Annales Mathematicae et Informaticae
55 (2022) pp. 137–157
DOI: https://doi.org/10.33039/ami.2022.09.002
URL: https://ami.uni-eszterhazy.hu

Two illustrating examples for comparison of uniform and proximal spaces using relators

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Abstract. We introduce (generalized) proximities in the same way as (generalized) uniformities in paper of Weil. We prove the equivalence of our new definitions with classical ones.

Using these analog definitions, we compare the properties of (generalized) proximities and (generalized) uniformities. The main parts of this paper are examples of an $(X, \mathcal{R})$ relator space such that $\mathcal{R}^{\#}$ is uniformly (and proximally) transitive, but neither $\mathcal{R}$ nor $\mathcal{R}^{\Phi}$ is proximally (or uniformly) transitive.

For this, we summarize the essential properties of relators, using their theory from earlier works of Á. Száz.

Keywords: (generalized) uniformities, (generalized) proximities, relators

AMS Subject Classification: 54E05, 54E15, 54A05, 54G15, 54G20

1. Introduction

At the beginning of the 20th century some mathematicians tried to define abstract topological structures. The most relevant results are Poincaré 1895, Fréchet 1906, Hausdorff 1914, and Kuratowski 1922.

Uniform spaces in terms of relations were introduced by Weil in 1937 [13].

Proximities were first investigated by Riesz in 1909 [9], Effremovič, and Smirnov in 1952 [2] and [10].

After the works of Davis, Pervin, and Nakano [1], [8], and [4] in 1987, Száz [11] introduced the notion of relator and relator space in the following way.

Submitted: July 12, 2022
Accepted: September 13, 2022
Published online: September 24, 2022

Definition 1.1. A nonvoid family $\mathcal{R}$ of relations on a nonvoid set X is called a relator on X, and the ordered pair $(X, \mathcal{R})$ is called a relator space.

In the last decades, a few authors investigated the interpretation of well-known topological properties in terms of relators. In 2021, Pataki introduced the general definition for (generalized) uniformities and (generalized) proximities in [5], moreover quasi-uniformities in [6].

For more details, see, for instance [5–7, 12], but for the readers' convenience, we summarize the necessary notions and notations.

Remark 1.2. With the usual notations, $\mathcal{R}$ is a relator on X means that

$$X \neq \emptyset, \quad \emptyset \neq \mathcal{R} \subset \operatorname{Exp}(X^2),$$

where $\operatorname{Exp}(X)$ is the power set of X, and $X^2 = X \times X$.

If R is a relation on X, $x \in X$, and $A \subset X$, then the sets

$$R(x) = \{y \in X : (x, y) \in R\}, \quad \text{and} \quad R[A] = \bigcup_{x \in A} R(x)$$

are called the images of x and A under R, respectively.

2. Preliminary concepts

Definition 2.1. If R and S are relations on X, then the composition of R and S can be defined, such that $(R \circ S)(x) = R[S(x)]$ for all $x \in X$.

Moreover, let $R^{-1} = \{(y, x) : (x, y) \in R\}$, $R^0 = \Delta_X = \{(x, x) : x \in X\}$ and $R^n = R \circ R^{n-1}$, for all $n = 1, 2, \ldots$. Finally, we say that R is

- reflexive if $R^0 \subset R$,
- symmetric if $R^{-1} \subset R$,
- transitive if $R^2 \subset R$.

Lemma 2.2. *If R is a relation on X, and $A, B \subset X$, then*

$$R[A] \subset B \iff R^{-1}[X \setminus B] \subset X \setminus A.$$

Definition 2.3. If $\mathcal{R}$ is a relator on X, then the relators

$$\mathcal{R}^* = \{S \subset X^2 : \exists R \in \mathcal{R} : R \subset S\},$$

and

$$\mathcal{R}^\# = \{S \subset X^2 : \forall A \subset X : \exists R \in \mathcal{R} : R[A] \subset S[A]\},$$

are called the uniform and the proximal refinements of $\mathcal{R}$, respectively. For more details, see [7].

Moreover, for all $n = -1, 0, 1, 2, \ldots$, we define

$$\mathcal{R}^n = \{R^n : R \in \mathcal{R}\}.$$

Remark 2.4. $*$ and $\#$ are really refinements as we defined in [7]. That is, they are self-increasing in the sense that

$$\mathcal{R} \subset \mathcal{S}^* \iff \mathcal{R}^* \subset \mathcal{S}^* \quad \text{and} \quad \mathcal{R} \subset \mathcal{S}^\# \iff \mathcal{R}^\# \subset \mathcal{S}^\#,$$

or equivalently, they are expansive, increasing, and idempotent, in the sense that

$$\mathcal{R} \subset \mathcal{R}^*,\ \mathcal{R} \subset \mathcal{S} \implies \mathcal{R}^* \subset \mathcal{S}^*,\ \mathcal{R}^{**} = \mathcal{R}^*$$

and

$$\mathcal{R} \subset \mathcal{R}^\#,\ \mathcal{R} \subset \mathcal{S} \implies \mathcal{R}^\# \subset \mathcal{S}^\#,\ \mathcal{R}^{\#\#} = \mathcal{R}^\#,$$

for all $\mathcal{R}$ and $\mathcal{S}$ relators on X.

Moreover, $\#$ is $*$-dominating, $*$-invariant, $*$-absorbing, and $*$-compatible, that is

$$\mathcal{R}^* \subset \mathcal{R}^\#, \quad \mathcal{R}^\# = \mathcal{R}^{\#*}, \quad \mathcal{R}^\# = \mathcal{R}^{*\#}, \quad \mathcal{R}^{\#*} = \mathcal{R}^{*\#}.$$

For all $n = -1, 0, 1, 2, \dots$ the mapping $\mathcal{R} \mapsto \mathcal{R}^n$ of relators on X is increasing.

Finally, $*$ and $\#$ are inversion-compatible, that is, for all $\mathcal{R}$ relators on X

$$\mathcal{R}^{*-1} = \mathcal{R}^{-1*} \quad \text{and} \quad \mathcal{R}^{\#-1} = \mathcal{R}^{-1\#}.$$

And we have that for all relators on X

$$\mathcal{R}^{2*} = \mathcal{R}^{*2*}.$$

The following example shows that the analog assertion is not true for $\#$.

