Topological loops with six-dimensional solvable multiplication groups having five-dimensional nilradical*

Ágota Figula, Kornélia Ficzere, Ameer Al-Abayechi

University of Debrecen, Institute of Mathematics, Hungary
figula@science.unideb.hu
ficzerelia@gmail.com
ameer@science.unideb.hu

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Abstract

Using connected transversals we determine the six-dimensional indecomposable solvable Lie groups with five-dimensional nilradical and their subgroups which are the multiplication groups and the inner mapping groups of three-dimensional connected simply connected topological loops. Together with this result we obtain that every six-dimensional indecomposable solvable Lie group which is the multiplication group of a three-dimensional topological loop has one-dimensional centre and two- or three-dimensional commutator subgroup.

Keywords: multiplication group of a topological loop, connected transversals, linear representations of solvable Lie algebras

MSC: 22E25, 17B30, 20N05, 57S20, 53C30

1. Introduction

The multiplication group \(\text{Mult}(L)\) and the inner mapping group \(\text{Inn}(L)\) of a loop \(L\) are important tools for the investigations in loop theory since there are strong

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relations between the structure of the normal subloops of $L$ and that of the normal subgroups of $Mult(L)$ (cf. [1, 2]). In [9] the authors have obtained necessary and sufficient conditions for a group $G$ to be the multiplication group of $L$. These conditions say that one can use special transversals $A$ and $B$ with respect to a subgroup $K$ of $G$. The subgroup $K$ plays the role of the inner mapping group of $L$ whereas the transversals $A$ and $B$ belong to the sets of left and right translations of $L$.

P. T. Nagy and K. Strambach in [8] investigate thoroughly topological and differentiable loops as continuous and differentiable sections in Lie groups. In this paper we follow their approach and study topological loops $L$ of dimension 3 having a solvable Lie group as their multiplication group. Applying the criteria of [9] we obtained in [3] all solvable Lie groups of dimension $\leq 5$ which are the multiplication group of a 3-dimensional connected simply connected topological proper loop. This classification has resulted only decomposable Lie groups as the group $Mult(L)$ of $L$. Hence we paid our attention to 6-dimensional solvable indecomposable Lie groups. If their Lie algebras have a 4-dimensional nilradical, then among the 40 isomorphism classes of Lie algebras there is only one class depending on a real parameter which consists of the Lie algebras of the group $Mult(L)$ of $L$ (cf. [4]). This result has confirmed the observation that the condition for the multiplication group of a topological loop to be a (finite-dimensional) Lie group is strong. Since the 6-dimensional solvable indecomposable Lie algebras have 4 or 5-dimensional nilradical it remains to deal with the 99 classes of solvable Lie algebras having 5-dimensional nilradical (cf. [7, 10]). In [5] we proved that among them there are 20 classes of Lie algebras which satisfy the necessary conditions to be the Lie algebra of the group $Mult(L)$ of a 3-dimensional loop $L$. We determined there also the possible subalgebras of the corresponding inner mapping groups.

The purpose of this paper is to determine the indecomposable solvable Lie groups of dimension 6 which have 5-dimensional nilradical and which are the multiplication group of a 3-dimensional connected simply connected topological loop. To find a suitable linear representation of the simply connected Lie groups for the 20 classes of solvable Lie algebras given in [5] is the first step to achieve this classification (cf. Theorem 3.1). Applying the method of connected transversals we show that only those Lie groups $G$ in Theorem 3.1 which have 2- or 3-dimensional commutator subgroup allow continuous left transversals $A$ and $B$ in the group $G$ with respect to the subgroup $K$ given in Theorem 3.1 such that $A$ and $B$ are $K$-connected and $A \cup B$ generates $G$ (cf. Proposition 3.2 and Theorem 3.3). An arbitrary left transversal $A$ to the 3-dimensional abelian subgroup $K$ of $G$ depends on three continuous real functions with three variables. The condition that the left transversals $A$ and $B$ are $K$-connected is formulated by functional equations. Summarizing the results of Theorem in [6], of Theorem 16 in [4] and of Theorem 3.3 we obtain that each 6-dimensional solvable indecomposable Lie group which is the multiplication group of a 3-dimensional topological loop has 1-dimensional centre and two- or three-dimensional commutator subgroup.
2. Preliminaries

A loop is a binary system \((L, \cdot)\) if there exists an element \(e \in L\) such that \(x = e \cdot x = x \cdot e\) holds for all \(x \in L\) and the equations \(x \cdot a = b\) and \(a \cdot y = b\) have precisely one solution \(x = b/a\) and \(y = a \backslash b\). A loop is proper if it is not a group.

The left and right translations \(\lambda_a = y \mapsto a \cdot y : L \to L\) and \(\rho_a : y \mapsto y \cdot a : L \to L\), \(a \in L\), are bijections of \(L\). The permutation group \(Mult(L) = \langle \lambda_a, \rho_a ; a \in L \rangle\) is called the multiplication group of \(L\). The stabilizer of the identity element \(e \in L\) in \(Mult(L)\) is called the inner mapping group \(Inn(L)\) of \(L\).

Let \(G\) be a group, let \(K \leq G\), and let \(A\) and \(B\) be two left transversals to \(K\) in \(G\). We say that \(A\) and \(B\) are \(K\)-connected if \(a^{-1}b^{-1}ab \in K\) for every \(a \in A\) and \(b \in B\). The core \(Co_G(K)\) of \(K\) in \(G\) is the largest normal subgroup of \(G\) contained in \(K\). If \(L\) is a loop, then \(\Lambda(L) = \{\lambda_a ; a \in L\}\) and \(R(L) = \{\rho_a ; a \in L\}\) are \(Inn(L)\)-connected transversals in the group \(Mult(L)\) and the core of \(Inn(L)\) in \(Mult(L)\) is trivial. In [9], Theorem 4.1, the following necessary and sufficient conditions are established for a group \(G\) to be the multiplication group of a loop \(L\):

**Proposition 2.1.** A group \(G\) is isomorphic to the multiplication group of a loop if and only if there exists a subgroup \(K\) with \(Co_G(K) = 1\) and \(K\)-connected transversals \(A\) and \(B\) satisfying \(G = \langle A, B \rangle\).

