

# Surfaces with constant extrinsically Gaussian curvature in the Heisenberg group

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## Abstract

In this work we study constant extrinsically Gaussian curvature translation surfaces in the 3-dimensional Heisenberg group which are invariant under the 1-parameter groups of isometries.

*Keywords:* Constant extrinsically Gaussian curvature Surfaces, Homogeneous group.

*MSC:* 49Q20 53C22.

## 1. Introduction

In 1982, W. P. Thurston formulated a geometric conjecture for three dimensional manifolds, namely every compact orientable 3-manifold admits a canonical decomposition into pieces, each of them having a canonical geometric structure from the following eight maximal and simply connected homogenous Riemannian spaces:  $\mathbb{E}^3$ ,  $\mathbb{S}^3$ ,  $\mathbb{H}^3$ ,  $\mathbb{S}^2 \times \mathbb{R}$ ,  $\mathbb{H}^2 \times \mathbb{R}$ ,  $SL(2, \mathbb{R})$ ,  $\mathbb{H}_3$  and  $Sol_3$ . See e.g. [34].

During the recent years, there has been a rapidly growing interest in the geometry of surfaces in three homogenous spaces focusing on flat and constant Gaussian curvature surfaces. Many works are studying the geometry of surfaces in homogeneous 3-manifolds. See for example [2–4, 9, 12, 14–16, 21, 22, 24, 36].

The concept of translation surfaces in  $\mathbb{R}^3$  can be generalized the surfaces in the three dimensional Lie group, in particular, homogeneous manifolds. In Euclidean

3-space, every cylinder is flat. Conversely, complete flat surfaces in  $\mathbb{E}^3$  are cylinders over complete curves. See [20]. López and Munteanu [17] studied invariant surfaces with constant mean curvature and constant Gaussian curvature in  $Sol_3$  space. Yoon and Lee [37] studied translation surfaces in Heisenberg group  $\mathbb{H}_3$  whose position vector  $x$  satisfies the equation  $\Delta x = Ax$ , where  $\Delta$  is the Laplacian operator of the surface and  $A$  is a  $3 \times 3$ -real matrix.

Flat  $G_4$ -invariant surfaces are nothing but surfaces invariant under  $SO(2)$ -action, i.e. rotational surfaces. Flat rotational surfaces are classified by Caddeo, Piu and Ratto in [8].

In [14], J. I. Inoguchi give a classification of intrinsically flat  $G_1$ -invariant translation surfaces in Heisenberg group  $\mathbb{H}_3$ . Let  $M$  be a surface invariant under  $G_3$ , then  $M$  is locally expressed as

$$X(u, v) = (0, 0, v). (x(u), y(u), 0) = (x(u), y(u), v), \quad u \in I, \quad v \in \mathbb{R}.$$

Here  $I$  is an open interval and  $u$  is the arclength parameter. Note that  $(x, y, 0)$  and  $(0, 0, v)$  commute. Then the sectional curvature  $K(X_x \wedge X_y) = \frac{1}{4}$  and the extrinsically Gaussian curvature  $K_{ext} = -\frac{1}{4}$ . Direct computation show that  $M$  is flat. (cf. [12–14, 28]).

The paper is divided according the type of surfaces invariant under 1-parameter subgroups of isometries  $\{G_i\}_{i=1,2,3,4}$ . So, in section 3 we classify  $G_1$ -invariant surfaces of the Heisenberg group  $\mathbb{H}_3$  with constant extrinsically Gaussian curvature  $K_{ext}$ , including extrinsically flat  $G_1$ -invariant surfaces.

In section 4 we classify  $G_2$ -invariant surfaces of the Heisenberg group  $\mathbb{H}_3$  with constant extrinsically Gaussian curvature  $K_{ext}$ , including extrinsically flat  $G_2$ -invariant surfaces.