Example 2.5. Let $X = \{1, 2, 3, 4\}$, and

$$\mathcal{R} = \{\Delta_X \cup \{(1,2), (4,2), (2,1), (2,4)\}, \Delta_X \cup \{(1,3), (4,3), (3,1), (3,4)\}\}$$

is an elementwise reflexive and symmetric relator on X. Now, $\mathcal{R}^{\#2} \not\subset \mathcal{R}^{2\#}$, since $R = X^2 \setminus \{(1,4), (4,1)\} \in \mathcal{R}^{\#2}$ however $R \notin \mathcal{R}^{2\#}$.

Note, that $\mathcal{R} = \Big\{$, $\Big\}$, and $R =$.

Definition 2.6. Let $\mathcal{R}$ be a relator on X, and $\square \in \{*, \#\}$ is a refinement for relators on X. We define the followings.

- $\mathcal{R}$ is $\square$-reflexive, if $\mathcal{R} \subset \mathcal{R}^{0\square}$;
- $\mathcal{R}$ is $\square$-symmetric, if $\mathcal{R} \subset \mathcal{R}^{-1\square}$;
- $\mathcal{R}$ is $\square$-transitive, if $\mathcal{R} \subset \mathcal{R}^{2\square}$;
- $\mathcal{R}$ is $\square$-fine, if $\mathcal{R} = \mathcal{R}^\square$.

For instance, we say that $\mathcal{R}$ is uniformly symmetric or proximally transitive instead of $*$-symmetric or $\#$-transitive.

Following Weil, we say that the relator $\mathcal{R}$ on X is a generalized uniformity/generalized proximity on X, and the ordered pair $(X, \mathcal{R})$ is a generalized uniform space/generalized proximal space if it is

- uniformly/proximally reflexive;
- uniformly/proximally symmetric;
- uniformly/proximally transitive;
- uniformly/proximally fine.

Following the notations of [3], in [6] we introduced quasi-uniformities, and now, we define generalized quasi-uniformities/generalized quasi-proximities.

We say that the relator $\mathcal{R}$ on X is a generalized quasi-uniformity/generalized quasi-proximity on X, and the ordered pair $(X, \mathcal{R})$ is a generalized quasi-uniform space/generalized quasi-proximal space if it is

- uniformly/proximally reflexive;
- uniformly/proximally transitive;
- uniformly/proximally fine.

Definition 2.7. Let $\mathcal{A}$ be a family of sets, or equivalently $\mathcal{A} \subset \mathrm{Exp}(X)$ for some set X. We call

$$\Phi(\mathcal{A}) = \left\{ \bigcap \mathcal{B} : \emptyset \neq \mathcal{B} \subset \mathcal{A}, \text{ and } \mathcal{B} \text{ is finite} \right\}$$

the filtered family of sets generated by $\mathcal{A}$.

Moreover, we say that $\mathcal{A}$ is filtered if $\Phi(\mathcal{A}) = \mathcal{A}$.

Remark 2.8. Since Φ is a refinement for relators on X, we write $\mathcal{R}^{\Phi}$ instead of $\Phi(\mathcal{R})$, if $\mathcal{R}$ is a relator on X. Note that $\mathcal{R}$ is filtered iff $\mathcal{R}^{\Phi} \subset \mathcal{R}$.

Moreover, note that Φ is an inversion-compatible refinement for relators on X. That is if $\mathcal{R}$ is a relator on X, then $\mathcal{R}^{-1\Phi} = \mathcal{R}^{\Phi-1}$.

Finally, if $\mathcal{R}$ is finite, then $\mathcal{R}^{\Phi*} = \{\bigcap \mathcal{R}\}^*$.

Lemma 2.9. *If $\mathcal{R}$ is a relator on X, then $\mathcal{R}^{*\Phi} = \mathcal{R}^{\Phi*}$, $\mathcal{R}^{2\Phi} \subset \mathcal{R}^{\Phi 2*}$, and $\mathcal{R}^{\#2\#} \subset \mathcal{R}^{\Phi 2\#}$.*

Definition 2.10. If $\square$ is a refinement for relators on X, then we say that the $\mathcal{R}$ relator on X is $\square$-filtered if there exists an $\mathcal{S}$ relator on X such that $\mathcal{S}^{\Phi} \subset \mathcal{S}^{\square} = \mathcal{R}^{\square}$.

We use the uniformly filtered and proximally filtered notions instead of $*$-filtered and $\#$-filtered.

Definition 2.11. If the generalized uniformity/generalized proximity (generalized quasi-uniformity/generalized quasi-proximity) $\mathcal{R}$ on X is also

- uniformly/proximally filtered,

then we say that $\mathcal{R}$ is a uniformity/proximity (quasi-uniformity/quasi-proximity) on X, and $(X, \mathcal{R})$ is a uniform space/proximal space (quasi-uniform space/quasi-proximal space).

3. Main example

Example 3.1. For all $i \in \mathbb{N}$ let $X_i = \{3i-2, 3i-1, 3i\}$ and $\pi_{i,0}, \pi_{i,1}, \ldots, \pi_{i,5}$ are the all bijections of $\{1,2,3\}$ to X_i such that $\pi_{i,0}$ is the only increasing one of them.

