, we give the sharp bounds for the distance signless Laplacian spectral radius of $\mathcal{P}(\mathbb{Z}_n)$ and characterize the power graphs attaining such bounds. In Section 3, we find the distance signless Laplacian spectrum of the power graphs of the dihedral and the dicyclic groups for some special cases. We also obtain the bounds for the distance signless Laplacian spectral radius for these graphs.

2. Distance Laplacian spectral radius of the power graphs of finite cyclic group $\mathbb{Z}_n$

The first result gives the bounds for the largest distance signless Laplacian eigenvalue of the power graph of the finite cyclic group $\mathbb{Z}_n$.

Theorem 2.1. *Let $\mathcal{P}(\mathbb{Z}_n)$ be the power graph of order $n \geq 3$. Then the distance signless Laplacian spectral radius ρ_1 of $\mathcal{P}(\mathbb{Z}_n)$ satisfies the following*

$$\frac{n-2+r_{\min}+\sqrt{D}}{2} \leq \rho_1 \leq \frac{n-2+r_{\max}+\sqrt{D'}}{2},$$

where $D = r_{\min}^2 - (2n-\phi(n))r_{\min} + n^2 + 8n\phi(n) + 4n - 8\phi(n) - 4$, $D' = r_{\max}^2 - (2n-\phi(n))r_{\max} + n^2 + 8n\phi(n) + 4n - 8\phi(n) - 4$, $r_{\min}$ and $r_{\max}$ are the minimum and maximum row sums of $\mathcal{A}$, which is the block matrix of (2.1). Equality occurs if and only if n is a prime power. (Note that D and D' are positive, since they are the roots of the spectral eigenequation of a real symmetric matrix).

Proof. We list the vertices of $\mathcal{P}(\mathbb{Z}_n)$ first by those vertices which are adjacent to every vertex and then by others. Under this labelling, the distance signless Laplacian matrix of $\mathcal{P}(\mathbb{Z}_n)$ can be partitioned as

$$\mathcal{Q}\big(\mathcal{P}(\mathbb{Z}_n)\big)_{n\times n} = \begin{pmatrix} \big((n-2)I+J\big)_{\phi(n)+1} & J_{(\phi(n)+1)\times(n-\phi(n)-1)} \\ J_{(n-\phi(n)-1)\times(\phi(n)+1)} & \mathcal{A}_{n-\phi(n)-1} \end{pmatrix}, \tag{2.1}$$

where I is the identity matrix and J is the matrix of all ones. Clearly, the constant row sum of $\big((n-2)I+J\big)_{\phi(n)+1}$, $J_{(\phi(n)+1)\times(n-\phi(n)-1)}$ and $J_{(n-\phi(n)-1)\times(\phi(n)+1)}$ are $n+\phi(n)+1$, $n-\phi(n)-1$ and $\phi(n)+1$, respectively. Let $r_{\min}$ and $r_{\max}$ be the minimum and the maximum row sums of the matrix $\mathcal{A}_{n-\phi(n)-1}$. Then, we know that they are bounded below by the constant row sum of $J_{(n-\phi(n)-1)\times(\phi(n)+1)}$ and we take $r_{\min}-\phi(n)-1$ and $r_{\max}-\phi(n)-1$ as the minimum and the maximum row sums of $\mathcal{Q}\big(\mathcal{P}(\mathbb{Z}_n)\big)$. As $\mathcal{Q}\big(\mathcal{P}(\mathbb{Z}_n)\big)_{n-\phi(n)-1}$ is an irreducible matrix, so by Perron–Frobenius theorem, the signless Laplacian spectral radius is simple and its corresponding eigenvector, say X, has positive entries. Let $X = (x_1, x_2, \ldots, x_n)^\top$ and assume that $x_i = \min_{1\leq k\leq \phi(n)+1} x_k$ and $x_j = \min_{\phi(n)+1<k\leq n} x_k$. Therefore, taking the i^{th} eigenvalue equation of $\mathcal{Q}\big(\mathcal{P}(\mathbb{Z}_n)\big)X = \rho_1 X$ and using the fact that