A loop \(L\) is called topological if \(L\) is a topological space and the binary operations \((x, y) \mapsto x \cdot y, (x, y) \mapsto x \backslash y, (x, y) \mapsto y/x : L \times L \to L\) are continuous. In general the multiplication group of a topological loop \(L\) is a topological transformation group that does not have a natural (finite dimensional) differentiable structure. In this paper we deal with 3-dimensional connected simply connected topological loops \(L\). We assume that the multiplication group of \(L\) is a 6-dimensional solvable indecomposable Lie group \(G\) such that its Lie algebra has 5-dimensional nilradical. Then \(L\) is homeomorphic to \(\mathbb{R}^3\) (cf. [3, Lemma 5]). Since it has nilpotency class 2 (cf. [5, Theorem 3.1]) by Theorem 8 A in [2] the subgroup \(K\) in Proposition 2.1 is a 3-dimensional abelian Lie subgroup of \(G\) which does not contain any non-trivial normal subgroup of \(G\), \(A\) and \(B\) are continuous \(K\)-connected left transversals to \(K\) in \(G\) such that \(A \cup B\) generates \(G\).

3. Six-dimensional solvable Lie multiplication groups with five-dimensional nilradical

Using necessary conditions we found in [5], Theorems 3.6, 3.7, those 6-dimensional solvable indecomposable Lie algebras with 5-dimensional nilradical which can occur as the Lie algebra \(g\) of the multiplication group of a 3-dimensional topological loop \(L\). We obtained also the Lie subalgebras \(k\) of the inner mapping group of \(L\). With the notation in [10] they are the following:

\(g_{1} := g_{6,14}^{a=b=0}, k_{1,1} = \langle e_{2}, e_{4} + e_{1}, e_{5} \rangle, k_{1,2} = \langle e_{3}, e_{4} + e_{1}, e_{5} \rangle;\)
\[ g_2 := g_{6,22}^a = 0, \quad k_2 = \langle e_3, e_4 + e_1, e_5 \rangle, \]
\[ g_3 := g_{6,17}^\delta, a = 0, e = 0, k_{3,1} = \langle e_3, e_4, e_5 + e_1 \rangle, k_{3,2} = \langle e_2, e_4, e_5 + e_1 \rangle; \]
\[ g_4 := g_{6,54}^a = b = 0, k_5 = \langle e_1 + e_2, e_3 + a_2 e_2, e_4 \rangle, a_2 \in \mathbb{R}; \]
\[ g_5 := g_{6,63}^a = b = 0, k_6 = \langle e_1 + e_2, e_3 + a_2 e_2, e_4 \rangle, a_2 \in \mathbb{R}; \]
\[ g_7 := g_{6,25}^a = b = 0, k_7 = \langle e_1 + e_5, e_2 + e_5 e_4, e_4 \rangle, e = 0, 1; \]
\[ g_8 := g_{6,15}^a = 0, k_8 = \langle e_1 + e_5, e_2 + a_2 e_5, e_4 + a_3 e_5, a_3 \in \mathbb{R} \setminus \{0\}, a_2 \in \mathbb{R}; \]
\[ g_9 := g_{6,21}^{a_0 < b \leq 1}, k_9 = \langle e_3, e_4 + e_1, e_5 + e_1 \rangle; \]
\[ g_{10} := g_{6,24}, k_{10} = \langle e_3, e_4, e_5 + e_1 \rangle; \]
\[ g_{11} := g_{6,30}, k_{11} = \langle e_3, e_4 + a_2 e_1, e_5 + e_1 \rangle, a_2 \in \mathbb{R}; \]
\[ g_{12} := g_{6,36}^a = b = 0, k_{12,1} = \langle e_3, e_4, e_5 + e_1 \rangle, k_{12,2} = \langle e_3, e_4 + e_1, e_5 + a_3 e_1 \rangle, a_3 \in \mathbb{R}; \]
\[ g_{13} := g_{6,16}, k_{13} = \langle e_1 + e_5, e_2 + a_2 e_5, e_4 + a_3 e_5, a_2, a_3 \in \mathbb{R}; \]
\[ g_{14} := g_{6,27}^a = b = \delta = 0, k_{14} = \langle e_1 + e_5, e_2 + a_2 e_5, e_4 \rangle, a_2 \in \mathbb{R}; \]
\[ g_{15} := g_{6,49}^a = b = 0, k_{15} = \langle e_1 + a_1 e_3, e_2 + e_3, e_4 + a_3 e_3 \rangle, a_1, a_3 \in \mathbb{R}; \]
\[ g_{16} := g_{6,52}^a = b = 0, k_{16} = \langle e_1 + a_1 e_2, e_3 + e_2, e_4 \rangle, a_1 \in \mathbb{R}; \]
\[ g_{17} := g_{6,57}^a = b = 0, k_{17} = \langle e_1 + e_2, e_3 + a_2 e_2, e_4 \rangle, a_2 \in \mathbb{R}; \]
\[ g_{18} := g_{6,59}^a = b = \delta = 0, k_{18} = \langle e_1 + e_2, e_3 + a_2 e_2, e_4 \rangle, a_2 \in \mathbb{R}; \]
\[ g_{19} := g_{6,17}^a = b = 0, k_{19} = \langle e_1 + e_4, e_2 + a_2 e_4, e_5 + e_4 \rangle, a_2 \in \mathbb{R}; \]
\[ g_{20} := g_{6,17}^a = b = 0, k_{20} = \langle e_1 + e_4, e_2 + a_2 e_4, e_5 + a_3 e_4 \rangle, a_2, a_3 \in \mathbb{R}. \]

In [11] a single matrix \( M \) is established depending on six variables such that the span of the matrices engenders the given Lie algebra in the list \( g_i, i = 1, \ldots, 20 \). To obtain the matrix Lie group \( G_i \) of the Lie algebra \( g_i \) we exponentiate the space of matrices spanned by the matrix \( M \). Simplifying the obtained exponential image we get a suitable simple form of a matrix Lie group such that by differentiating and evaluating at the identity its Lie algebra is isomorphic to the Lie algebra \( g_i \). In case of the Lie algebras \( g_j, j = 1, 2, 8, 9, 16 \), we take in order the exponential image of the matrices:

\[
M_1 = \begin{pmatrix}
0 & -s_3 & s_2 & 0 & -s_6 & 2s_1 \\
0 & 0 & 0 & 0 & 0 & s_2 \\
0 & 0 & 0 & 0 & 0 & s_3 \\
0 & 0 & -s_6 & 0 & 0 & s_4 \\
0 & 0 & 0 & 0 & 0 & 2s_5 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad s_i \in \mathbb{R}, i = 1, \ldots, 6,
\]
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\[
M_2 = \begin{pmatrix}
0 & -s_3 & s_2 & 0 & -s_6 & 2s_1 \\
0 & 0 & 0 & 0 & 0 & s_2 \\
0 & -s_6 & 0 & 0 & 0 & s_3 \\
0 & 0 & 0 & -s_6 & 0 & s_4 \\
0 & 0 & 0 & 0 & 0 & 2s_5 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad s_i \in \mathbb{R}, i = 1, \ldots, 6,
\]