Moreover, let

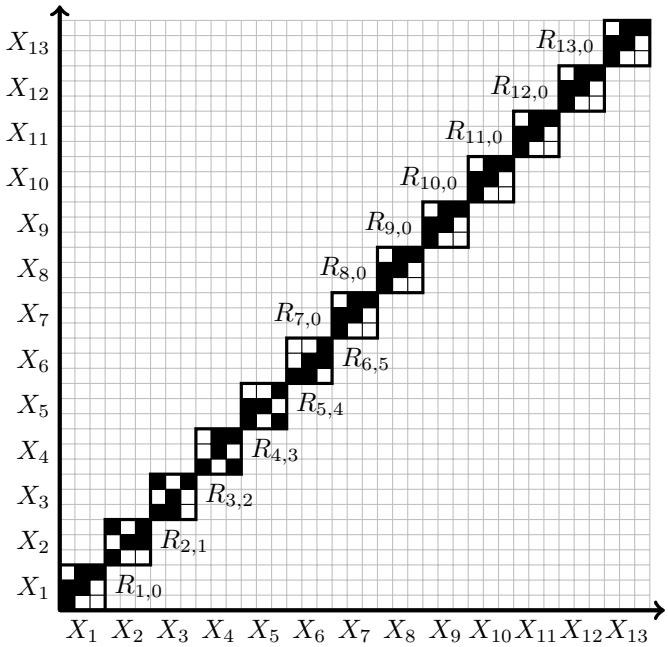
$$R_{i,k} = \{(\pi_{i,k}(1), \pi_{i,k}(2)), (\pi_{i,k}(2), \pi_{i,k}(3))\} \cup \Delta_{X_i}$$

for all $i \in \mathbb{N}$ and $k \in \{0, \ldots, 5\}$.

Furthermore, for all $n \in \mathbb{N}$ let $S_n = \bigcup_{i \in \mathbb{N}} R_{i,\nu_i}$, where ν_i is the ith digits of $(n-1)$ in a positional base 6 numeral system, that is $n - 1 = \sum_{i \in \mathbb{N}} \nu_i \cdot 6^{i-1}$.

The following figure shows the main part of the graph of S_{44791}, where $44790 = \sum_{i=0}^{5} i \cdot 6^i$ and

$\pi_{2,1}(1) = 4$, $\pi_{2,1}(2) = 6$, $\pi_{3,2}(1) = 8$, $\pi_{3,2}(2) = 7$, $\pi_{4,3}(1) = 11$,
$\pi_{4,3}(2) = 12$, $\pi_{5,4}(1) = 15$, $\pi_{5,4}(2) = 13$, $\pi_{6,5}(1) = 18$, $\pi_{6,5}(2) = 17$.



Finally, let $X = \mathbb{N} = \bigcup_{i \in \mathbb{N}} X_i$ and $\mathcal{R} = \{S_n : n \in \mathbb{N}\}$ is an elementwise reflexive relator on X. Then $\mathcal{R}^{\#} = \{\Delta_X\}^*$ is a uniformly transitive relator on X, but neither $\mathcal{R}$ nor $\mathcal{R}^{\Phi}$ is proximally transitive. Namely,

$$S_1 = \Delta_X \cup \bigcup_{i \in X \setminus 3X} \{(i, i+1)\} \in \mathcal{R} \subset \mathcal{R}^{\Phi},$$

but we show that if $A = \{3i-2 : i \in X\}$, then $Q^2[A] \not\subset S_1[A] = A \cup (A+1)$ for all $Q \in \mathcal{R}^{\Phi}$, that is $S_1 \notin \mathcal{R}^{\Phi 2\#}$ and hence $S_1 \notin \mathcal{R}^{2\#}$.

To prove this let $\emptyset \neq \mathcal{S} \subset \mathcal{R}$ finite, such that $Q = \bigcap \mathcal{S}$. Then there exists a greatest $m \in X$ such that $S_m \in \mathcal{S}$. If i is large enough then $\nu_i = 0$ for all elements of $\mathcal{S}$, therefore $R_{i,0} \subset \bigcap \mathcal{S} = Q$, and then $3i \in R_{i,0}^2(3i-2) \subset Q^2(3i-2) \subset Q^2[A]$.

We can change the above example to be symmetric.

Example 3.2. Namely, for all $i \in \mathbb{N}$ let $X_i = \{5i-4, 5i-3, 5i-2, 5i-1, 5i\}$ and $\pi_{i,0}, \pi_{i,1}, \ldots, \pi_{i,119}$ are the all bijections of $\{1,2,3,4,5\}$ to X_i such that $\pi_{i,0}$ is the only increasing one of them. Moreover, let

$$R_{i,k} = \{(\pi_{i,k}(1), \pi_{i,k}(2)), (\pi_{i,k}(2), \pi_{i,k}(1)), (\pi_{i,k}(2), \pi_{i,k}(3)), (\pi_{i,k}(3), \pi_{i,k}(2))\} \cup \Delta_{X_i}$$

for all $i \in \mathbb{N}$ and $k \in \{0, \ldots, 119\}$. Furthermore for all $n \in \mathbb{N}$ let $S_n = \bigcup_{i \in \mathbb{N}} R_{i,\nu_i}$, where ν_i is the ith digits of $(n-1)$ in a positional base 120 numeral system, that is $n - 1 = \sum_{i \in \mathbb{N}} \nu_i \cdot 120^{i-1}$. Now $\mathcal{R} = \{S_n : n \in \mathbb{N}\}$ is an elementwise reflexive, elementwise symmetric relator on $X = \mathbb{N} = \bigcup_{i \in \mathbb{N}} X_i$ with the same properties.

4. A finite example

Lemma 4.1. *Let $\mathcal{R}$ be a finite relator on X. Now, $\mathcal{R}^\Phi$ is proximally transitive iff the relation $\bigcap \mathcal{R}$ is transitive.*

Proof. If $\mathcal{R}$ is finite, then Remark 2.4 and 2.8 yield that

$$\mathcal{R}^{\Phi 2*} = \mathcal{R}^{\Phi * 2*} = \left\{\bigcap \mathcal{R}\right\}^{*2*} = \left\{\bigcap \mathcal{R}\right\}^{2*}$$

therefore

$$\mathcal{R}^{\Phi 2\#} = \mathcal{R}^{\Phi 2 * \#} = \left\{\bigcap \mathcal{R}\right\}^{2*\#} = \left\{\bigcap \mathcal{R}\right\}^{2\#} = \left\{\bigcap \mathcal{R}\right\}^{2*} = \mathcal{R}^{\Phi 2*}.$$