$$q_{ik} = \begin{cases} n-1 & \text{if } i = k, \\ 1 & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} \rho_1 x_i &= q_{i1}x_1 + q_{i2}x_2 + \cdots + q_{i(\phi(n)+1)}x_{(\phi(n)+1)} \\ &\quad + q_{i(\phi(n)+2)}x_{(\phi(n)+2)} + \cdots + q_{in}x_n \\ &\geq \phi(n)x_i + (n-1)x_i + (n-\phi(n)-1)x_j, \end{aligned}$$

which is equivalent to

$$(\rho_1 - \phi(n) - n + 1)x_i \geq (n - \phi(n) - 1)x_j. \tag{2.2}$$

Also, taking the j^{th} eigenvalue equation, we have

$$\begin{aligned}\rho_1 x_j &= q_{j1}x_1 + q_{j2}x_2 + \cdots + q_{j(\phi(n)+1)}x_{(\phi(n)+1)} \\ &\quad + q_{j(\phi(n)+2)}x_{(\phi(n)+2)} + \cdots + q_{jn}x_n \\ &\geq (\phi(n)+1)x_i + (r_{\min} - \phi(n) - 1)x_j,\end{aligned}$$

which implies that

$$(\rho_1 - r_{\min} + \phi(n) + 1)x_j \geq (\phi(n)+1)x_i. \tag{2.3}$$

Thus, from Equations (2.2) and (2.3), we obtain

$$\rho_1^2 - (n + r_{\min} - 2)\rho_1 + r_{\min}(n + \phi(n) - 1) - 2n\phi(n) - 2n + 2\phi(n) + 2 \geq 0.$$

So, the lower bound follows

$$\rho_1 \geq \frac{n - 2 + r_{\min} + \sqrt{r_{\min}^2 - (2n - \phi(n))r_{\min} + n^2 + 8n\phi(n) + 4n - 8\phi(n) - 4}}{2}.$$

Again, letting $x_i = \max_{1\leq k\leq \phi(n)+1} x_k$ and $x_j = \max_{\phi(n)+1<k\leq n} x_k$, and proceeding as above, we have

$$\rho_1^2 - (n + r_{\max} - 2)\rho_1 + r_{\max}(n + \phi(n) - 1) - 2n\phi(n) - 2n + 2\phi(n) + 2 \leq 0$$

and the upper bound for ρ_1 of $\mathcal{Q}(\mathcal{P}(\mathbb{Z}_n))$ follows.

Now, equality occurs in both the cases if and only $r_{\min} = r_{\max}$, which is possible if and only if $G \cong K_n$ and hence the equality holds if and only if $n = p^{n_1}$, where p is prime and n_1 is a positive integer. □

The following result [13] gives a relation between the eigenvalues of a symmetric matrix and its principal submatrix.

Theorem 2.2 (Interlacing Theorem)**.** *Let $M \in \mathbb{M}_n(\mathbb{R})$ be the real symmetric matrix and A be its principal submatrix of order m, $(m \leq n)$, respectively. Then the eigenvalues of M and A satisfy the following relation*

$$\lambda_{i+n-m}(M) \leq \lambda_i(A) \leq \lambda_i(M), \quad \textit{with } 1 \leq i \leq m.$$

The next result gives the lower bounds for the largest and the second largest distance signless Laplacain eigenvalues of $\mathcal{P}(\mathbb{Z}_n)$ in terms of the maximum transmission degree and the second maximum transmission degree.