## 2. Preliminaries

The 3-dimensional Heisenberg group  $\mathbb{H}_3$  is the simply connected and connected 2-step nilpotent Lie group. Which has the following standard representation in  $GL(3, \mathbb{R})$

$$\begin{pmatrix} 1 & r & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$$

with  $r, s, t \in \mathbb{R}$ . The Lie algebra  $\mathfrak{h}_3$  of  $\mathbb{H}_3$  is given by the matrices

$$A = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$$

with  $x, y, z \in \mathbb{R}$ . The exponential map  $\exp : \mathfrak{h}_3 \rightarrow \mathbb{H}_3$  is a global diffeomorphism, and is given by

$$\exp(A) = I + A + \frac{A^2}{2} = \begin{pmatrix} 1 & x & z + \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

The Heisenberg group  $\mathbb{H}_3$  is represented as the cartesian 3-space  $\mathbb{R}^3(x, y, z)$  with group structure:

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) := \left( x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}x_1y_2 - \frac{1}{2}x_2y_1 \right).$$

We equip  $\mathbb{H}_3$  with the following left invariant Riemannian metric

$$g := dx^2 + dy^2 + \left( dz + \frac{1}{2}(ydx - xdy) \right)^2.$$

The identity component  $I^\circ(\mathbb{H}_3)$  of the full isometry group of  $(\mathbb{H}_3, g)$  is the semi-direct product  $SO(2) \ltimes \mathbb{H}_3$ . The action of  $SO(2) \ltimes \mathbb{H}_3$  is given explicitly by

$$\begin{aligned} A &= \left( \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ \frac{1}{2}(a \sin \theta - b \cos \theta) & \frac{1}{2}(a \cos \theta + b \sin \theta) & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \end{aligned}$$

In particular, rotational around the  $z$ -axis and translations:

$$(x, y, z) \rightarrow (x, y, z + a), a \in \mathbb{R}$$

along the  $z$ -axis are isometries of  $\mathbb{H}_3$ .

The Lie algebra  $\mathfrak{h}_3$  of  $I^\circ(\mathbb{H}_3)$  is generated by the following Killing vector fields:

$$\begin{aligned} F_1 &= \frac{\partial}{\partial x} + \frac{y}{2} \frac{\partial}{\partial z}, & F_2 &= \frac{\partial}{\partial y} - \frac{x}{2} \frac{\partial}{\partial z}, \\ F_3 &= \frac{\partial}{\partial z}, & F_4 &= -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}. \end{aligned}$$

One can check that  $F_1, F_2, F_3$  are infinitesimal transformations of the 1-parameter groups of isometries defined by

$$G_1 = \{(t, 0, 0) | t \in \mathbb{R}\}, \quad G_2 = \{(0, t, 0) | t \in \mathbb{R}\}, \quad G_3 = \{(0, 0, t) | t \in \mathbb{R}\},$$

respectively. Here this groups acts on  $\mathbb{H}_3$  by the left translation. The vector field  $F_4$  generates the group of rotations around the  $z$ -axis. Thus  $G_4$  is identified with  $SO(2)$ .

**Definition 2.1.** A surface  $\Sigma$  in the Heisenberg space  $\mathbb{H}_3$  is said to be invariant surface if it is invariant under the action of the 1-parameter subgroups of isometries  $\{G_i\}$ , with  $i \in \{1, 2, 3, 4\}$ .

The Lie algebra  $\mathfrak{h}_3$  of  $\mathbb{H}_3$  has an orthonormal basis  $\{E_1, E_2, E_3\}$  defined by

$$E_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}, \quad E_2 = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}, \quad E_3 = \frac{\partial}{\partial z}.$$

The Levi-Civita connection  $\nabla$  of  $g$ , in terms of the basis  $\{E_i\}_{i=1,2,3}$  is explicitly given as follows

$$\begin{cases} \nabla_{E_1} E_1 = 0, \nabla_{E_1} E_2 = \frac{1}{2} E_3, \nabla_{E_1} E_3 = -\frac{1}{2} E_2 \\ \nabla_{E_2} E_1 = -\frac{1}{2} E_3, \nabla_{E_2} E_2 = 0, \nabla_{E_2} E_3 = \frac{1}{2} E_1 \\ \nabla_{E_3} E_1 = -\frac{1}{2} E_2, \nabla_{E_3} E_2 = \frac{1}{2} E_1, \nabla_{E_3} E_3 = 0 \end{cases}$$

The Riemannian curvature tensor  $R$  is a tensor field on  $\mathbb{H}_3$  defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

The components  $\{R_{ijk}^l\}$  are computed as

$$R_{212}^1 = -\frac{3}{4}, \quad R_{313}^1 = \frac{1}{4}, \quad R_{323}^2 = \frac{1}{4}.$$

Let us denote  $K_{ij} = K(E_i, E_j)$  the sectional curvature of the plane spanned by  $E_i$  and  $E_j$ . Then we get easily the following:

$$K_{12} = -\frac{3}{4}, \quad K_{13} = -\frac{1}{4}, \quad K_{23} = -\frac{1}{4}.$$