\[
M_8 = \begin{pmatrix}
-s_6 & -s_3 & -s_2 & 0 & 0 & 2s_1 \\
0 & -s_6 & 0 & 0 & 0 & s_2 \\
0 & 0 & 0 & 0 & 0 & -s_3 \\
0 & -s_6 & 0 & -s_6 & 0 & s_4 \\
0 & 0 & -s_6 & 0 & 0 & -s_5 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad s_i \in \mathbb{R}, i = 1, \ldots, 6,
\]

\[
M_9 = \begin{pmatrix}
0 & -s_3 & s_2 & 0 & 0 & 2s_1 \\
0 & 0 & 0 & 0 & 0 & s_2 \\
0 & -s_6 & 0 & 0 & 0 & s_3 \\
0 & 0 & 0 & -s_6 & 0 & s_4 \\
0 & 0 & 0 & 0 & -b s_6 & s_5 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad s_i \in \mathbb{R}, i = 1, \ldots, 6,
\]

\[
M_{16} = \begin{pmatrix}
-s_6 & 0 & 0 & 0 & 0 & s_3 \\
0 & 0 & 2s_5 & -\varepsilon s_6 & \varepsilon s_4 & 2s_2 \\
0 & 0 & 0 & s_5 & 0 & -s_1 \\
0 & 0 & 0 & 0 & s_5 & s_4 \\
0 & 0 & 0 & 0 & 0 & s_6 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad s_i \in \mathbb{R}, \varepsilon = 0, \pm 1, i = 1, \ldots, 6.
\]

This procedure yields the following

**Theorem 3.1.** The simply connected Lie group \( G_i \) and its subgroup \( K_i \) of the Lie algebra \( g_i \) and its subalgebra \( k_i \), \( i = 1, \ldots, 20 \), is isomorphic to the linear group of matrices the multiplication of which is given by:

For \( i = 1 \):

\[
g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + y_1 + x_2 y_3 - x_3 y_2 - x_6 y_5, x_2 + y_2, x_3 + y_3, x_4 + y_4 e^{-x_6}, x_5 + y_5, x_6 + y_6),
\]

\[
K_{1,1} = \{g(u_1, u_3, 0, u_1, u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},
\]

\[
K_{1,2} = \{g(u_1, 0, u_3, u_1, u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},
\]

for \( i = 2 \):

\[
g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + y_1 + x_2 y_3 - x_3 y_2 - x_6 (y_5 + x_2 y_2),
\]

\[
x_2 + y_2, x_3 + y_3 - x_6 y_2, x_4 + y_4 e^{-x_6}, x_5 + y_5, x_6 + y_6),
\]

\[
K_2 = \{g(u_1, 0, u_3, u_1, u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},
\]
for $i = 3$:
\[
g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) \\
= g(x_1 + y_1 - x_6y_4 + (\frac{1}{2}x_6^2 + x_3)y_2, \\
x_2 + y_2, x_3 + y_3, x_4 + y_4 - x_6y_2, x_5 + y_5e^{-x_6}, x_6 + y_6), \\
K_{3,1} = \{g(u_2, u_3, 0, u_1, u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, \\
K_{3,2} = \{g(u_2, 0, u_3, u_1, u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},
\]

for $i = 4$:
\[
g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) \\
= g(x_1 + y_1 + x_5y_4, x_2 + y_2 + x_5y_1 + \varepsilon x_4y_6 + \frac{1}{2}x_6^2y_4, \\
x_3 + y_3e^{-x_6}, x_4 + y_4, x_5 + y_5, x_6 + y_6), \varepsilon = \pm 1, \\
K_4 = \{g(u_1, a_1u_1 + u_2, u_2, u_3, 0, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_1 \in \mathbb{R},
\]

for $i = 5$:
\[
g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) \\
= g(x_1 + (y_1 + x_5y_3)e^{-x_6}, x_2 + y_2 + x_5y_4, x_3 + y_3e^{-x_6}, x_4 + y_4, x_5 + y_5, x_6 + y_6), \\
K_5 = \{g(u_1, u_1 + a_2u_2, u_2, u_3, 0, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_2 \in \mathbb{R},
\]

for $i = 6$:
\[
g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) \\
= g(x_1 + (y_1 + y_3x_5)e^{-x_6}, \\
x_2 + y_2 - (x_5 + x_6)y_4, x_3 + y_3e^{-x_6}, x_4 + y_4, x_5 + y_5, x_6 + y_6), \\
K_6 = \{g(u_1, u_1 + a_2u_2, u_2, u_3, 0, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_2 \in \mathbb{R},
\]

for $i = 7$:
\[
g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) \\
= g(x_1 + (y_1 + y_2x_3)e^{-x_6}, x_2 + y_2e^{-x_6}, x_3 + y_3, x_4 + y_4, x_5 + y_5 - x_4y_6, x_6 + y_6), \\
K_7 = \{g(u_1, u_2, 0, u_3, u_1 + \varepsilon u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, \varepsilon = 0, 1,
\]

for $i = 8$:
\[
g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) \\
= g(x_1 + (y_1 + y_2x_3)e^{-x_6} - y_3x_2, \\
x_2 + y_2e^{-x_6}, x_3 + y_3, x_4 + (y_4 - y_2x_6)e^{-x_6}, x_5 + y_5 - x_6y_3, x_6 + y_6), \\
K_8 = \{g(u_1, u_2, 0, u_3, u_1 + a_2u_2 + a_3u_3, 0); u_i \in \mathbb{R}, i=1, 2, 3\}, a_3 \in \mathbb{R} \setminus \{0\}, a_2 \in \mathbb{R},
\]
for $i = 9$:

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + y_1 + x_2 y_3 - (x_3 + x_2 x_6) y_2, x_2 + y_2,$$

$$x_3 + y_3 - x_6 y_2, x_4 + y_4 - x_6 y_3 + \frac{y_2^2}{x_6} y_2, x_5 + y_5 e^{-x_6}, x_6 + y_6), \quad 0 < |b| \leq 1,$$

$$K_9 = \{g(u_1 + u_2, 0, u_3, u_1, u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$

for $i = 10$:

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + y_1 - 2 x_6 y_4 + \frac{y_2^2}{x_6} y_2 - \frac{1}{3} x_6^2 - x_2 x_6 - x_3) y_2, x_2 + y_2,$$

$$x_3 + y_3 - x_6 y_2, x_4 + y_4 - x_6 y_3 + \frac{y_2^2}{x_6} y_2, x_5 + y_5 e^{-x_6}, x_6 + y_6),$$

$$K_{10} = \{g(u_2, 0, u_3, u_1, u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$

for $i = 11$:

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + y_1 + x_2 y_3 - \frac{1}{3} x_6^2 y_6, x_2 + y_2, x_3 + y_3 - x_2 y_6, x_4 + y_4 e^{-x_6}, x_5 + y_5 e^{-x_6} - x_4 y_6, x_6 + y_6),$$

$$K_{11} = \{g(a_2 u_1 + u_2, 0, u_3, u_1, u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_2 \in \mathbb{R},$$

for $i = 12$:

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + y_1 - x_2 y_3 + y_2 (x_3 + x_2 x_6), x_2 + y_2, x_3 + y_3 - x_6 y_2, x_4 + y_4 e^{-x_6} \cos x_6 + y_5 e^{bx_6} \sin x_6, x_5 - y_4 e^{-bx_6} \sin x_6 + y_5 e^{-bx_6} \cos x_6, x_6 + y_6), b \geq 0,$$

$$K_{12,1} = \{g(u_2, 0, u_3, u_1, u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$

$$K_{12,2} = \{g(u_1 + a_3 u_2, 0, u_3, u_1, u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_3 \in \mathbb{R},$$

for $i = 13$:

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + [y_1 - y_4 x_6 + y_2 (\frac{1}{2} x_6^2 + x_3)] e^{-x_6} - x_2 y_3, x_2 + y_2 e^{-x_6}, x_3 + y_3, x_4 + (y_4 - y_2 x_6) e^{-x_6}, x_5 + y_5 - x_6 y_3, x_6 + y_6),$$

$$K_{13} = \{g(u_1, u_2, 0, u_3, u_1 + a_2 u_2 + a_3 u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_2, a_3 \in \mathbb{R},$$

for $i = 14$:

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + y_1 e^{-x_6} + x_2 y_3, x_2 + y_2 e^{-x_6}, x_3 + y_3, x_4 + y_4 - x_6 y_3, x_5 + y_5 - x_6 y_4 + \frac{y_2^2}{x_6} y_2, x_6 + y_6),$$
\[ K_{14} = \{g(u_1, u_2, 0, u_3, u_1 + a_2 u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_2 \in \mathbb{R}, \]

for \( i = 15: \)
\[ g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + y_1 e^{-x_6} + x_4 y_5, x_2 + (y_2 - 2\varepsilon y_4 x_6 - y_1 x_5) e^{-x_6} + (x_1 - x_4 x_5) y_5, \]
\[ x_3 + y_3 - x_6 y_5, x_4 + y_4 e^{-x_6}, x_5 + y_5, x_6 + y_6), \varepsilon = 0, \pm 1, \]
\[ K_{15} = \{g(u_1, u_2, a_1 u_1 + u_2 + a_3 u_3, u_3, 0, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_1, a_3 \in \mathbb{R}, \]

for \( i = 16: \)
\[ g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + y_1 + x_5 y_4 + \frac{1}{2} x_5^2 y_6, \]
\[ x_2 + y_2 + 2x_5 y_1 + (x_5^2 - \varepsilon x_6) y_4 + (\frac{1}{2} x_5^3 + \varepsilon (x_4 - x_5 x_6)) y_6, \]
\[ x_3 + y_5 e^{-x_6}, x_4 + y_4 + x_5 y_6, x_5 + y_5, x_6 + y_6), \varepsilon = 0, \pm 1, \]
\[ K_{16} = \{g(u_1, u_2, a_1 u_1 + u_2, u_2, u_3, 0, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_1 \in \mathbb{R}, \]

for \( i = 17: \)
\[ g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + (y_1 + x_5 y_3) e^{-x_6}, x_2 + y_2 + x_5 y_4 - \frac{1}{2} x_5^2 y_6, \]
\[ x_3 + y_5 e^{-x_6}, x_4 + y_4 - x_5 y_6, x_5 + y_5, x_6 + y_6), \]
\[ K_{17} = \{g(u_1, u_1 + a_2 u_2, u_2, u_3, 0, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_2 \in \mathbb{R}, \]

for \( i = 18: \)
\[ g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + (y_1 + x_3 y_5) e^{-x_6}, x_2 + y_2 - (x_5 + x_6) y_4 - \frac{1}{2} (x_5 + x_6)^2 y_5, \]
\[ x_3 + y_5 e^{-x_6}, x_4 + y_4 + (x_5 + x_6) y_5, x_5 + y_5, x_6 + y_6), \]
\[ K_{18} = \{g(u_1, u_1 + a_2 u_2, u_2, u_3, 0, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_2 \in \mathbb{R}, \]

for \( i = 19: \)
\[ g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + y_1 e^{-a x_6} + x_3 y_2, x_2 + y_2, x_2 + y_3 e^{-a x_6}, \]
\[ x_4 + y_4 - x_5 y_2, x_5 + y_5 e^{-x_6}, x_6 + y_6), a \in \mathbb{R} \setminus \{0\}, \]
\[ K_{19} = \{g(u_1, 0, u_2, u_1 + a_2 u_2 + u_3, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_2 \in \mathbb{R}, \]

for \( i = 20: \)
\[ g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + (y_1 - x_6 y_5 + y_2 x_3) e^{-x_6}, x_2 + y_2 e^{-x_6}, \]
\[ x_3 + y_3, x_4 + y_4 - x_3 y_6, x_5 + y_5 e^{-x_6}, x_6 + y_6), \]
\[ K_{20} = \{g(u_1, u_2, 0, u_1 + a_2 u_2 + a_3 u_3, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_2, a_3 \in \mathbb{R}. \]
Among the Lie groups in Theorem 3.1 only the group $G_1$ has 2-dimensional commutator subgroup and the groups $G_i$, $i = 2, \ldots, 7$, have 3-dimensional commutator subgroup. We show that among the 6-dimensional solvable indecomposable Lie groups with 5-dimensional nilradical precisely these Lie groups are the multiplication groups of three-dimensional connected simply connected topological loops.