Theorem 2.3. *Let $\mathcal{P}(\mathbb{Z}_n)$ be the power graph of $\mathbb{Z}_n$ having the maximum transmission degree $\mathrm{Tr}_{\max}$ and the second maximum transmission degree $\mathrm{Tr}^2_{\max}$. Then*

$$\rho_1 \geq \frac{1}{2}\left(\mathrm{Tr}_{\max} + \mathrm{Tr}^2_{\max} + \sqrt{(\mathrm{Tr}_{\max} - \mathrm{Tr}^2_{\max})^2 + 4}\right)$$

and

$$\rho_1 \geq \frac{1}{2}\left(\mathrm{Tr}_{\max} + \mathrm{Tr}^2_{\max} + \sqrt{(\mathrm{Tr}_{\max} - \mathrm{Tr}^2_{\max})^2 + 16}\right),$$

according as the two vertices of maximum and second maximum transmission degree are adjacent or non-adjacent.

Proof. Assume that $n \geq 3$ and let v_1 and v_2 be the vertices having the maximum transmission degree $\mathrm{Tr}_{\max}$ and the second maximum transmission degree $\mathrm{Tr}^2_{\max}$, respectively. We have the following two possibilities.

(i). Suppose that v_1 and v_2 are adjacent. Then it is clear that $d(v_1, v_2) = 1$. Now, consider the principal 2×2 submatrix

$$A = \begin{pmatrix} \mathrm{Tr}_{\max} & 1 \\ 1 & \mathrm{Tr}^2_{\max} \end{pmatrix}.$$

By using Theorem 2.2, we have

$$\rho_1(\mathcal{P}(\mathbb{Z}_n)) \geq \rho_1(A) = \frac{1}{2}\left(\mathrm{Tr}_{\max} + \mathrm{Tr}^2_{\max} + \sqrt{(\mathrm{Tr}_{\max} - \mathrm{Tr}^2_{\max})^2 + 4}\right).$$

(ii). If v_1 and v_2 are not adjacent, then as power graphs of finite groups are of diameter at most two, so $d(v_1, v_2) = 2$. Again, consider the principal 2×2 submatrix

$$B = \begin{pmatrix} \mathrm{Tr}_{\max} & 2 \\ 2 & \mathrm{Tr}^2_{\max} \end{pmatrix}.$$

Thus, by Theorem 2.2, we obtain

$$\rho_1(\mathcal{P}(\mathbb{Z}_n)) \geq \rho_1(B) = \frac{1}{2}\left(\mathrm{Tr}_{\max} + \mathrm{Tr}^2_{\max} + \sqrt{(\mathrm{Tr}_{\max} - \mathrm{Tr}^2_{\max})^2 + 16}\right).$$

With the same notations and procedure as in Theorem 2.3, we see that the second largest distance signless Laplacian eigenvalues are bounded below by

$$\frac{1}{2}\left(\mathrm{Tr}_{\max} + \mathrm{Tr}^2_{\max} - \sqrt{(\mathrm{Tr}_{\max} - \mathrm{Tr}^2_{\max})^2 + 4}\right)$$

and

$$\frac{1}{2}\left(\mathrm{Tr}_{\max} + \mathrm{Tr}^2_{\max} - \sqrt{(\mathrm{Tr}_{\max} - \mathrm{Tr}^2_{\max})^2 + 16}\right). \qquad \square$$

3. Distance signless Laplacian eigenvalues of dihedral and dicyclic groups

Let $M \in \mathbb{M}_n(\mathbb{R})$ be partitioned in the blocks matrices B_j and let Q be the new matrix whose ij^{th} entry is the average row sum of B_i block. Then Q is called the quotient matrix, and the eigenvalues of M interlace the eigenvalues of Q. In case row sums of each block are some constants, the partition is said to be equitable, and in such a situation, each eigenvalue of Q is an eigenvalue of M.

Let G be any graph of order n and let $G_i(V_i, E_i)$ be graphs of order m_i, where $i = 1, \dots, n$. The *joined union* of graphs $G_1, G_2, \dots, G_n$, denoted by $G[G_1, G_2, \dots, G_n]$, is the union of graphs $G_1, G_2, \dots, G_n$ together with the edges from every vertex of G_i to each vertex of G_j whenever v_i and v_j are adjacent in G.

The next result gives the distance signless Laplacian spectrum of $G[G_1, \dots, G_n]$ together with the eigenvalues of the quotient matrix, where G_i is an r_i regular graph.

