A Marcinkiewicz–Zygmund type strong law of large numbers for non-negative random variables with multidimensional indices

Tibor Tómács*

Institute of Mathematics and Informatics
Eszterházy Károly University, Eger, Hungary
tomacs.tibor@uni-eszterhazy.hu

Submitted: September 2, 2019
Accepted: December 4, 2019
Published online: December 5, 2019

Abstract

In this paper a Marcinkiewicz–Zygmund type strong law of large numbers is proved for non-negative random variables with multidimensional indices, furthermore we give its an application for multi-index sequence of non-negative random variables with finite variances.

Keywords: Marcinkiewicz–Zygmund type strong law of large numbers, almost sure convergence, non-negative random variables, multidimensional indices

MSC: 60F15

1. Introduction

The Kolmogorov theorem and the Marcinkiewicz–Zygmund theorem are two famous theorems on the strong law of large numbers for $X_n$ ($n \in \mathbb{N}$) sequence of independent identically distributed random variables (see e.g. Loève [8]). By Kolmogorov theorem, there exists a constant $b$ such that $\lim_{n \to \infty} S_n/n = b$ almost surely if and only if $E|X_1| < \infty$, where $S_n = \sum_{k=1}^{n} X_k$. If the latter condition is satisfied then $b = E X_1$. By Marcinkiewicz–Zygmund theorem, if $0 < r < 2$ then

*The author’s research was supported by the grant EFOP-3.6.1-16-2016-00001 (“Complex improvement of research capacities and services at Eszterhazy Karoly University”).
\[
\lim_{n \to \infty} (S_n - b_n)/n^{1/r} = 0 \text{ almost surely if and only if } E |X_1|^r < \infty, \text{ where } b = 0 \text{ if } 0 < r < 1, \text{ and } b = E X_1 \text{ if } 1 \leq r < 2.
\]

Etemadi [1] proved that the Kolmogorov theorem holds for identically distributed and pairwise independent random variables, furthermore Kruglov [7] extended the Marcinkiewicz–Zygmund theorem for pairwise independent case if \( r < 1 \).

Several papers are devoted to the study of the strong law of large numbers for multi-index sequence of random variables (see e.g. Gut [4], Klesov [5, 6], Fazekas [2], Fazekas, Tómács [3]). For example, Theorem 3.1 of Fazekas, Tómács [3] extends Theorem 2 of Kruglov [7] for multi-index case.

In this paper the main result is Theorem 3.1, which is a Marcinkiewicz–Zygmund type strong law of large numbers for non-negative random variables with multidimensional indices. It is a generalization of Theorem 3.1 of Fazekas, Tómács [3] in case \( n \to \infty \). Furthermore we give an application (see Theorem 4.1) for multi-index sequence of non-negative random variables with finite variances. A special case of this result gives Theorem of Petrov [9].

2. Notation

Let \( \mathbb{N}^d \) be the positive integer \( d \)-dimensional lattice points, where \( d \) is a positive integer. For \( n, m \in \mathbb{N}^d \), \( n \leq m \) is defined coordinate-wise, \( (n, m) = (n_1, m_1) \times \cdots \times (n_d, m_d) \) is a \( d \)-dimensional rectangle and \( |n| = n_1 n_2 \cdots n_d \), where \( n = (n_1, n_2, \ldots, n_d) \), \( m = (m_1, m_2, \ldots, m_d) \). \( \sum_n \) will denote the summation for all \( n \in \mathbb{N}^d \). We also use \( 1 = (1, 1, \ldots, 1) \in \mathbb{N}^d \) and \( 2 = (2, 2, \ldots, 2) \in \mathbb{N}^d \). Denote the integer part of \( x \) real number by \( [x] \).

We shall say that \( \lim_{n \to \infty} a_n = 0 \), where \( a_n (n \in \mathbb{N}^d) \) are real numbers, if for all \( \delta > 0 \) there exists \( N \in \mathbb{N}^d \) such that \( |a_n| < \delta \forall n \geq N \).

We shall assume that random variables \( X_n (n \in \mathbb{N}^d) \) are defined on the same probability space \( (\Omega, \mathcal{F}, P) \). E and Var stand for the expectation and the variance.

Remark that a sum or a minimum over the empty set will be interpreted as zero (i.e. \( \sum_{n \in H} a_n = \min_{n \in H} a_n = 0 \) if \( H = \emptyset \)).