The Ricci curvature  $Ric$  is defined by

$$Ric(X, Y) = trace\{Z \rightarrow R(Z, X)Y\}.$$

The components  $\{R_{ij}\}$  of  $Ric$  are defined by

$$Ric(E_i, E_j) = R_{ij} = \sum_{k=1}^3 \langle R(E_i, E_k)E_k, E_j \rangle.$$

The components  $\{R_{ij}\}$  are computed as

$$R_{11} = -\frac{1}{2}, \quad R_{12} = R_{13} = R_{23} = 0, \quad R_{22} = -\frac{1}{2}, \quad R_{33} = \frac{1}{2}.$$

The scalar curvature  $S$  of  $\mathbb{H}_3$  is constant and we have

$$S = tr Ric = \sum_{i=1}^3 Ric(E_i, E_i) = -\frac{1}{2}.$$

### 3. Constant extrinsically Gaussian curvature $G_1$ -invariant translation surfaces in Heisenberg group $\mathbb{H}_3$

#### 3.1.

In this subsection we study complete extrinsically flat translation surfaces  $\Sigma$  in Heisenberg group  $\mathbb{H}_3$  which are invariant under the one parameter subgroup  $G_1$ . Clearly, such a surface is generated by a curve  $\gamma$  in the totally geodesic plane  $\{x = 0\}$ . Discarding the trivial case of a vertical plane  $\{y = y_0\}$ . Thus  $\gamma$  is given by  $\gamma(y) = (0, y, v(y))$ . Therefore the generated surface is parameterized by

$$X(x, y) = (x, 0, 0) \cdot (0, y, v(y)) = (x, y, v(y) + \frac{xy}{2}), \quad (x, y) \in \mathbb{R}^2.$$

We have an orthogonal pair of vector fields on  $(\Sigma)$ , namely,

$$e_1 := X_x = (1, 0, \frac{y}{2}) = E_1 + yE_3.$$

and

$$e_2 := X_y = (0, 1, v' + \frac{x}{2}) = E_2 + v'E_3.$$

The coefficients of the first fundamental form are:

$$E = \langle e_1, e_1 \rangle = 1 + y^2, \quad F = \langle e_1, e_2 \rangle = yv', \quad G = \langle e_2, e_2 \rangle = 1 + v'^2.$$

As a unit normal field we can take

$$N = \frac{-y}{\sqrt{1 + y^2 + v'^2}}E_1 - \frac{v'}{\sqrt{1 + y^2 + v'^2}}E_2 + \frac{1}{\sqrt{1 + y^2 + v'^2}}E_3$$

The covariant derivatives are

$$\begin{aligned} \tilde{\nabla}_{e_1} e_1 &= -yE_2 \\ \tilde{\nabla}_{e_1} e_2 &= \frac{y}{2}E_1 - \frac{v'}{2}E_2 + \frac{1}{2}E_3 \\ \tilde{\nabla}_{e_2} e_2 &= v'E_1 + v''E_3. \end{aligned}$$

The coefficients of the second fundamental form are

$$\begin{aligned} l &= \langle \tilde{\nabla}_{e_1} e_1, N \rangle = \frac{yv'}{\sqrt{1 + y^2 + v'^2}} \\ m &= \langle \tilde{\nabla}_{e_1} e_2, N \rangle = \frac{-\frac{y^2}{2} + \frac{v'^2}{2} + \frac{1}{2}}{\sqrt{1 + y^2 + v'^2}} \\ n &= \langle \tilde{\nabla}_{e_2} e_2, N \rangle = \frac{-yv' + v''}{\sqrt{1 + y^2 + v'^2}}. \end{aligned}$$

Let  $K_{ext}$  be the extrinsic Gauss curvature of  $\Sigma$ ,

$$K_{ext} = \frac{ln - m^2}{EG - F^2} = \frac{-y^2v'^2 + yv'v'' - \left(-\frac{y^2}{2} + \frac{v'^2}{2} + \frac{1}{2}\right)^2}{(1 + y^2 + v'^2)^2}.$$