Theorem 3.1 ([25]). *Let G be a graph of order n having vertex set $V(G) = \{v_1, \dots, v_n\}$. Let G_i be r_i regular graphs of order n_i having adjacency eigenvalues $\lambda_{i1} = r_i \geq \lambda_{i2} \geq \dots \geq \lambda_{in_i}$, where $i = 1, 2, \dots, n$. The distance signless Laplacian spectrum of the joined union graph $G[G_1, \dots, G_n]$ of order $N = \sum_{i=1}^{n} n_i$ consists of the eigenvalues $2n_i + n'_i - r_i - \lambda_{ik} - 4$ for $i = 1, \dots, n$ and $k = 2, 3, \dots, n_i$, where $n'_i = \sum_{k=1, k\neq i}^{n} n_k d_G(v_i, v_k)$. The remaining n eigenvalues are given by the equitable quotient matrix*

$$Q = \begin{pmatrix} 4n_1 + n'_1 - 2r_1 - 4 & n_2 d_G(v_1, v_2) & \dots & n_n d_G(v_1, v_n) \\ n_1 d_G(v_2, v_1) & 4n_2 + n'_2 - 2r_2 - 4 & \dots & n_n d_G(v_2, v_n) \\ \vdots & \vdots & \ddots & \vdots \\ n_1 d_G(v_n, v_1) & n_2 d_G(v_n, v_2) & \dots & 4n_n + n'_n - 2r_n - 4 \end{pmatrix}.$$

Next, we find the distance Laplacian spectrum of the dihedral group and the dicyclic group for some particular values of n. The dihedral group of order $2n$ and the dicyclic group of order $4n$ are denoted and presented as follows:

$$D_{2n} = \langle a, b \mid a^n = b^2 = e, bab = a^{-1} \rangle,$$
$$Q_n = \langle a, b \mid a^{2n} = e, b^2 = a^n, ab = ba^{-1} \rangle.$$

If n is a power of 2, then Q_n is called the *generalized quaternion group* of order $4n$.

Now, we obtain the distance signless Laplacian spectrum of the power graph of the dihedral and the dicyclic group for some special cases and obtain bounds for the spectral radius.

Proposition 3.2. *If n is a prime power, then the distance signless Laplacian spectrum of $\mathcal{P}(D_{2n})$ is*

$$\Big\{(3n-2)^{[n-2]}, (4n-5)^{[n-1]}, x_1 \geq x_2 \geq x_3\Big\},$$

where x_i, for $i = 1, 2, 3$ are the zeros of the following polynomial

$$x^3 - (12n - 9)x^2 + \left(44n^2 - 106n + 24\right)x - 48n^3 + 188n^2 - 140n + 20.$$

Proof. As $\langle a \rangle$ generates the cyclic group of order n, its power graph behaves as that of $\mathcal{P}(\mathbb{Z}_n)$. The other n elements of D_{2n} in $\mathcal{P}(D_{2n})$ form a star graph with identity as the vertex of maximum degree. Therefore, the power graph of D_{2n} can be obtained from the power graph $\mathcal{P}(\mathbb{Z}_n)$ by adding the n pendent vertices at the identity vertex e. If $n = p^{m_1}$, where m_1 is a positive integer, then

$$\mathcal{P}(D_{2n}) = P_3[K_{n-1}, K_1, \overline{K}_n],$$

that is, $\mathcal{P}(D_{2p^{m_1}})$ is the *pineapple graph*, the graph obtained from K_n by appending vertices of degree 1 at some vertex of K_n. Now, the value of n_i''s are given by $n_1' = 2n + 1$, $n_2' = 2n - 1$ and $n_3' = 2n - 1$. Thus, by Theorem 3.1, the distance signless Laplacian spectrum of $\mathcal{P}(D_{2n})$ consists of the eigenvalue

$$2n_i + n_i' + r_i + \lambda_{1k} - 4 = 2(n-1) + 2n + 1 - n + 2 + 1 - 4 = 3n - 2,$$

with multiplicity $n - 2$. Similarly, the other distance signless Laplacian eigenvalue is $4n - 5$ with multiplicity $n - 1$ and the remaining three distance signless Laplacian eigenvalues are the eigenvalues of the following matrix

$$\begin{pmatrix} 4n-3 & 1 & 2n \\ n-1 & 2n-11 & n \\ 2n-2 & 1 & 6n-5 \end{pmatrix}. \qquad \square$$

The following lemma gives an equivalent method for finding determinant (det) of a matrix.