**Proposition 3.2.** There does not exist 3-dimensional connected topological proper loop $L$ such that the Lie algebra $g$ of the multiplication group of $L$ is one of the Lie algebras $g_i$, $i = 8, \ldots, 20$.

**Proof.** If $L$ exists, then there exists its universal covering loop $\tilde{L}$ which is homeomorphic to $\mathbb{R}^3$. The pairs $(G_i, K_i)$ in Theorem 3.1 can occur as the group $\text{Mult}(\tilde{L})$ and the subgroup $\text{Inn}(\tilde{L})$. We show that none of the groups $G_i$, $i = 8, \ldots, 20$, satisfies the condition that there exist continuous left transversals $A$ and $B$ to $K_i$ in $G_i$ such that for all $a \in A$ and $b \in B$ one has $a^{-1}b^{-1}ab \in K_i$. By Proposition 2.1 the groups $G_i$, $i = 8, \ldots, 20$, are not the multiplication group of a loop $L$. Hence no proper loop $\tilde{L}$ exists which yields that also no proper loop $L$ exists. This proves the assertion.

Two arbitrary left transversals to the group $K_i$ in $G_i$ are:

For $i = 9, 10, 11, 12$,

$$A = \{g(u, v, h_1(u, v, w), h_2(u, v, w), h_3(u, v, w), w); u, v, w \in \mathbb{R}\},$$
$$B = \{g(k, l, f_1(k, l, m), f_2(k, l, m), f_3(k, l, m), m); k, l, m \in \mathbb{R}\},$$

for $i = 8, 13, 14, 15$,

$$A = \{g(h_1(u, v, w), h_2(u, v, w), u, h_3(u, v, w), v, w); u, v, w \in \mathbb{R}\},$$
$$B = \{g(f_1(k, l, m), k, f_2(k, l, m), k, f_3(k, l, m), l, m); k, l, m \in \mathbb{R}\},$$

for $i = 16, 17, 18$,

$$A = \{g(h_1(u, v, w), u, h_2(u, v, w), h_3(u, v, w), v, w); u, v, w \in \mathbb{R}\},$$
$$B = \{g(f_1(k, l, m), k, f_2(k, l, m), f_3(k, l, m), l, m); k, l, m \in \mathbb{R}\},$$

for $i = 19$

$$A = \{g(h_1(u, v, w), u, h_2(u, v, w), v, h_3(u, v, w), w); u, v, w \in \mathbb{R}\},$$
$$B = \{g(f_1(k, l, m), k, f_2(k, l, m), l, f_3(k, l, m), m); k, l, m \in \mathbb{R}\},$$

for $i = 20$

$$A = \{g(h_1(u, v, w), h_2(u, v, w), u, v, h_3(u, v, w), w); u, v, w \in \mathbb{R}\},$$
$$B = \{g(f_1(k, l, m), f_2(k, l, m), k, l, f_3(k, l, m), m); k, l, m \in \mathbb{R}\},$$

where $h_i(u, v, w) : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $f_i(k, l, m) : \mathbb{R}^3 \rightarrow \mathbb{R}$, $i = 1, 2, 3$, are continuous functions with $f_i(0, 0, 0) = h_i(0, 0, 0) = 0$. Taking in $G_i$, $i = 9, 11, 12$, the elements

$$a = g(0, v, h_1(0, v, 0), h_2(0, v, 0), h_3(0, v, 0), 0) \in A,$$
\[ b = g(0, 0, f_1(0, 0, m), f_2(0, 0, m), f_3(0, 0, m), m) \in B \]

and in \( G_{17} \) the elements
\[ a = g(h_1(0, v, 0), 0, h_2(0, v, 0), h_3(0, v, 0), v, 0) \in A, \]
\[ b = g(f_1(0, 0, m), 0, f_2(0, 0, m), f_3(0, 0, m), 0, m) \in B \]