3. The result

The following result is a generalization of Theorem 3.1 of Fazekas, Tómács [3] in case \( n \to \infty \).

**Theorem 3.1.** Let \( X_n (n \in \mathbb{N}^d) \) be a sequence of non-negative random variables, let \( b_n (n \in \mathbb{N}^d) \) be a sequence of non-negative numbers, \( B_n = \sum_{k \leq n} b_k \), \( S_n = \sum_{k \leq n} X_k \), \( c > 0 \), \( K \in \mathbb{N} \) and \( 0 < r \leq 1 \). If

\[
B_n - B_m \leq c(|n| - |m|) \quad \forall n, m \in \mathbb{N}^d, n \geq m, |n| - |m| \geq K
\]

(3.1)
and
\[
\sum_n 1 \left\lfloor n \right\rfloor P \left( |S_n - B_n| > \varepsilon |n|^{1/r} \right) < \infty \quad \forall \varepsilon > 0,
\] (3.2)

then
\[
\lim_{n \to \infty} \frac{S_n - B_n}{|n|^{1/r}} = 0 \quad \text{almost surely.}
\]

Proof. Let \( \delta > 0, 1 < \alpha < \left( \frac{\delta}{2\varepsilon} + 1 \right)^{1/3d} \) and \( 0 < \varepsilon < \frac{\delta}{2} \left( \frac{\delta}{2\varepsilon} + 1 \right)^{-1/r} \), which imply
\[
\varepsilon \alpha^{3d/r} + c(\alpha^{3d} - 1) < \delta.
\] (3.3)

Let \( k_n = [\alpha^n] \ (n \in \mathbb{N}) \) and \( k_n = (k_{n1}, k_{n2}, \ldots, k_{nd}) \), where \( n = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d \). It follows from the inequalities

\[
\sum_n \frac{1}{|n|} P \left( |S_n - B_n| > \varepsilon |n|^{1/r} \right)
\geq \sum_n \sum_{h \in (k_n, k_{n+1}]} \frac{1}{|h|} P \left( |S_h - B_h| > \varepsilon |h|^{1/r} \right)
\leq \sum_n \sum_{h \in (k_n, k_{n+1}]} \frac{1}{|k_{n+1}|} \min_{k \in (k_n, k_{n+1}]} P \left( |S_k - B_k| > \varepsilon |k|^{1/r} \right)
\]

and condition (3.2) that
\[
\sum_n \frac{|k_{n+1} - k_n|}{|k_{n+1}|} \min_{k \in (k_n, k_{n+1}]} P \left( |S_k - B_k| > \varepsilon |k|^{1/r} \right) < \infty. \quad (3.4)
\]

Since \( \lim_{n \to \infty} \left( 1 - \frac{1}{\alpha^{n+1}} - \frac{1}{\alpha} \right) = 1 - \frac{1}{\alpha} > 0 \), so \( \left( 1 - \frac{1}{\alpha^{n+1}} - \frac{1}{\alpha} \right) > \frac{\alpha - 1}{2\alpha} \) except for finitely many \( n \in \mathbb{N} \). This implies that there exists \( N_0 \in \mathbb{N}^d \) such that
\[
0 < \left( \frac{\alpha - 1}{2\alpha} \right)^d < \prod_{i=1}^d \left( 1 - \frac{1}{\alpha^{n_i+1}} - \frac{1}{\alpha} \right) = \prod_{i=1}^d \frac{\alpha^{n_i+1} - 1 - \alpha^{n_i}}{\alpha^{n_i+1}}
\leq \prod_{i=1}^d \frac{[\alpha^{n_i+1}] - [\alpha^{n_i}]}{[\alpha^{n_i+1}]} = \frac{|k_{n+1} - k_n|}{|k_{n+1}|} \quad \forall n = (n_1, n_2, \ldots, n_d) \geq N_0.
\]

Hence
\[
\left( \frac{\alpha - 1}{2\alpha} \right)^d \sum_{n \geq N_0} \min_{k \in (k_n, k_{n+1}]} P \left( |S_k - B_k| > \varepsilon |k|^{1/r} \right)
\leq \sum_{n \geq N_0} \frac{|k_{n+1} - k_n|}{|k_{n+1}|} \min_{k \in (k_n, k_{n+1}]} P \left( |S_k - B_k| > \varepsilon |k|^{1/r} \right).
\]
By this inequality and (3.4), it follows that
\[
\sum_{n \geq N_0} \min_{k \in (k_n, k_n+1]} P \left( |S_k - B_k| > \varepsilon |k|^{1/r} \right) < \infty. \tag{3.5}
\]