Thus  $\Sigma$  is extrinsically flat invariant surface in Heisenberg group  $\mathbb{H}_3$  if and only if

$$K_{ext} = 0,$$

that is, if and only if

$$-y^2v'^2 + yv'v'' - \left(-\frac{y^2}{2} + \frac{v'^2}{2} + \frac{1}{2}\right)^2 = 0 \quad (3.1)$$

to classify extrinsically flat invariant surfaces must solve the equation (3.1). We can writes equation (3.1) as

$$y^2 + yv'v'' - \left(\frac{y^2}{2} + \frac{v'^2}{2} + \frac{1}{2}\right)^2 = 0 \quad (3.2)$$

we assume that  $z = \frac{y^2}{2} + \frac{v'^2}{2} + \frac{1}{2}$ . Then

$$\begin{cases} z' = y + v'v'' \\ v'v'' = z' - y \\ v'^2 = 2z - y^2 - 1. \end{cases} \quad (3.3)$$

Therefore equation (3.2) becomes

$$yz' - z^2 = 0. \quad (3.4)$$

equation (3.4) implies that

$$-\frac{z'}{z^2} = -\frac{1}{y}. \quad (3.5)$$

and equation (3.5) implies that

$$z = \frac{1}{-\ln(y) + \alpha}. \quad (3.6)$$

where  $\alpha \in \mathbb{R}$ , and if  $y \neq e^\alpha$ .

From (3.3) and (3.6), we have

$$\begin{aligned} v'^2 &= 2z - y^2 - 1 \\ &= \frac{2}{-\ln(y) + \alpha} - y^2 - 1. \end{aligned}$$

Thus

$$v' = \sqrt{\frac{2}{-\ln(y) + \alpha} - y^2 - 1}.$$

As conclusion, we have

**Theorem 3.1.** • *The only non-extendable extrinsically flat translation surfaces in the 3-dimensional Heisenberg group  $\mathbb{H}_3$  invariant under the 1-parameter subgroup  $G_1 = \{(t, 0, 0) \in \mathbb{H}_3 / t \in \mathbb{R}\}$ , are the surfaces whose parametrization is  $X(x, y) = (x, y, v(y) + \frac{xy}{2})$  where  $y$  and  $v$  satisfy*

$$v(y) = \int \sqrt{\frac{2}{-\ln(y) + \alpha} - y^2 - 1} dy.$$

where  $\alpha \in \mathbb{R}$ , and  $y \neq e^\alpha$ .

• *There are no complete extrinsically flat translation surfaces in the 3-dimensional Heisenberg group  $\mathbb{H}_3$  invariant under the 1-parameter subgroup  $G_1 = \{(t, 0, 0) \in \mathbb{H}_3 / t \in \mathbb{R}\}$ .*

*Remark 3.2.* Let  $\Sigma$  be a  $G_1$ -invariant translation surfaces in the 3-dimensional Heisenberg space. Then  $\Sigma$  is locally expressed as

$$X(x, y) = (0, y, v(y)) \cdot (x, 0, 0) = \left(x, y, v(y) - \frac{xy}{2}\right).$$

Then the extrinsically Gaussian curvature  $K_{ext}$  of  $\Sigma$  is computed as

$$K_{ext} = \frac{((v' - x)^2 - 1)^2}{4(1 + (v' - x)^2)^2}.$$

Thus  $\Sigma$  can not be of constant extrinsically Gaussian curvature.

### 3.2.

In this subsection we study complete constant extrinsically Gaussian curvature translation surfaces  $\Sigma$  in Heisenberg group  $\mathbb{H}_3$  which are invariant under the one parameter subgroup  $G_1$ . Clearly, such a surface is generated by a curve  $\gamma$  in the totally geodesic plane  $\{x = 0\}$ . Discarding the trivial case of a vertical plane  $\{y = y_0\}$ . Thus  $\gamma$  is given by  $\gamma(y) = (0, y, v(y))$ . Therefore the generated surface is parameterized by

$$X(x, y) = (x, 0, 0) \cdot (0, y, v(y)) = \left(x, y, v(y) + \frac{xy}{2}\right), \quad (x, y) \in \mathbb{R}^2.$$

**Theorem 3.3.** • *The  $G_1$ -invariant constant extrinsically Gaussian curvature translation surfaces in the 3-dimensional Heisenberg group  $\mathbb{H}_3$ , are:*

1.  $K_{ext} = -\frac{1}{4}$ .

*The surfaces of equation*

$$z = v(y) + \frac{xy}{2} = \frac{xy}{2} + \frac{1}{2}y\sqrt{2\beta - y^2} + \arctan\left(\frac{y}{\sqrt{\beta - y^2}}\right),$$

where  $\beta \in \mathbb{R}$ .

2.  $K_{ext} \neq -\frac{1}{4}$ .