By using the self-increasingness of $*$ we have that

$$\left(\bigcap \mathcal{R}\right)^2 \subset \bigcap \mathcal{R} \iff \left\{\bigcap \mathcal{R}\right\} \subset \left\{\bigcap \mathcal{R}\right\}^{2*} \iff \left\{\bigcap \mathcal{R}\right\}^{*} \subset \left\{\bigcap \mathcal{R}\right\}^{2*} \iff$$
$$\iff \mathcal{R}^{\Phi *} \subset \mathcal{R}^{\Phi 2*} \iff \mathcal{R}^{\Phi} \subset \mathcal{R}^{\Phi 2*} \iff \mathcal{R}^{\Phi} \subset \mathcal{R}^{\Phi 2\#}. \qquad \square$$

Theorem 4.2. *If $\mathcal{R}$ is a finite relator on X, then uniform transitivity of $\mathcal{R}^\#$ implies proximal transitivity of $\mathcal{R}^\Phi$.*

Proof. Assume to the contrary that $\mathcal{R}^\Phi$ is not proximally transitive that is, by using the above Lemma, there exists an $(x, z) \in (\bigcap \mathcal{R})^2 \setminus \bigcap \mathcal{R}$. It means that there exist $x, y, z \in X$ such that

(a) $y \in (\bigcap \mathcal{R})(x)$;

(b) $z \in (\bigcap \mathcal{R})(y)$;

(c) $z \notin (\bigcap \mathcal{R})(x)$.

For all $S \in \mathcal{R}^{\#}$ there exists an $R \in \mathcal{R}$ such that $R(x) \subset S(x)$. For such an $R \in \mathcal{R}$, by using (a), it is easy to see, that $y \in (\bigcap \mathcal{R})(x) \subset R(x)$.

In a similar way (b) implies that $z \in S(y)$ for all $S \in \mathcal{R}^{\#}$ and (c) implies that $z \notin R(x)$ for some $R \in \mathcal{R}$.

In summary we have that

$$\exists R \in \mathcal{R} : \forall S \in \mathcal{R}^{\#} : S^2 \not\subset R$$

that is $\mathcal{R} \not\subset \mathcal{R}^{\#2*}$. This is a contradiction, because uniform transitivity of $\mathcal{R}^{\#}$ means that $\mathcal{R} \subset \mathcal{R}^{\#} \subset \mathcal{R}^{\#2*}$. □

The following examples show that uniform transitivity of $\mathcal{R}^{\#}$ does not imply proximal transitivity of $\mathcal{R}$ even if the space X is finite.

The appendix shows the graphs of all 24 R_π elements of the $\mathcal{R}$ relator in the following example.

Moreover, we can see the graphs of R_π^2 and S_n elements of $\mathcal{S}$, which is the smallest relator on X such that $\mathcal{S}^* = \mathcal{R}^{\#}$.

Finally, in appendix we can also see $S_k^2 \in \mathcal{S}^2$ examples for all $S_n \in \mathcal{S}$ such that $S_k^2 \subset S_n$.

Example 4.3. Let $X = \{1, 2, 3, 4\}$ and

$$R_\pi = \{(\pi(1), \pi(1)), (\pi(1), \pi(2)), (\pi(2), \pi(2)), (\pi(2), \pi(3))\}$$

for all π permutation of X. Moreover, let $\mathcal{R} = \{R_\pi : \pi \text{ is a permutation of } X\}$.

At first, we investigate $\mathcal{R}^{\#}$. For this end, let $Q \in \mathcal{R}^{\#}$ be arbitrary. By definition, there exists a π permutation of X, such that $\pi[\{1, 2, 3\}] = R_\pi[X] \subset Q[X]$, that is $Q[X]$ has at least 3 elements. Moreover, if $A \subset X$ has 3 elements, then $\pi(1) \in A$ or $\pi(2) \in A$ when π is a permutation of X such that $R_\pi[A] \subset Q[A]$, that is $Q[A]$ has at least 2 elements.

In summary, a Q relation on X is an element of $\mathcal{R}^{\#}$ iff it satisfies at least one of the following conditions.

- There exists α and β permutations of X such that $\beta(i) \in Q(\alpha(i))$ for all $i \in \{1, 2, 3\}$.
- There exists α and β permutations of X such that $\{\beta(i)\} \subsetneq Q(\alpha(i))$ for all $i \in \{1, 2\}$ and $\beta(3) \in Q(\alpha(1))$.

Now, we show that $\mathcal{R}^{\#}$ is uniformly transitive. For this end, let $Q \in \mathcal{R}^{\#}$ be arbitrary. This is possible in the following ways.

<u>$Q \subset \Delta_X$</u>:

(1) There exists an α permutation of X such that $(\alpha(1), \alpha(1)), (\alpha(2), \alpha(2)) \in Q$. In this case

$$P = \{(\alpha(1), \alpha(1)), (\alpha(2), \alpha(2)), (\alpha(3), \alpha(4))\} \in \mathcal{R}^{\#} \text{ and } P^2 \subset Q.$$

$\underline{Q \subset X^2 \setminus \Delta_X}$:

(2) There exists an α permutation of X such that $(\alpha(1), \alpha(2))$, $(\alpha(1), \alpha(3)) \in Q$. In this case

$$P = \{(\alpha(1), \alpha(2)), (\alpha(1), \alpha(4)), (\alpha(4), \alpha(2)), (\alpha(4), \alpha(3))\} \in \mathcal{R}^{\#} \text{ and } P^2 \subset Q.$$

Or

(3) there exists an α permutation of X such that $(\alpha(1), \alpha(2))$, $(\alpha(3), \alpha(4)) \in Q$. In this case

$$P = \{(\alpha(1), \alpha(3)), (\alpha(2), \alpha(4)), (\alpha(3), \alpha(2))\} \in \mathcal{R}^{\#} \text{ and } P^2 \subset Q.$$

Or

(4) there exists an α permutation of X such that $(\alpha(1), \alpha(2))$, $(\alpha(2), \alpha(3))$, $(\alpha(3), \alpha(1)) \in Q$. In this case

$$P = \{(\alpha(1), \alpha(3)), (\alpha(2), \alpha(1)), (\alpha(3), \alpha(2))\} \in \mathcal{R}^{\#} \text{ and } P^2 \subset Q.$$