one has \( a^{-1}b^{-1}ab \in K_i \) if and only if
for \( i = 9 \)
\[ mv^2 - 2vf_1(0, 0, m) = h_2(0, v, 0)(1 - e^m) + h_3(0, v, 0)(1 - e^{bm}), \quad (3.1) \]
for \( i = 11 \)
\[ \frac{1}{2}mv^2 + vf_1(0, 0, m) = (e^m - 1)(h_3(0, v, 0) + a_2h_2(0, v, 0)) - e^m mh_2(0, v, 0), \quad (3.2) \]
for \( i = 12 \) and for \( K_{12,1} \)
\[ 2vf_1(0, 0, m) - mv^2 = (1 - e^{bm} \cos m)h_3(0, v, 0) - e^{bm} \sin mh_2(0, v, 0), \quad (3.3) \]
for \( i = 12 \) and for \( K_{12,2} \)
\[ 2vf_1(0, 0, m) - mv^2 = (1 - e^{bm} \cos m)(h_2(0, v, 0) + a_3h_3(0, v, 0)) + e^{bm} \sin m(h_3(0, v, 0) - a_3h_2(0, v, 0)), \quad (3.4) \]
for \( i = 17 \)
\[ -\frac{1}{2}mv^2 - vf_3(0, 0, m) = (1 - e^m)[h_1(0, v, 0) + (a_2 - v)h_2(0, v, 0)] - e^m vf_2(0, 0, m) \quad (3.5) \]
is satisfied for all \( m, v \in \mathbb{R} \). On the left hand side of equations (3.1), (3.2), (3.3), (3.4), (3.5) is the term \( mv^2 \) hence there does not exist any function \( f_i(0, 0, m) \) and \( h_i(0, v, 0) \), \( i = 1, 2, 3 \), satisfying these equations. Taking in \( G_{10} \) the elements
\[ a = g(0, v, h_1(0, v, w), h_2(0, v, w), h_3(0, v, w), w) \in A \]
\[ b = g(0, 0, f_1(0, 0, m), f_2(0, 0, m), f_3(0, 0, m), m) \in B, \]
respectively in \( G_{18} \) the elements
\[ a = g(h_1(0, v, w), 0, h_2(0, v, w), h_3(0, v, w), v, w) \in A, \]
\[ b = g(f_1(0, 0, m), 0, f_2(0, 0, m), f_3(0, 0, m), 0, m) \in B, \]
respectively in \( G_{16} \) the elements
\[ a = g(h_1(0, v, 0), 0, h_2(0, v, 0), h_3(0, v, 0), v, 0) \in A, \]
\[ b = g(f_1(0, l, m), 0, f_2(0, l, m), f_3(0, l, m), l, m) \in B \]
we obtain that $a^{-1}b^{-1}ab \in K_i$ if and only if in case $i = 10$ the equation
\[
e^w (1 - e^m)h_3(0, v, w) + e^m (e^w - 1)f_3(0, 0, m) \\
= (w^2 + 2v + 2mw)f_1(0, 0, m) + 2wf_2(0, 0, m) \\
- (m^2 + 2wm)h_1(0, v, w) - 2mh_2(0, v, w) \\
- m^2wv - w^2mv - mv^2 - \frac{1}{3}vm^3,
\]
respectively in case $i = 18$ the equation
\[
e^m (e^w - 1)(f_1(0, 0, m) + a_2 f_2(0, 0, m)) \\
+ e^w(1 - e^m)[h_1(0, v, w) + (a_2 - v)h_2(0, v, w)] \\
= e^{m+w}vf_2(0, 0, m) + (w + v)f_3(0, 0, m) \\
- mh_3(0, v, w) + v^2m + \frac{1}{2}m^2v + wvm,
\]
respectively in case $i = 16$ the equation
\[
- \frac{1}{3}v^3m - v^2lm - l^2vm - \frac{1}{2}a_1v^2m - \varepsilon m^2v - a_1vlm \\
= (1 - e^m)h_2(0, v, 0) - 2lh_1(0, v, 0) + (l^2 + 2vl + a_1l + 2\varepsilon m)h_3(0, v, 0) \\
+ 2vf_1(0, l, m) - (v^2 + 2vl + a_1v)f_3(0, l, m)
\]
holds for all $m, l, v, w \in \mathbb{R}$. Substituting into (3.6)
\[
f_2(0, 0, m) = f'_2(0, 0, m) - mf_1(0, 0, m), h_2(0, v, w) = h'_2(0, v, w) - wh_1(0, v, w),
\]
respectively into (3.7)
\[
f_1(0, 0, m) = f'_1(0, 0, m) - a_2f_2(0, 0, m), h_1(0, v, w) = h'_1(0, v, w) + (v - a_2)h_2(0, v, w),
\]
respectively into (3.8)
\[
h_1(0, v, 0) = h'_1(0, v, 0) + (v + \frac{1}{2}a_1)h_3(0, v, 0), \\
f_1(0, l, m) = f'_1(0, l, m) + (l + \frac{1}{2}a_1)f_3(0, l, m),
\]
we get in case $i = 10$
\[
e^w(1 - e^m)h_3(0, v, w) + e^m (e^w - 1)f_3(0, 0, m) \\
= (w^2 + 2v)f_1(0, 0, m) - m^2h_1(0, v, w) + 2wf_2(0, 0, m) \\
- 2mh_2(0, v, w) - m^2wv - w^2mv - mv^2 - \frac{1}{3}vm^3,
\]
respectively in case $i = 18$
\[
e^m (e^w - 1)f'_1(0, 0, m) - e^{m+w}vf_2(0, 0, m) + e^w(1 - e^m)h'_1(0, v, w) \\
= (w + v)f_3(0, 0, m) - mh_3(0, v, w) + v^2m + \frac{1}{2}m^2v + wvm,
\]
respectively in case $i = 16$
\[
(1 - e^m)h_2(0, v, 0) + (l^2 + 2\varepsilon m)h_3(0, v, 0) - v^2f_3(0, l, m) - 2lh'_1(0, v, 0) + 2vf'_1(0, l, m) = \ldots + a_2h_2(0, 0, w) + e^m(e^w - 1)(f_3(k, 0, m) + a_2f_2(k, 0, m)) - e^m + wk h_2(0, 0, w), \tag{3.14}
\]

Since on the right hand side of (3.9), respectively (3.10), respectively (3.11) there is the term $-\frac{1}{3}v^2m^3$, respectively $\frac{1}{2}m^2v$, respectively $-\frac{1}{3}v^3m$ there does not exist any function $f_i(0, 0, m)$ and $h_i(0, v, w)$, $i = 1, 2, 3$, respectively $f_i(0, l, m)$, $i = 1, 3$, and $h_j(0, v, 0)$, $j = 1, 2, 3$, satisfying equation (3.9), respectively (3.10), respectively (3.11).

Taking in $G_i$, $i = 8, 13, 14$, the elements
\[
a = g(h_1(0, 0, w), h_2(0, 0, w), 0, h_3(0, 0, w), 0, w) \in A,
b = g(f_1(k, 0, m), f_2(k, 0, m), k, f_3(k, 0, m), 0, m) \in B,
\]
respectively in $G_{19}$ the elements
\[
a = g(h_1(0, 0, w), h_2(0, 0, w), 0, h_3(0, 0, w), w) \in A,
b = g(f_1(k, 0, m), f_2(k, 0, m), k, f_3(k, 0, m), m) \in B,
\]
respectively in $G_{20}$ the elements
\[
a = g(h_1(0, 0, w), h_2(0, 0, w), 0, h_3(0, 0, w), w) \in A,
b = g(f_1(k, 0, m), f_2(k, 0, m), k, f_3(k, 0, m), m) \in B
\]
we have $a^{-1}b^{-1}ab \in K_i$ precisely if for $i = 8$ the equation
\[
wk = e^w(1 - e^m)[(a_2 + a_3w)h_2(0, 0, w) + a_3h_3(0, 0, w) + h_1(0, 0, w)] + e^m(e^w - 1)[(a_3m + a_2 - k)f_2(k, 0, m) + a_3f_3(k, 0, m) + f_1(k, 0, m)]
\]
\[
+ e^{m+w}[a_3wf_2(k, 0, m) + (2k - a_3m)h_2(0, 0, w)], \tag{3.12}
\]
for $i = 13$ the equation
\[
wk = e^w(1 - e^m)[(\frac{1}{2}w^2 + a_2 + a_3w)h_2(0, 0, w) + (a_3 + w)h_3(0, 0, w) + h_1(0, 0, w)]
\]
\[
+ e^m(e^w - 1)[(\frac{1}{2}m^2 - k + a_3m + a_2)f_2(k, 0, m)
\]
\[
+ (m + a_3)f_3(k, 0, m) + f_1(k, 0, m)] + e^{m+w}[(m + a_3)w + \frac{1}{2}w^2)f_2(k, 0, m) + (2k - \frac{1}{2}m^2 - (w + a_3)m)h_2(0, 0, w)
\]
\[
+ e^{m+w}(wf_3(k, 0, m) - mh_3(0, 0, w)), \tag{3.13}
\]
for $i = 14$ the equation
\[
\frac{1}{2}w^2k + mwk + wf_3(k, 0, m) - mh_3(0, 0, w)
\]
\[
= e^w(1 - e^m)(h_1(0, 0, w) + a_2h_2(0, 0, w))
\]
\[
+ e^m(e^w - 1)(f_1(k, 0, m) + a_2f_2(k, 0, m)) - e^{m+w}kh_2(0, 0, w), \tag{3.14}
\]
for $i = 19$ the equation