Then  $y$  and  $v$  satisfy

$$v(y) = \int \sqrt{\frac{1}{-2(K_{ext} + \frac{1}{4}) \ln(y) + \gamma} - y^2 - 1} dy.$$

where  $\gamma \in \mathbb{R}$ , and  $y \neq e^{\frac{\gamma}{2(K_{ext} + \frac{1}{4})}}$ .

• There are no complete constant extrinsically Gaussian curvature translation surfaces in the 3-dimensional Heisenberg group  $\mathbb{H}_3$  invariant under the 1-parameter subgroup  $G_1$ .

*Proof.* From (4.1) and (3.2) we have

$$K_{ext} = \frac{ln - m^2}{EG - F^2} = \frac{y^2 + yv'v'' - \frac{1}{4}(1 + y^2 + v'^2)^2}{(1 + y^2 + v'^2)^2}. \quad (3.7)$$

1. If  $K_{ext} = -\frac{1}{4}$ . Then equation (3.7) becomes

$$y^2 + yv'v'' = 0 \quad (3.8)$$

We note that  $y$  equal zero is solution of the equation(3.8).

If  $y$  is different to zero ( $y \neq 0$ ), equation (3.8) becomes

$$v'v'' = -y.$$

Integration gives us

$$v(y) = \frac{1}{2}y\sqrt{2\beta - y^2} + \arctan\left(\frac{y}{\sqrt{\beta - y^2}}\right),$$

where  $\beta \in \mathbb{R}$ .

2. If  $K_{ext} \neq -\frac{1}{4}$ . Then equation (3.7) becomes

$$y^2 + yv'v'' = (K_{ext} + \frac{1}{4})(1 + y^2 + v'^2)^2.$$

In fact, put  $z = 1 + y^2 + v'^2$ . Then  $z$  satisfies

$$\frac{1}{2}yz' = (K_{ext} + \frac{1}{4})z^2.$$

Hence we have

$$z = \frac{1}{-2(K_{ext} + \frac{1}{4})y + \gamma},$$

where  $\gamma \in \mathbb{R}$ , and  $y \neq e^{\frac{\gamma}{2(K_{ext} + \frac{1}{4})}}$ . Using the equation  $z = 1 + y^2 + v'^2$ , we get

$$v'^2 = \frac{1}{-2(K_{ext} + \frac{1}{4})y + \gamma} - y^2 - 1. \quad \square$$



## 4. Constant extrinsically Gaussian curvature $G_2$ -invariant translation surfaces in Heisenberg group $\mathbb{H}_3$

In this section we study constant complete extrinsically flat translation surfaces  $\Sigma$  in Heisenberg group  $\mathbb{H}_3$  which are invariant under the one parameter subgroup  $G_2$ . Clearly, such a surface is generated by a curve  $\gamma$  in the totally geodesic plane  $\{y = 0\}$ . Discarding the trivial case of a vertical plane  $\{x = x_0\}$ . Thus  $\gamma$  is given by  $\gamma(x) = (x, 0, f(x))$ . Therefore the generated surface is parameterized by

$$X(x, y) = (0, y, 0) \cdot (x, 0, f(x)) = (x, y, f(x) - \frac{xy}{2}), \quad (x, y) \in \mathbb{R}^2.$$

We have an orthogonal pair of vector fields on  $(\Sigma)$ , namely,

$$e_1 := X_x = (1, 0, f' - \frac{y}{2}) = E_1 + f'E_3.$$

and

$$e_2 := X_y = (0, 1, -\frac{x}{2}) = E_2 - xE_3.$$

The coefficients of the first fundamental form are:

$$E = \langle e_1, e_1 \rangle = 1 + f'^2, \quad F = \langle e_1, e_2 \rangle = -xf', \quad G = \langle e_2, e_2 \rangle = 1 + x^2.$$

As a unit normal field we can take

$$N = \frac{-f'}{\sqrt{1+x^2+f'^2}}E_1 + \frac{x}{\sqrt{1+x^2+f'^2}}E_2 + \frac{1}{\sqrt{1+x^2+f'^2}}E_3.$$

The covariant derivatives are

$$\begin{aligned} \tilde{\nabla}_{e_1}e_1 &= -f'E_2 + f''E_3 \\ \tilde{\nabla}_{e_1}e_2 &= \frac{f'}{2}E_1 + \frac{x}{2}E_2 - \frac{1}{2}E_3 \\ \tilde{\nabla}_{e_2}e_2 &= -xE_1. \end{aligned}$$