$\underline{Q \not\subset \Delta_X \text{ and } Q \not\subset X^2 \setminus \Delta_X}$:

(5) There exists an α permutation of X such that $(\alpha(1), \alpha(1))$, $(\alpha(1), \alpha(2)) \in Q$. In this case

$$P = \{(\alpha(1), \alpha(1)), (\alpha(1), \alpha(3)), (\alpha(2), \alpha(4)), (\alpha(3), \alpha(4))\} \in \mathcal{R}^{\#} \text{ and } P^2 \subset Q.$$

Or

(6) there exists an α permutation of X such that $(\alpha(1), \alpha(1))$, $(\alpha(2), \alpha(3)) \in Q$. In this case

$$P = \{(\alpha(1), \alpha(1)), (\alpha(2), \alpha(4)), (\alpha(4), \alpha(3))\} \in \mathcal{R}^{\#} \text{ and } P^2 \subset Q.$$

Finally, we show that $\mathcal{R}$ is not proximally transitive. For this, note that for any π permutation of X

$$R_\pi^2[\{1,3,4\}] \not\subset \{1,2\} = R_{\Delta_X}[\{1,3,4\}]$$

Note that $R_{\Delta_X} =$.

Unfortunately, the above relator is neither reflexive nor symmetric. We can make a reflexive one.

Example 4.4. Let $X = \{1,2,3,4\}$, $Y = \{1,2,3,4,5,6,7,8\}$ and

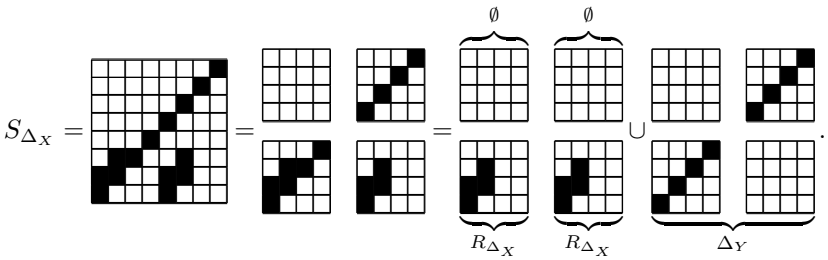
$$R_\pi = \{(\pi(1), \pi(1)), (\pi(1), \pi(2)), (\pi(2), \pi(2)), (\pi(2), \pi(3))\}$$

and

$$S_\pi = R_\pi \cup \{(\pi(1)+4, \pi(1)), (\pi(1)+4, \pi(2)), (\pi(2)+4, \pi(2)), (\pi(2)+4, \pi(3))\} \cup \Delta_Y$$

for all π permutation of X.

Moreover, let $\mathcal{S} = \{S_\pi : \pi \text{ is a permutation of } X\}$ be an elementwise reflexive relator on Y. Now $\mathcal{S}^\#$ is uniformly transitive, but $\mathcal{S}$ is not proximally transitive. Note that



We do not know whether if a symmetric finite $\mathcal{R}$ relator is not proximally transitive, then can $\mathcal{R}^\#$ be uniformly transitive.

References

[1] A. S. Davis: *Indexed Systems of Neighborhoods for General Topological Spaces*, The American Mathematical Monthly 68.9 (1961), pp. 886–893, ISSN: 00029890, 19300972, DOI: https://doi.org/10.2307/2311686, URL: http://www.jstor.org/stable/2311686 (visited on 09/13/2022).

[2] V. A. Efremovič: *The geometry of proximity. I.* Matematicheskii Sbornik 73.1 (1952), pp. 189–200.

[3] L. Nachbin: *Topology and order*, Princetown: D. Van Nostrand, 1965.

[4] H. Nakano, K. Nakano: *Connector theory*, Pacific J. Math. 56.1 (1975), pp. 195–213.

[5] G. Pataki: *Investigation of proximal spaces using relators*, Axioms 10.3 (2021), pp. 1–10, DOI: https://doi.org/10.3390/axioms10030143.

[6] G. Pataki: *Investigation of topological spaces using relators*, Applied General Topology 23.1 (2022), pp. 45–54, DOI: https://doi.org/10.4995/agt.2022.16128.

[7] G. Pataki, Á. Száz: *A unified treatment of well-chainedness and connectedness properties*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) 19 (2003), pp. 101–166.

[8] W. J. Pervin: *Quasi-uniformization of topological spaces*, Math. Ann. 147 (1962), pp. 316–317, DOI: https://doi.org/10.1007/BF01440953.

[9] F. Riesz: *Stetigkeit und abstrakte Mengenlehre*, Rom. 4. Math. Kongr. 2 (1909), pp. 18–24.

[10] Y. M. Smirnov: *On proximity spaces*, Matematicheskii Sbornik 73.3 (1952), pp. 543–574.

[11] Á. Száz: *Basic tools and mild continuities in relator spaces*, Acta Math. Hungar. 50.3-4 (1987), pp. 177–201, DOI: https://doi.org/10.1007/bf01903935.

[12] Á. Száz: *Minimal structures, generalized topologies, and ascending systems should not be studied without generalized uniformities*, Filomat 21.1 (2007), pp. 87–97, DOI: https://doi.org/10.2298/FIL0701087S.

[13] A. Weil: *Sur les espaces a structure uniforme at sur la topologie générale*, Paris: Herman and Cie, 1937, DOI: https://doi.org/10.1007/978-1-4757-1705-1_27.

Appendix

List of elements of $\mathcal{R}$ of Example 4.3, with using the notation $\pi(1)\pi(2)\pi(3)\pi(4)$ for the π permutation of X.

$R_{1234} =$, $R_{1243} =$, $R_{1324} =$, $R_{1342} =$,

$R_{1423} =$, $R_{1432} =$, $R_{2134} =$, $R_{2143} =$,

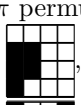
$R_{2314} =$, $R_{2341} =$, $R_{2413} =$, $R_{2431} =$,

$R_{3124} =$, $R_{3142} =$, $R_{3214} =$, $R_{3241} =$,

$R_{3412} =$, $R_{3421} =$, $R_{4123} =$, $R_{4132} =$,

$R_{4213} =$, $R_{4231} =$, $R_{4312} =$, $R_{4321} =$.