If \( n \geq N_0 \) then there exists \( m_n \in \mathbb{N}^d \) such that \( m_n \in (k_n, k_n+1] \) and
\[
P \left( |S_{m_n} - B_{m_n}| > \varepsilon |m_n|^{1/r} \right) = \min_{k \in (k_n, k_n+1]} P \left( |S_k - B_k| > \varepsilon |k|^{1/r} \right).
\]

Therefore, by (3.5) we have
\[
\sum_{n \geq N_0} P \left( |S_{m_n} - B_{m_n}| > \varepsilon |m_n|^{1/r} \right) < \infty. \tag{3.6}
\]

By the Borel–Cantelli lemma, (3.6) implies that there exist \( N_1 \in \mathbb{N}^d \) and \( A \in \mathcal{F} \) such that \( N_1 \geq N_0 \), \( P(A) = 1 \) and
\[
\left| S_{m_n}(\omega) - B_{m_n} \right| \leq \varepsilon \quad \forall n \geq N_1, \forall \omega \in A. \tag{3.7}
\]

Henceforward let \( \omega \in A \) be fixed.

If \( n \geq N_1 \) and \( t \in (k_n+1, k_n+2] \), then by \( t \in (m_n, m_{n+2}] \), (3.7) and
\[
|m_{n+2}|^{1/r} \geq |m_n|^{1/r} \geq |m_n|
\]
we have
\[
\frac{S_t(\omega) - B_t}{|t|^{1/r}} \geq \frac{S_{m_n}(\omega) - B_{m_{n+2}}}{|m_{n+2}|^{1/r}} = \frac{S_{m_n}(\omega) - B_{m_n}}{|m_n|^{1/r}} \frac{|m_n|^{1/r}}{|m_{n+2}|^{1/r}} - \frac{B_{m_{n+2}} - B_{m_n}}{|m_{n+2}|^{1/r}} \\
\geq -\varepsilon - \frac{B_{m_{n+2}} - B_{m_n}}{|m_n|}. \tag{3.8}
\]

If \( n = (n_1, n_2, \ldots, n_d) \geq N_0 \) and \( m_n = (m_n^{(1)}, m_n^{(2)}, \ldots, m_n^{(d)}) \) then
\[
[\alpha^{n_i}] < m_n^{(i)} \leq [\alpha^{n_i+1}].
\]

On the other hand \( m_n^{(i)} \in \mathbb{N} \), hence we get
\[
\alpha^{n_i} < m_n^{(i)} \leq \alpha^{n_i+1}. \tag{3.9}
\]

This inequality implies
\[
|m_{n+2} - m_n| \geq \prod_{i=1}^{d} \alpha^{n_{i+1} - n_i} - \prod_{i=1}^{d} \alpha^{n_{i+1}}
\]
\[
= (\alpha^d - 1) \prod_{i=1}^{d} \alpha^{n_i+1}
\]

\[
> (\alpha^d - 1) \alpha^{n_1} \quad \forall n = (n_1, n_2, \ldots, n_d) \geq N_0.
\]

Since \( \lim_{n \to \infty} \alpha^n = \infty \), therefore \( \alpha^n \geq K(\alpha^d - 1)^{-1} \) except for finitely many values of \( n \in \mathbb{N} \). Hence there exists \( N_2 \in \mathbb{N}^d \) such that \( N_2 \geq N_1 \) and

\[
|m_{n+2}| - |m_n| > (\alpha^d - 1) \frac{K}{\alpha^d - 1} = K \quad \forall n \geq N_2.
\]

This inequality implies by (3.1), that

\[
B_{m_{n+2}} - B_{m_n} \leq c(|m_{n+2}| - |m_n|) \quad \forall n \geq N_2.
\]

Using (3.9) we have

\[
\frac{|m_{n+2}|}{|m_n|} \leq \frac{d}{i=1} \frac{\alpha^{n_i+3}}{\alpha^{n_i}} = \alpha^{3d} \quad \forall n = (n_1, n_2, \ldots, n_d) \geq N_2. \tag{3.11}
\]