Lemma 3.3 ([18])**.** *Let M_1, M_2, M_3 and M_4 be respectively $p \times p$, $p \times q$, $q \times p$ and $q \times q$ matrices with M_1 and M_4 invertible. Then*

$$\begin{aligned} \det\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} &= \det(M_1)\det(M_4 - M_3 M_1^{-1} M_2) \\ &= \det(M_4)\det(M_1 - M_2 M_4^{-1} M_3), \end{aligned}$$

where $M_4 - M_3M_1^{-1}M_2$ and $M_1 - M_2M_4^{-1}M_3$ are called Schur complement of M_1 and M_4, respectively.

The next result gives the distance signless Laplacian spectrum of the generalized quaternions.

Proposition 3.4. *Let $n = 2^{m_1}$, where m_1 is a positive integer. Then the distance signless Laplacian eigenvalues of $\mathcal{P}(Q_n)$ are the simple eigenvalue $4n-2$, the eigenvalue $6n - 2$ with multiplicity $2n - 3$, the eigenvalue $8n - 4$ with multiplicity n, the eigenvalue $8n - 6$ and the two zeros of the polynomial*

$$\det(M_4)\det(M_1 - M_2 M_4^{-1} M_3).$$

Proof. The identity element is always adjacent to every other vertex of $\mathcal{P}(Q_n)$. In particular, if n is a power of 2, then it can be seen that a^n is also adjacent to all other vertices of $\mathcal{P}(Q_n)$. By using these observations, the power graph $\mathcal{P}(Q_n)$ can be written as

$$\mathcal{P}(Q_n) = S[K_2, K_{2n-2}, \underbrace{K_2, K_2, \dots, K_2}_{n}],$$

where $S = K_{1,n+1}$. Using Theorem 3.1, we see that $2n_1 + n_1' - n_1 + 1 + 1 - 4 = n_1 + n_1' - 2 = 4n - 2 + 2 - 2 = 4n - 2$ is the simple distance signless Laplacian eigenvalue of $\mathcal{P}(Q_n)$. Similarly, $6n-2$ and $8n-6$ are the distance signless Laplacian eigenvalues with multiplicity $2n-3$ and n, respectively. The remaining distance signless Laplacian eigenvalues of $\mathcal{P}(Q_n)$ are the eigenvalues of following matrix

$$\begin{pmatrix} 4n & 2n-2 & 2 & \dots & 2 & 2 \\ 2 & 8n-4 & 2 & \dots & 2 & 2 \\ 2 & 4n-4 & 8n-4 & \dots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 4n-4 & 2 & \dots & 8n-4 & 2 \\ 2 & 4n-4 & 2 & \dots & 2 & 8n-4 \end{pmatrix}.$$

Let

$$M_1 = \begin{pmatrix} 4n & 2n-2 \\ 2 & 8n-2 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 2 & \dots & 2 & 2 \\ 2 & \dots & 2 & 2 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 2 & \dots & 2 & 2 \\ 4n-4 & \dots & 4n-4 & 4n-4 \end{pmatrix}^{\top},$$

$$M_4 = \begin{pmatrix} 8n-4 & \dots & 2 & 2 \\ \vdots & \ddots & \vdots & \vdots \\ 2 & \dots & 8n-4 & 2 \\ 2 & \dots & 2 & 8n-4 \end{pmatrix}.$$