List of elements of $\mathcal{R}^2$ of Example 4.3, with using the notation $\pi(1)\pi(2)\pi(3)\pi(4)$ for the π permutation of X.

$R^2_{1234} =$  , $R^2_{1243} =$, $R^2_{1324} =$, $R^2_{1342} =$,

$R^2_{1423} =$, $R^2_{1432} =$, $R^2_{2134} =$, $R^2_{2143} =$,

$R^2_{2314} =$, $R^2_{2341} =$, $R^2_{2413} =$, $R^2_{2431} =$,

$R^2_{3124} =$, $R^2_{3142} =$, $R^2_{3214} =$, $R^2_{3241} =$,

$R^2_{3412} =$, $R^2_{3421} =$, $R^2_{4123} =$, $R^2_{4132} =$,

$R^2_{4213} =$, $R^2_{4231} =$, $R^2_{4312} =$, $R^2_{4321} =$.

In Example 4.3, $\mathcal{S}$ is the smallest relator on X, such that $\mathcal{S}^* = \mathcal{R}^{\#}$

We draw the graph of elements of $\mathcal{S}$ with marked the required member of $\mathcal{S}$, which implies the $\mathcal{S} \subset \mathcal{S}^{2*}$ inclusion.

$S^2_{27} =$ $\subset S_1 =$,

$S^2_{90} =$ $\subset S_2 =$,

$S^2_{29} =$ $\subset S_3 =$,

$S^2_{82} =$ $\subset S_4 =$,

$S^2_{28} =$ $\subset S_5 =$,

$S^2_{95} =$ $\subset S_6 =$,

$S^2_{191} =$ $\subset S_7 =$,

$S^2_{29} =$ $\subset S_8 =$,

$S^2_{80} =$ $\subset S_9 =$,

$S^2_{28} =$ $\subset S_{10} =$,

$S^2_{15} =$ $\subset S_{11} =$,

$S^2_{7} =$ $\subset S_{12} =$,

$S^2_{13} =$ $\subset S_{13} =$,

$S^2_{15} =$ $\subset S_{14} =$,

$S^2_{15} =$ $\subset S_{15} =$,

$S^2_{83} =$ $\subset S_{16} =$,

$S^2_{27} =$ $\subset S_{17} =$,

$S^2_{84} =$ $\subset S_{18} =$,

$S^2_{91} =$ $\subset S_{19} =$,

$S^2_{28} =$ $\subset S_{20} =$,

$S^2_{25} =$ $\subset S_{21} =$,

$S^2_{9} =$ $\subset S_{22} =$,

$S^2_{25} =$ $\subset S_{23} =$,

$S^2_{19} =$ $\subset S_{24} =$,

$S^2_{25} =$ $\subset S_{25} =$,

$S^2_{10} =$ $\subset S_{26} =$,

$S^2_{10} =$ $\subset S_{27} =$,

$S^2_{10} =$ $\subset S_{28} =$,

$S^2_{10} =$ $\subset S_{29} =$,

$S^2_{10} =$ $\subset S_{30} =$,

$S^2_{131} =$ $\subset S_{31} =$,

$S^2_{82} =$ $\subset S_{32} =$,

$S^2_{37} =$ $\subset S_{33} =$,

$S^2_{80} =$ $\subset S_{34} =$,

$S^2_{38} =$ $\subset S_{35} =$,

$S^2_{38} =$ $\subset S_{36} =$,

$S^2_{33} =$ $\subset S_{37} =$,

$S^2_{35} =$ $\subset S_{38} =$,

$S^2_{147} =$ $\subset S_{39} =$,

$S^2_{35} =$ $\subset S_{40} =$,

$S^2_{83} =$ $\subset S_{41} =$,

$S^2_{84} =$ $\subset S_{42} =$,

$S^2_{27} =$ $\subset S_{43} =$,

$S^2_{163} =$ $\subset S_{44} =$,

$S^2_{29} =$ $\subset S_{45} =$,

$S^2_{35} =$ $\subset S_{46} =$,

$S^2_{28} =$ $\subset S_{47} =$,

$S^2_{163} =$ $\subset S_{48} =$,

$S^2_{15} =$ $\subset S_{49} =$,

$S^2_{38} =$ $\subset S_{50} =$,

$S^2_{25} =$ $\subset S_{51} =$,

$S^2_{10} =$ $\subset S_{52} =$,

$S^2_{128} =$ $\subset S_{53} =$,

$S^2_{58} =$ $\subset S_{54} =$,

$S^2_{31} =$ $\subset S_{55} =$,

$S^2_{40} =$ $\subset S_{56} =$,

$S^2_{36} =$ $\subset S_{57} =$,

$S^2_{54} =$ $\subset S_{58} =$,

$S^2_{36} =$ $\subset S_{59} =$,

$S^2_{66} =$ $\subset S_{60} =$,

$S^2_{29} =$ $\subset S_{61} =$,

$S^2_{58} =$ $\subset S_{62} =$,

$S^2_{28} =$ $\subset S_{63} =$,

$S^2_{40} =$ $\subset S_{64} =$,

$S^2_{15} =$ $\subset S_{65} =$,

$S^2_{38} =$ $\subset S_{66} =$,

$S^2_{25} =$ $\subset S_{67} =$,

$S^2_{10} =$ $\subset S_{68} =$,

$S^2_{36} =$ $\subset S_{69} =$,

$S^2_{27} =$ $\subset S_{70} =$,

$S^2_{29} =$ $\subset S_{71} =$,

$S^2_{35} =$ $\subset S_{72} =$,

$S^2_{28} =$ $\subset S_{73} =$,

$S^2_{40} =$ $\subset S_{74} =$,

$S^2_{72} =$ $\subset S_{75} =$,

$S^2_{54} =$ $\subset S_{76} =$,

$S^2_{25} =$ $\subset S_{77} =$,

$S^2_{10} =$ $\subset S_{78} =$,

$S^2_{34} =$ $\subset S_{79} =$,

$S^2_{34} =$ $\subset S_{80} =$,