$$w^k = e^w(1 - e^m)h_3(0, 0, w) - e^m(1 - e^w)f_3(k, 0, m) - e^{a(m+w)}kh_2(0, 0, w)$$
$$+ e^{aw}(1 - e^{am})(h_1(0, 0, w) + a_2h_2(0, 0, w))$$
$$- e^{am}(1 - e^{aw})(f_1(k, 0, m) + a_2f_2(k, 0, m)), \tag{3.15}$$

for $i = 20$ the equation

$$-w^k = e^w(1 - e^m)(h_1(0, 0, w) + a_2h_2(0, 0, w) + (w + a_3)h_3(0, 0, w))$$
$$+ e^m(1 - e^w)((k - a_2)f_2(k, 0, m) - f_1(k, 0, m) - (m + a_3)f_3(k, 0, m))$$
$$+ e^{m+w}(kh_2(0, 0, w) - mh_3(0, 0, w) + wf_3(k, 0, m)) \tag{3.16}$$

is satisfied for all $k, m, w \in \mathbb{R}, a_2, a_3 \in \mathbb{R}$. Putting into (3.12)

$$h_1(0, 0, w) = h'_1(0, 0, w) - (a_3w + a_2)h_2(0, 0, w) - a_3h_3(0, 0, w),$$
$$f_1(k, 0, m) = f'_1(k, 0, m) + (k - a_3m - a_2)f_2(k, 0, m) - a_3f_3(k, 0, m),$$

respectively into (3.13)

$$h_1(0, 0, w) = h'_1(0, 0, w) - (\frac{1}{2}w^2 + a_3w + a_2)h_2(0, 0, w) - (a_3 + w)h_3(0, 0, w),$$
$$f_1(k, 0, m) = f'_1(k, 0, m) + (k - \frac{1}{2}m^2 - a_3m - a_2)f_2(k, 0, m) - (m + a_3)f_3(k, 0, m),$$
$$f_3(k, 0, m) = f'_3(k, 0, m) - (m + a_3)f_2(k, 0, m),$$
$$h_3(0, 0, w) = h'_3(0, 0, w) - (w + a_3)h_2(0, 0, w),$$

respectively into (3.14)

$$h_1(0, 0, w) = h'_1(0, 0, w) - a_2h_2(0, 0, w),$$
$$f_3(k, 0, m) = f'_3(k, 0, m) - mk,$$
$$f_1(k, 0, m) = f'_1(k, 0, m) - a_2f_2(k, 0, m),$$

respectively into (3.15)

$$h_1(0, 0, w) = h'_1(0, 0, w) - a_2h_2(0, 0, w),$$
$$f_1(k, 0, m) = f'_1(k, 0, m) - a_2f_2(k, 0, m),$$

respectively into (3.16)

$$h_1(0, 0, w) = h'_1(0, 0, w) - a_2h_2(0, 0, w) - (w + a_3)h_3(0, 0, w),$$
$$f_1(k, 0, m) = f'_1(k, 0, m) + (k - a_2)f_2(k, 0, m) - (m + a_3)f_3(k, 0, m)$$

in order equations (3.12), (3.13), (3.14), (3.15), (3.16) reduce in case $i = 8$ to

$$w^k = e^w(1 - e^m)h'_1(0, 0, w) + e^m(e^w - 1)f'_1(k, 0, m)$$
$$+ e^{m+w}[a_3wf_2(k, 0, m) + (2k - a_3m)h_2(0, 0, w)], \tag{3.17}$$
in case \( i = 13 \) to
\[
wk = e^w(1 - e^m)h_1'(0,0,w) + e^m(e^w - 1)f_1'(k,0,m) + e^{m+w}\left[\frac{1}{2}w^2f_2(k,0,m) + (2k - \frac{1}{2}m^2)h_2(0,0,w) + wf_3'(k,0,m) - mh_3'(0,0,w)\right],
\]
(3.18)

in case \( i = 14 \) to
\[
\frac{1}{2}w^2k + wf_3'(k,0,m) - mh_3(0,0,w) = e^w(1 - e^m)h_1'(0,0,w) + e^m(e^w - 1)f_1'(k,0,m) - e^{m+w}kh_2(0,0,w),
\]
(3.19)

in case \( i = 19 \) to
\[
wk = e^w(1 - e^m)h_3(0,0,w) - e^m(1 - e^w)f_3(k,0,m) - e^{a(m+w)}kh_2(0,0,w) + e^{aw}(1 - e^{am})f_1'(k,0,m),
\]
(3.20)

and in case \( i = 20 \) to
\[
-wk = e^w(1 - e^m)h_1'(0,0,w) + e^m(e^w - 1)f_1'(k,0,m) + e^{m+w}(kh_2(0,0,w) - mh_3(0,0,w) + wf_3(k,0,m)).
\]
(3.21)

Since on the left hand side of (3.17), (3.18), (3.20), (3.21), respectively of (3.19) is the term \( wk \), respectively \( \frac{1}{2}w^2k \) there does not exist any function \( f_i(k,0,m) \), \( h_i(0,0,w) \), \( i = 1,2,3 \), satisfying equation (3.17), (3.18), (3.20), (3.21), respectively (3.19).