Now, by Lemma 3.3, it is easy to verify that the polynomial $\det(M_4 - xI)$ has a zero $8n-6$ with multiplicity n. The remaining two distance signless Laplacian eigenvalues are the zeros of the following polynomial

$$\det(M_4)\det(M_1 - M_2 M_4^{-1} M_3). \qquad \square$$

The distance signless Laplacian matrix of $\mathcal{P}(D_{2n})$ can be written as

$$\mathcal{Q}\big(\mathcal{P}(D_{2n})\big) = \begin{pmatrix} \mathcal{Q}\big(\mathcal{P}(\mathbb{Z}_n) + \mathcal{A}\big) & B_n \\ B_n' & C_n \end{pmatrix},$$

where

$$\mathcal{A} = \begin{pmatrix} n & 0 & \cdots & 0 \\ 0 & 2n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2n \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \cdots & 2 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 4n-3 & 2 & \cdots & 2 \\ 2 & 4n-3 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \cdots & 4n-3 \end{pmatrix}$$

As C_n is invertible, so by Schur's Lemma 3.3,

$$\det(C - xI) \det((\mathcal{Q}\big(\mathcal{P}(\mathbb{Z}_n) + \mathcal{A}\big) - xI) - (B - xI) \det(C - xI)^{-1}(B - xI)$$

gives the characteristic polynomial of the matrix $\mathcal{Q}\big(\mathcal{P}(Q_{2n})\big)$. Clearly, $x = 4n - 5$ is a zero of the characteristic polynomial $\det(C - xI)$ with multiplicity n.

4. Conclusion

In general, to find all the distance signless Laplacian eigenvalues of a power graph of any group is difficult. So in this regard, we have obtained the bounds on the largest distance signless Laplacian eigenvalue of the power graph of the finite cyclic group $\mathbb{Z}_n$. However to find the bounds for other eigenvalues of such power graphs remains open. Also, we find some distance signless Laplacian eigenvalues (including bounds) of the power graphs of D_{2n} and Q_n, for some special cases. Though in general, the distance signless Laplacian eigenvalues of these graphs remain challenging, we need to devise more techniques and information about the structure of the power graphs, so that more distance signless Laplacian eigenvalues (if not all) need to be obtained. For the remaining distance signless Laplacian eigenvalues, best possible bounds need to be established. All other distance signless Laplacian spectral parameters like distance signless Laplacian energy, distance signless Laplacian spread, distance signless Laplacian Estrada index and others can be discussed for power graphs of finite groups.

Acknowledgements. We are highly grateful to the anonymous referee for his useful comments. This research is supported by SERB-DST, New Delhi under the research project number CRG/2020/000109.

References

[1] J. Abawajy, A. Kelarev, M. Chowdhury: *Power graphs: A survey*, Electronic J. Graph Theory Appl. 1.2 (2013), pp. 125–147.

[2] A. Alhevaz, M. Baghipur, Y. Alizadeh, S. Pirzada: *On eigenvalues of the reciprocal distance signless Laplacian matrix of graphs*, Asian European J. Mathematics 14.10 (2021), p. 2150176.

[3] A. Alhevaz, M. Baghipur, H. A. Ganie, Y. Shang: *Bounds for the generalized distance eigenvalues of a graph*, Symmetry 11.12 (2019), p. 1529, doi: https://doi.org/10.3390/sym11121529.

[4] A. Alhevaz, M. Baghipur, S. Pirzada: *On distance signless Laplacian Estrada index and energy of graphs*, Kragujevac J. Mathematics 45.6 (2021), pp. 837–858.

[5] A. Alhevaz, M. Baghipur, S. Pirzada, Y. Shang: *Some inequalities involving the distance signless Laplacian eigenvalues of graphs*, Transactions on Combinatorics 10.1 (2021), pp. 9–29.

[6] A. Alhevaz, M. Baghipur, Y. Shang: *Merging the spectral theories of distance Estrada and distance signless Laplacian Estrada indices of graphs*, Mathematics 7.10 (2019), p. 995, doi: https://doi.org/10.3390/math7100995.