$S^2_{32} =$ $\subset S_{81} =$,

$S^2_{32} =$ $\subset S_{82} =$,

$S^2_{41} =$ $\subset S_{83} =$,

$S^2_{42} =$ $\subset S_{84} =$,

$S^2_{41} =$ $\subset S_{85} =$,

$S^2_{27} =$ $\subset S_{86} =$,

$S^2_{42} =$ $\subset S_{87} =$,

$S^2_{42} =$ $\subset S_{88} =$,

$S^2_{28} =$ $\subset S_{89} =$,

$S^2_{32} =$ $\subset S_{90} =$,

$S^2_{34} =$ $\subset S_{91} =$,

$S^2_{15} =$ $\subset S_{92} =$,

$S^2_{25} =$ $\subset S_{93} =$,

$S^2_{10} =$ $\subset S_{94} =$,

$S^2_{41} =$ $\subset S_{95} =$,

$S^2_{27} =$ $\subset S_{96} =$,

$S^2_{42} =$ $\subset S_{97} =$,

$S^2_{29} =$ $\subset S_{98} =$,

$S^2_{28} =$ $\subset S_{99} =$,

$S^2_{32} =$ $\subset S_{100} =$,

$S^2_{34} =$ $\subset S_{101} =$,

$S^2_{15} =$ $\subset S_{102} =$,

$S^2_{34} =$ $\subset S_{103} =$,

$S^2_{10} =$ $\subset S_{104} =$,

$S^2_{36} =$ $\subset S_{105} =$,

$S^2_{27} =$ $\subset S_{106} =$,

$S^2_{42} =$ $\subset S_{107} =$,

$S^2_{29} =$ $\subset S_{108} =$,

$S^2_{42} =$ $\subset S_{109} =$,

$S^2_{32} =$ $\subset S_{110} =$,

$S^2_{34} =$ $\subset S_{111} =$,

$S^2_{15} =$ $\subset S_{112} =$,

$S^2_{34} =$ $\subset S_{113} =$,

$S^2_{25} =$ $\subset S_{114} =$,

$S^2_{169} =$ $\subset S_{115} =$,

$S^2_{82} =$ $\subset S_{116} =$,

$S^2_{133} =$ $\subset S_{117} =$,

$S^2_{80} =$ $\subset S_{118} =$,

$S^2_{134} =$ $\subset S_{119} =$,

$S^2_{84} =$ $\subset S_{120} =$,

$S^2_{149} =$ $\subset S_{121} =$,

$S^2_{35} =$ $\subset S_{122} =$,

$S^2_{133} =$ $\subset S_{123} =$,

$S^2_{149} =$ $\subset S_{124} =$,

$S^2_{144} =$ $\subset S_{125} =$,

$S^2_{58} =$ $\subset S_{126} =$,

$S^2_{40} =$ $\subset S_{127} =$,

$S^2_{54} =$ $\subset S_{128} =$,

$S^2_{42} =$ $\subset S_{129} =$,

$S^2_{173} =$ $\subset S_{130} =$,

$S^2_{32} =$ $\subset S_{131} =$,

$S^2_{133} =$ $\subset S_{132} =$,

$S^2_{117} =$ $\subset S_{133} =$,

$S^2_{119} =$ $\subset S_{134} =$,

$S^2_{170} =$ $\subset S_{135} =$,

$S^2_{80} =$ $\subset S_{136} =$,

$S^2_{83} =$ $\subset S_{137} =$,

$S^2_{84} =$ $\subset S_{138} =$,

$S^2_{117} =$ $\subset S_{139} =$,

$S^2_{129} =$ $\subset S_{140} =$,

$S^2_{129} =$ $\subset S_{141} =$,

$S^2_{38} =$ $\subset S_{142} =$,

$S^2_{36} =$ $\subset S_{143} =$,

$S^2_{58} =$ $\subset S_{144} =$,

$S^2_{125} =$ $\subset S_{145} =$,

$S^2_{54} =$ $\subset S_{146} =$,

$S^2_{41} =$ $\subset S_{147} =$,

$S^2_{117} =$ $\subset S_{148} =$,

$S^2_{34} =$ $\subset S_{149} =$,

$S^2_{179} =$ $\subset S_{150} =$,

$S^2_{27} =$ $\subset S_{151} =$,

$S^2_{117} =$ $\subset S_{152} =$,

$S^2_{29} =$ $\subset S_{153} =$,

$S^2_{97} =$ $\subset S_{154} =$,

$S^2_{28} =$ $\subset S_{155} =$,

$S^2_{133} =$ $\subset S_{156} =$,

$S^2_{15} =$ $\subset S_{157} =$,

$S^2_{91} =$ $\subset S_{158} =$,

$S^2_{25} =$ $\subset S_{159} =$,

$S^2_{10} =$ $\subset S_{160} =$,

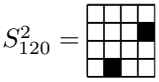
$S^2_{66} =$ $\subset S_{161} =$,

$S^2_{72} =$ $\subset S_{162} =$,

$S^2_{50} =$ $\subset S_{163} =$,

$S^2_{46} =$ $\subset S_{164} =$,

$S^2_{109} =$ $\subset S_{165} =$,

$S^2_{120} =$  $\subset S_{166} =$,

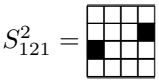
$S^2_{120} =$ $\subset S_{167} =$,

$S^2_{79} =$ $\subset S_{168} =$,

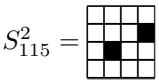
$S^2_{82} =$ $\subset S_{169} =$,

$S^2_{83} =$ $\subset S_{170} =$,

$S^2_{121} =$ $\subset S_{171} =$,

$S^2_{121} =$  $\subset S_{172} =$,

$S^2_{130} =$ $\subset S_{173} =$,

$S^2_{115} =$  $\subset S_{174} =$,

$S^2_{38} =$ $\subset S_{175} =$,

$S^2_{135} =$ $\subset S_{176} =$,

$S^2_{35} =$ $\subset S_{177} =$,