Taking in \( G_{15} \) the elements
\[
a = g(h_1(0,0,w), h_2(0,0,w), 0, h_3(0,0,w), 0, w) \in A,
b = g(f_1(0,l,m), f_2(0,l,m), 0, f_3(0,l,m), l, m) \in B
\]
the product \( a^{-1}b^{-1}ab \) lies in \( K_{15} \) if and only if the equation
\[
wl = e^w(1 - e^m)[h_2(0,0,w) + (a_3 + 2w\epsilon)h_3(0,0,w) + a_1h_1(0,0,w)] + e^m(e^w - 1)[f_2(0,l,m) + (l + a_1)f_1(0,l,m) + (a_3 + 2m\epsilon)f_3(0,l,m)] + e^{m+w}[2w\epsilon f_3(0,l,m) - 2h_1(0,0,w) - (l^2 + 2m\epsilon + a_1l)h_3(0,0,w)]
\]
(3.22)
is satisfied for all \( m, l, w \in \mathbb{R} \). Substituting into (3.22)
\[
h_1(0,0,w) = h_1'(0,0,w) - \frac{1}{2}a_1h_3(0,0,w),
h_2(0,0,w) = h_2'(0,0,w) - a_1h_1(0,0,w) - (a_3 + 2w\epsilon)h_3(0,0,w),
f_2(0,l,m) = f_2'(0,l,m) - (l + a_1)f_1(0,l,m) - (a_3 + 2m\epsilon)f_3(0,l,m),
\]
we obtain
\[
wl = e^w(1 - e^m)h_2'(0,0,w) + e^m(e^w - 1)f_2'(0,l,m)
\]
\[ + e^{m+w}[2w\varepsilon f_3(0, l, m) - 2l h'_1(0, 0, w) - (l^2 + 2m\varepsilon)h_3(0, 0, w)]. \] (3.23)

On the left hand side of equation (3.23) is the term \(wl\) hence there does not exist any function \(f_i(0, l, m), i = 2, 3,\) and \(h_j(0, 0, w), j = 1, 2, 3\) such that equation (3.23) holds.

**Theorem 3.3.** Let \(L\) be a connected simply connected topological proper loop of dimension 3 such that its multiplication group is a 6-dimensional solvable indecomposable Lie group having 5-dimensional nilradical. Then the pairs of Lie groups \((G_i, K_i), i = 1, \ldots, 7,\) are the multiplication groups \(\text{Mult}(L)\) and the inner mapping groups \(\text{Inn}(L)\) of \(L.\)

**Proof.** The sets

\[ A = \{g(k, 1 - e^m, l, me^{-m}, 2l, m); k, l, m \in \mathbb{R}\}, \]

\[ B = \{g(u, w, v, 2ve^{-w}, 1 - e^w, w); u, v, w \in \mathbb{R}\}, \]

respectively

\[ C = \{g(k, 1 - e^m, me^{-m}, -2l, m); k, l, m \in \mathbb{R}\}, \]

\[ D = \{g(u, v, w, -2ve^{-w}, 1 - e^w, w); u, v, w \in \mathbb{R}\} \]

are \(K_{1,1^{-}}\), respectively \(K_{1,2^{-}}\)-connected left transversals in \(G_1.\) The sets

\[ A = \{g(k, l, l, me^{-m}, l^2 - 1 + e^m, m); k, l, m \in \mathbb{R}\}, \]

\[ B = \{g(u, v, v, -ve^{-w}, v^2 + 1 - e^w, w); u, v, w \in \mathbb{R}\} \]

are \(K_{2^{-}}\)-connected left transversals in \(G_2.\) The sets

\[ A = \{g(k, l, \frac{1}{2}m^2 + l, e^m - 1 - m(\frac{1}{2}m^2 - l), me^{-m}, m); k, l, m \in \mathbb{R}\}, \]

\[ B = \{g(u, \frac{1}{2}w^2 - v, v, 1 - e^w - w(\frac{1}{2}w^2 - v), -we^{-w}, w); u, v, w \in \mathbb{R}\}, \]

respectively

\[ C = \{g(k, l, \frac{1}{2}m^2 + e^m - 1, -lm + m, le^{-m}, m); k, l, m \in \mathbb{R}\}, \]

\[ D = \{g(u, v, \frac{1}{2}w^2 - e^w + 1, -vw + w, -ve^{-w}, w); u, v, w \in \mathbb{R}\} \]

are \(K_{3,1^{-}}\), respectively \(K_{3,2^{-}}\)-connected left transversals in \(G_3.\) The sets

\[ A = \{g((l + a_1)(1 - e^m) + l, k, -e^{-m}(\frac{1}{2}l^2 + m), 1 - e^m, l, m); k, l, m \in \mathbb{R}\}, \]

\[ B = \{g((v + a_1)(e^w - 1) + v, u, e^{-u}(\frac{1}{2}v^2 + m), e^w - 1, v, w); u, v, w \in \mathbb{R}\} \]

are \(K_{4^{-}}\)-connected left transversals in \(G_4.\) The sets

\[ A = \{g(le^{-k}(a_2 - l + 1), m, -le^{-k}, 1 - le^k - e^k, l, k); k, l, m \in \mathbb{R}\}, \]

\[ B = \{g(ve^{-u}(v - 1 - a_2), w, ve^{-u}, ve^u + e^u - 1, v, w); u, v, w \in \mathbb{R}\} \]
are $K_5$-connected left transversals in $G_5$. The sets
\[
A = \{g((l-a_2)l+(l+m)e^{-m},k,l,e^m-1,l,m); k,l,m \in \mathbb{R}\},
\]
\[
B = \{g((v-a_2)v-(v+w)e^{-w},u,v,1-e^w,v,w); u,v,w \in \mathbb{R}\}
\]
are $K_6$-connected left transversals in $G_6$. The sets
\[
A = \{g((\varepsilon-k)me^{-m},-me^{-m},k,-ke^m,l,m),k,l,m \in \mathbb{R}\},
\]
\[
B = \{g((u-\varepsilon)we^{-w},we^{-w},u,ue^w,v,w),u,v,w \in \mathbb{R}\}
\]
are $K_7$-connected left transversals in $G_7$. For all $i = 1, \ldots, 7$, the sets $A, B$, respectively $C, D$ generate the group $G_i$. According to Proposition 2.1 the pairs $(G_i,K_i)$, $i = 1, \ldots, 7$, are multiplication groups and inner mapping groups of $L$ which proves the assertion.

**Corollary 3.4.** Each 3-dimensional connected topological proper loop $L$ having a solvable indecomposable Lie group of dimension 6 as the group $\text{Mult}(L)$ of $L$ has 1-dimensional centre and 2- or 3-dimensional commutator subgroup.

**Proof.** If $L$ has a 6-dimensional indecomposable nilpotent Lie group as its multiplication group, then the assertion follows from case b) of Theorem in [6]. If it has a 6-dimensional indecomposable solvable Lie group with 4-dimensional nilradical, then the assertion is proved in Theorem 16 in [4]. If it has a 6-dimensional indecomposable solvable Lie group with 5-dimensional nilradical, then Theorems 3.6 and 3.7 in [5] and Theorem 3.3 give the assertion.

**References**