[7] A. Alhevaz, M. Baghipur, Y. Shang: *On generalized distance Gaussian Estrada index of graphs*, Symmetry 11.10 (2019), p. 1276, doi: https://doi.org/10.3390/sym11101276.

[8] M. Aouchiche, P. Hansen: *Distance spectra of graphs: a survey,* Linear Algebra Appl. 458 (2014), pp. 301–386.

[9] M. Aouchiche, P. Hansen: *On the distance signless Laplacian of a graph*, Linear Multilinear Algebra 64 (2016), pp. 1113–1123.

[10] M. Aouchiche, P. Hansen: *Two Laplacians for the distance matrix of a graph*, Linear Algebra Appl. 439 (2013), pp. 21–33.

[11] M. Baghipur, M. Ghorbani, H. A. Ganie, Y. Shang: *On the second-largest reciprocal distance signless Laplacian eigenvalue*, Mathematics 9.5 (2021), pp. 1113–1123, doi: https://doi.org/10.3390/math9050512.

[12] S. Banerjee, A. Adhikari: *Signless Laplacian spectra of power graphs of certain finite groups*, AKCE Int. J. Graphs Combinatorics 17.1 (2020), pp. 356–366.

[13] A. E. Brouwer, W. H. Haemers: *Spectra of Graphs*, New York: Springer, 2010.

[14] P. J. Cameron, S. Ghosh: *The power graphs of a finite group*, Discrete Math. 311.1 (2011), pp. 1220–1222.

[15] I. Chakrabarty, M. Ghosh, M. K. Sen: *Undirected power graph of semigroups*, Semigroup Forum 78 (2009), pp. 410–426.

[16] S. Chattopadhyay, P. Panigrahi: *On Laplacian spectrum of power graphs of finite cyclic and dihedral groups*, Linear Multilinear Algebra 63.7 (2015), pp. 1345–1355.

[17] A. Hamzeh: *Signless and normalized Laplacian spectrums of the power graph and its supergraphs of certain finite groups*, J. Indonesian Math. Soc. 24.1 (2018), pp. 61–69.

[18] R. Horn, C. Johnson: *Matrix Analysis*, London: Cambridge University Press, 2013.

[19] A. V. Kelarev, S. J. Quinn: *Directed graphs and combinatorial properties of semigroups*, J. Algebra 251 (2002), pp. 16–26.

[20] A. V. Kelarev, S. J. Quinn: *Graph algebras and automata*, New York: Marcel Dekker, 2003.

[21] Z. Mehranian, A. Gholami, A. R. Ashrafi: *The spectra of power graphs of certain finite groups*, Linear Multilinear Algebra 65.5 (2016), pp. 1003–1010.

[22] R. P. Panda: *Laplacian spectra of power graphs of certain finite groups*, Graphs Combinatorics 35.5 (2019), pp. 1209–1223.

[23] S. Pirzada: *An Introduction to Graph Theory*, Hyderabad: Universities Press, Orient Blackswan, 2012.

[24] S. Pirzada, B. A. Ganie, A. Alhevaz, M. Baghipur: *On sum of the powers of distance signless Laplacian eigenvalues of graphs*, J. Pure and Applied Mathematics 51.3 (2020), pp. 1143–1163.

[25] S. Pirzada, B. A. Rather, M. Aijaz, T. A. Chishti: *Distance signless Laplacian spectrum of graphs and spectrum of zero divisor graphs of $\mathbb{Z}_n$*, Linear MultiLinear Algebra (2020), doi: https://doi.org/10.1080/03081087.2020.1838425.

[26] B. A. Rather, S. Pirzada, Z. Guofei: *On distance Laplacian spectra of power graphs of finite groups*, Acta Math. Sinica (2022), to appear.

[27] B. A. Rather, S. Pirzada, T. A. Naikoo: *On distance signless Laplacian spectra of power graphs of the integer modulo group*, Art Discrete Appl. Math. (2022), doi: https://doi.org/10.26493/2590-9770.1393.2be.