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Diophantine triples in a Lucas-Lehmer sequence

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Abstract

In this paper, we define a Lucas-Lehmer type sequence denoted by $(L_n)_{n=0}^{\infty}$, and show that there are no integers $0 < a < b < c$ such that $ab + 1$, $ac + 1$ and $bc + 1$ all are terms of the sequence.

Keywords: Diophantine triples, Lucas-Lehmer sequences

MSC: Primary 11B39; Secondary 11D99

1. Introduction

A diophantine m-tuple consists of m distinct positive integers such that the product of any two of them is one less than a square of an integer. Diophantus found the first four, but rational numbers 1/16, 33/16, 17/4, 105/16 with this property. Fermat gave 1, 3, 8, 120 as the first integer quadruple. Hoggatt and Bergum [8] provided infinitely many diophantine quadruples by $F_{2k}, F_{2k+2}, F_{2k+4}, 4F_{2k+1}F_{2k+2}F_{2k+3}$. The most outstanding result is due to Dujella [3], who proved that there are only finitely many quintuples. Recently He, Togbe, and Ziegler submitted a work which solved the longstanding problem of the non-existence of diophantine quintuples [7].

There are several variations of the basic problem, most of them replace the squares by a given infinite set of integers. For instance, Luca and Szalay studied the diophantine triples for the terms of binary recurrences. They proved that there are no integers $0 < a < b < c$ such that $ab+1$, $ac+1$ and $bc+1$ all are Fibonacci numbers (see [9]), further for the Lucas sequence there is only one such a triple: $a = 1, b = 2, c = 3$ (see [10]). Fuchs, Luca and Szalay [4] gave sufficient and necessary conditions to have infinitly many diophantine triples for a general second order sequence.

For ternary recurrences Fuchs et al. [5] justified that there exist only finitely many triples corresponding to Tribonacci sequence. This paper was generalized by Fuchs et al. [6]. Alp and Irmak were the first who investigated the existence of diophantine triples in a Lucas-Lehmer type sequence (see [2]). They showed that there are no diophantine triples for the so-called pellans sequence.

In this paper, we study another Lucas-Lehmer sequence and prove the non-existence of diophantine triples associated to it. Let $(L_n)_{n=0}^{\infty}$ be defined by the initial values $L_0 = 0$, $L_1 = 1$, $L_2 = 1$ and $L_3 = 3$, and by the recursive rule

$$L_n = 4L_{n-2} - L_{n-4}. \tag{1.1}$$

Our principal result is the following.

Theorem 1.1. *There exist no integers* $0 < a < b < c$ *such that*

$$ab + 1 = L_x, \quad ac + 1 = L_y, \quad bc + 1 = L_z \tag{1.2}$$

would hold for any positive integers x, y *and* z.

2. Preliminaries

The associate sequence of (L_n) is denoted by $(M_n)_{n=0}^{\infty}$, which according to the general theory of Lucas-Lehmer sequences satisfies $M_0 = 2$, $M_1 = 2$, $M_2 = 4$, $M_3 = 10$, and $M_n = 4M_{n-2} - M_{n-4}$. It is easy to see that L_n is divisible by 4 if and only if $4 \mid n$, otherwise L_n is odd. Using the recurrence relation (1.1), for negative subscripts $M_{-n} = (-1)^n M_n$ follows.

The zeros of the common characteristic polynomial $x^4 - 4x^2 + 1$ of (L_n) and (M_n) are $\omega = (\sqrt{3}+1)/\sqrt{2}$, $\psi = (-\sqrt{3}+1)/\sqrt{2}$, $-\omega$ and $-\psi$, further the initial values provide the explicit formulae

$$L_n = \frac{1+\sqrt{2}}{4\sqrt{3}}\left(\omega^n - \psi^n\right) + \frac{1-\sqrt{2}}{4\sqrt{3}}\left((-\omega)^n - (-\psi)^n\right),$$
$$M_n = \frac{1+\sqrt{2}}{2}\left(\omega^n + \psi^n\right) + \frac{1-\sqrt{2}}{2}\left((-\omega)^n + (-\psi)^n\right). \tag{2.1}$$

It's trivial from the recursive rules of both (L_n) and (M_n) that the subsequences of terms with even resp. odd indices form second order sequences by the same coefficients. The zeros of their companion polynomial are $\alpha = \omega^2 = 2 + \sqrt{3}$ and $\beta = \psi^2 = 2 - \sqrt{3}$, and the dominant root is α.