$S^2_{120} =$ $\subset S_{178} =$,

$S^2_{150} =$ $\subset S_{179} =$,

$S^2_{41} =$ $\subset S_{180} =$,

$S^2_{42} =$ $\subset S_{181} =$,

$S^2_{34} =$ $\subset S_{182} =$,

$S^2_{32} =$ $\subset S_{183} =$,

$S^2_{95} =$ $\subset S_{184} =$,

$S^2_{121} =$ $\subset S_{185} =$,

$S^2_{29} =$ $\subset S_{186} =$,

$S^2_{97} =$ $\subset S_{187} =$,

$S^2_{28} =$ $\subset S_{188} =$,

$S^2_{90} =$ $\subset S_{189} =$,

$S^2_{15} =$ $\subset S_{190} =$,

$S^2_{91} =$ $\subset S_{191} =$,

$S^2_{25} =$ $\subset S_{192} =$,

$S^2_{10} =$ $\subset S_{193} =$,

$S^2_{149} =$ $\subset S_{194} =$,

$S^2_{72} =$ $\subset S_{195} =$,

$S^2_{50} =$ $\subset S_{196} =$,

$S^2_{46} =$ $\subset S_{197} =$,

$S^2_{109} =$ $\subset S_{198} =$,

$S^2_{95} =$ $\subset S_{199} =$,

$S^2_{27} =$ $\subset S_{200} =$,

$S^2_{29} =$ $\subset S_{201} =$,

$S^2_{97} =$ $\subset S_{202} =$,

$S^2_{28} =$ $\subset S_{203} =$,

$S^2_{90} =$ $\subset S_{204} =$,

$S^2_{120} =$ $\subset S_{205} =$,

$S^2_{91} =$ $\subset S_{206} =$,

$S^2_{25} =$ $\subset S_{207} =$,

$S^2_{10} =$ $\subset S_{208} =$,

$S^2_{129} =$ $\subset S_{209} =$,

$S^2_{66} =$ $\subset S_{210} =$,

$S^2_{50} =$ $\subset S_{211} =$,

$S^2_{46} =$ $\subset S_{212} =$,

$S^2_{109} =$ $\subset S_{213} =$,

$S^2_{223} =$ $\subset S_{214} =$,

$S^2_{219} =$ $\subset S_{215} =$,

$S^2_{138} =$ $\subset S_{217} =$,

$S^2_{215} =$ $\subset S_{219} =$,

$S^2_{219} =$ $\subset S_{221} =$,

$S^2_{215} =$ $\subset S_{223} =$,

$S^2_{219} =$ $\subset S_{225} =$,

$S^2_{215} =$ $\subset S_{227} =$,

$S^2_{144} =$ $\subset S_{229} =$,

$S^2_{40} =$ $\subset S_{231} =$,

$S^2_{27} =$ $\subset S_{233} =$,

$S^2_{97} =$ $\subset S_{235} =$,

$S^2_{90} =$ $\subset S_{237} =$,

$S^2_{15} =$ $\subset S_{239} =$,

$S^2_{79} =$ $\subset S_{216} =$,

$S^2_{118} =$ $\subset S_{218} =$,

$S^2_{225} =$ $\subset S_{220} =$,

$S^2_{131} =$ $\subset S_{222} =$,

$S^2_{118} =$ $\subset S_{224} =$,

$S^2_{220} =$ $\subset S_{226} =$,

$S^2_{36} =$ $\subset S_{228} =$,

$S^2_{128} =$ $\subset S_{230} =$,

$S^2_{95} =$ $\subset S_{232} =$,

$S^2_{202} =$ $\subset S_{234} =$,

$S^2_{28} =$ $\subset S_{236} =$,

$S^2_{133} =$ $\subset S_{238} =$,

$S^2_{25} =$ $\subset S_{240} =$,

$S^2_{10} =$ $\subset S_{241} =$,

$S^2_{66} =$ $\subset S_{243} =$,

$S^2_{46} =$ $\subset S_{245} =$,

$S^2_{95} =$ $\subset S_{247} =$,

$S^2_{117} =$ $\subset S_{249} =$,

$S^2_{28} =$ $\subset S_{251} =$,

$S^2_{191} =$ $\subset S_{253} =$,

$S^2_{91} =$ $\subset S_{255} =$,

$S^2_{129} =$ $\subset S_{257} =$,

$S^2_{72} =$ $\subset S_{259} =$,

$S^2_{109} =$ $\subset S_{261} =$,

$S^2_{27} =$ $\subset S_{263} =$,

$S^2_{29} =$ $\subset S_{265} =$,

$S^2_{90} =$ $\subset S_{267} =$,

$S^2_{149} =$ $\subset S_{242} =$,

$S^2_{72} =$ $\subset S_{244} =$,

$S^2_{109} =$ $\subset S_{246} =$,

$S^2_{27} =$ $\subset S_{248} =$,

$S^2_{29} =$ $\subset S_{250} =$,

$S^2_{90} =$ $\subset S_{252} =$,

$S^2_{15} =$ $\subset S_{254} =$,

$S^2_{10} =$ $\subset S_{256} =$,

$S^2_{66} =$ $\subset S_{258} =$,

$S^2_{50} =$ $\subset S_{260} =$,

$S^2_{95} =$ $\subset S_{262} =$,

$S^2_{121} =$ $\subset S_{264} =$,

$S^2_{97} =$ $\subset S_{266} =$,

$S^2_{120} =$ $\subset S_{268} =$,

$S^2_{15} =$ $\subset S_{269} =$,

$S^2_{91} =$ $\subset S_{270} =$,

$S^2_{25} =$ $\subset S_{271} =$,

$S^2_{163} =$ $\subset S_{272} =$,

$S^2_{66} =$ $\subset S_{273} =$,

$S^2_{72} =$ $\subset S_{274} =$,

$S^2_{50} =$ $\subset S_{275} =$,

$S^2_{46} =$ $\subset S_{276} =$.