The coefficients of the second fundamental form are

$$\begin{aligned} l &= \langle \tilde{\nabla}_{e_1}e_1, N \rangle = \frac{-xf' + f''}{\sqrt{1+x^2+f'^2}} \\ m &= \langle \tilde{\nabla}_{e_1}e_2, N \rangle = \frac{-\frac{f'^2}{2} + \frac{x^2}{2} - \frac{1}{2}}{\sqrt{1+x^2+f'^2}} \\ n &= \langle \tilde{\nabla}_{e_2}e_2, N \rangle = \frac{-yv' + v''}{\sqrt{1+y^2+v'^2}}. \end{aligned}$$

Let  $K_{ext}$  be the extrinsic Gauss curvature of  $\Sigma$ ,

$$K_{ext} = \frac{ln - m^2}{EG - F^2} = \frac{x^2 + xf'f'' - \frac{1}{4}(x^2 + f'^2 + 1)^2}{(1 + x^2 + f'^2)^2}. \quad (4.1)$$

Thus  $\Sigma$  is extrinsically flat invariant surface in Heisenberg group  $\mathbb{H}_3$  if and only if

$$K_{ext} = 0,$$

that is, if and only if

$$x^2 + xf'f'' - \frac{1}{4}(x^2 + f'^2 + 1)^2 = 0. \quad (4.2)$$

to classify extrinsically flat invariant surfaces must solve the equation (4.2).

We remark that the equation (4.2) is similarly to the equation (3.1), It is sufficient to change  $y$  by  $x$  and  $v$  by  $f$ .

As conclusion, we have

**Theorem 4.1.** • *The only non-extendable extrinsically flat translation surfaces in the 3-dimensional Heisenberg group  $\mathbb{H}_3$  invariant under the 2-parameter subgroup  $G_2 = \{(0, t, 0) \in \mathbb{H}_3 / t \in \mathbb{R}\}$ , are the surfaces whose parametrization is  $X(x, y) = (x, y, f(x) - \frac{xy}{2})$  where  $x$  and  $f$  satisfy*

$$f(x) = \int \sqrt{\frac{2}{-\ln(x) + \alpha} - x^2 - 1} dy.$$

where  $\alpha \in \mathbb{R}$ , and  $x \neq e^\alpha$ .

• *There are no complete extrinsically flat translation surfaces in the 3-dimensional Heisenberg group  $\mathbb{H}_3$  invariant under the 1-parameter subgroup  $G_2 = \{(0, t, 0) \in \mathbb{H}_3 / t \in \mathbb{R}\}$ .*

*Remark 4.2.* Let  $\Sigma$  be a  $G_2$ -invariant translation surfaces in the 3-dimensional Heisenberg space. Then  $\Sigma$  is locally expressed as

$$X(x, y) = (x, 0, f(x)) \cdot (0, y, 0) = \left(x, y, f(x) + \frac{xy}{2}\right).$$

Then the extrinsically Gaussian curvature  $K_{ext}$  of  $\Sigma$  is computed as

$$K_{ext} = -\frac{((f' + y)^2 - 1)^2}{4(1 + (v' - x)^2)^2}.$$

Thus  $\Sigma$  can not be of constant extrinsically Gaussian curvature.

**Theorem 4.3.** • *The  $G_2$ -invariant constant extrinsically Gaussian curvature translation surfaces in the 3-dimensional Heisenberg group  $\mathbb{H}_3$ , are:*

1.  $K_{ext} = -\frac{1}{4}$ .

The surfaces of equation

$$z = f(x) - \frac{xy}{2} = -\frac{xy}{2} + \frac{1}{2}x\sqrt{2\beta - x^2} + \arctan\left(\frac{x}{\sqrt{\beta - x^2}}\right),$$

where  $\beta \in \mathbb{R}$ .

2.  $K_{ext} \neq -\frac{1}{4}$ .

Then  $x$  and  $f$  satisfy

$$f(x) = \int \sqrt{\frac{1}{-2(K_{ext} + \frac{1}{4})\ln(x) + \gamma} - x^2 - 1} dy.$$

where  $\gamma \in \mathbb{R}$ , and  $x \neq e^{\frac{\gamma}{2(K_{ext} + \frac{1}{4})}}$ .

- There are no complete constant extrinsically Gaussian curvature translation surfaces in the 3-dimensional Heisenberg group  $\mathbb{H}_3$  invariant under the 1-parameter subgroup  $G_2$ .

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