Hence (3.8), (3.10), (3.11) and (3.3) imply, that if \( n \geq N_2 \) and \( t \in (k_{n+1}, k_{n+2}] \), then

\[
\frac{S_t(\omega) - B_t}{|t|^{1/r}} \geq -\varepsilon - \frac{B_{m_{n+2}} - B_{m_n}}{|m_n|} \geq -\varepsilon - c \left( \frac{|m_{n+2}|}{|m_n|} - 1 \right)
\]

\[
\geq -\varepsilon - c(\alpha^{3d} - 1) \geq -\varepsilon \alpha^{3d/r} - c(\alpha^{3d} - 1) > -\delta. \tag{3.12}
\]

If \( n \geq N_2 \) and \( t \in (k_{n+1}, k_{n+2}] \), then by \( t \in (m_n, m_{n+2}], \ |m_n|^{1/r} \geq |m_n| \), (3.7), (3.11), (3.10) and (3.3), we have

\[
\frac{S_t(\omega) - B_t}{|t|^{1/r}} \leq \frac{S_{m_{n+2}}(\omega) - B_{m_n}}{|m_{n+2}|^{1/r}}
\]

\[
= \frac{S_{m_{n+2}}(\omega) - B_{m_{n+2}}}{|m_{n+2}|^{1/r}} \frac{|m_{n+2}|^{1/r}}{|m_n|^{1/r}} + \frac{B_{m_{n+2}} - B_{m_n}}{|m_n|^{1/r}}
\]

\[
\leq \frac{S_{m_{n+2}}(\omega) - B_{m_{n+2}}}{|m_{n+2}|^{1/r}} \frac{|m_{n+2}|^{1/r}}{|m_n|^{1/r}} + \frac{B_{m_{n+2}} - B_{m_n}}{|m_n|}
\]

\[
\leq \varepsilon \alpha^{3d/r} + c \left( \frac{|m_{n+2}|}{|m_n|} - 1 \right) \leq \varepsilon \alpha^{3d/r} + c(\alpha^{3d} - 1) < \delta.
\]

This inequality and (3.12) imply

\[
\frac{|S_t(\omega) - B_t|}{|t|^{1/r}} < \delta \quad \forall n \geq N_2, t \in (k_{n+1}, k_{n+2}]. \tag{3.13}
\]

If \( t \geq k_{N_2+1} + 1 \), then there exists \( n \geq N_2 \) such that \( t \in (k_{n+1}, k_{n+2}] \). Hence (3.13) implies

\[
\frac{|S_t(\omega) - B_t|}{|t|^{1/r}} < \delta \quad \forall t \geq k_{N_2+1} + 1.
\]

Therefore the statement is proved. \( \Box \)
In this section we give an application of Theorem 3.1. In case $d = r = 1$, this result gives Theorem of Petrov [9].

Theorem 4.1. Let $X_n \ (n \in \mathbb{N}^d)$ be a sequence of non-negative random variables with finite variances, $S_n = \sum_{k \leq n} X_k$, $c > 0$, $K \in \mathbb{N}$ and $0 < r \leq 1$. If

$$E(S_n - E S_n) \leq c(|n| - |m|) \quad \forall n, m \in \mathbb{N}^d, n \geq m, |n| - |m| \geq K \quad (4.1)$$

and

$$\sum_n \frac{\text{Var} S_n}{|n|^{1 + 2/r}} < \infty \quad (4.2)$$

then

$$\lim_{n \to \infty} \frac{S_n - E S_n}{|n|^{1/r}} = 0 \quad \text{almost surely.}$$

Proof. With notation $b_k = E X_k$ and $B_n = \sum_{k \leq n} b_k = E S_n$, (4.1) implies (3.1). On the other hand, if $\varepsilon > 0$, then the Chebyshev inequality and (4.2) imply

$$\sum_n \frac{1}{|n|} P \left( |S_n - B_n| > \varepsilon |n|^{1/r} \right) \leq \sum_n \frac{1}{|n|} \frac{\text{Var} S_n}{\varepsilon^2 |n|^{2/r}} = \varepsilon^{-2} \sum_n \frac{\text{Var} S_n}{|n|^{1 + 2/r}} < \infty.$$

Therefore (3.2) holds. Hence, using Theorem 3.1, we have that the statement is true. $\square$

References