Generally the Lucas-Lehmer sequences are union of two binary recursive sequences. Many properties, which are well known for binary sequences with initial values 0 and 1, hold for Lucas-Lehmer sequences too (may be by a little modification). So the research of Lucas-Lehmer sequences is a new feature in the investigations.

In the sequel, we prove a few lemma which will be useful in proving the main theorem.

Lemma 2.1. *If $n = mt$ and t is odd, then $M_m \mid M_n$.*

Proof. The statement is obvious for $t = 1$. Formula (2.1) admits

$$M_{6k} = M_{2k}(M_{4k} - 1), \tag{2.2}$$

$$M_{6k+3} = M_{2k+1}(M_{4k+2} + 1), \tag{2.3}$$

which proves the lemma for $t = 3$. It can be seen by induction on k that

$$M_{n+k} = \begin{cases} \frac{1}{2} M_n M_k + M_{n-k}, & \text{if } n \equiv k \equiv 1 \pmod 2, \\ M_n M_k - (-1)^k M_{n-k}, & \text{otherwise.} \end{cases} \tag{2.4}$$

Finally, using (2.4), we can prove the lemma by induction on t. □

Lemma 2.2. *If $n = mt$ and t is even, then* $\gcd(M_n, M_m) = 2$.

Proof. Put $m = 2k$. From (2.1) it follows that

$$M_{4k} = M_{2k}^2 - 2. \tag{2.5}$$

Subsequently, $\gcd(M_{2k}, M_{4k}) = 2$. It can be seen that $M_{2^l k}$ $(l \geq 3)$ can be expressed as a polynomial of M_{2k}, where the constant term is always 2. Thus $\gcd(M_{2k}, M_{2^l k}) = 2$ $(l \geq 2)$.

Now let $m = 2k + 1$. Again by (2.1) we see that

$$M_{4k+2} = M_{2k+1}^2/2 + 2 \tag{2.6}$$

holds. Putting $H_{2k+1} = M_{2k+1}^2/2$, it is trivial that H_{2k+1} and M_{2k+1} are divisible by the same primes, and the exponent of 2 is 1 in both integers. So $\gcd(H_{2k+1}, N) = 2$ and $\gcd(M_{2k+1}, N) = 2$ are equivalent for an arbitrary integer N. Hence we have $M_{4k+2} = H_{2k+1} + 2$, and it implies $\gcd(M_{4k+2}, H_{2k+1}) = 2$. By induction and (2.5) we can see that $M_{2^l(2k+1)}$ can be written as a polynomial of H_{2k+1} for any positive integer l, with constant term 2. Consequently, $\gcd(M_{2k+1}, M_{2^l(2k+1)}) = \gcd(H_{2k+1}, M_{2^l(2k+1)}) = 2$. Together with Lemma 2.1, it shows immediately, that $\gcd(M_m, M_{tm}) = 2$ for arbitrary even t. □

Lemma 2.3. *For any $n \geq 0$ we have*

$$L_n - 1 = \begin{cases} L_{\frac{n-1}{2}} M_{\frac{n+1}{2}}, & \text{if } n \equiv 1 \pmod 4, \\ L_{\frac{n+1}{2}} M_{\frac{n-1}{2}}, & \text{if } n \equiv 3 \pmod 4, \\ \frac{1}{2} L_{\frac{n+2}{2}} M_{\frac{n-2}{2}}, & \text{if } n \equiv 0 \pmod 4, \\ L_{\frac{n-2}{2}} M_{\frac{n+2}{2}}, & \text{if } n \equiv 2 \pmod 4. \end{cases} \tag{2.7}$$

Proof. To prove the statement one can use the explicit formulae for the terms appearing in (2.7). □

Lemma 2.4. *The greatest common divisors of the terms of* (L_n) *and* (M_n) *satisfy*

1. $\gcd(L_m, L_n) = L_{\gcd(m,n)}$;

2. $\gcd(M_m, M_n) = \begin{cases} M_{\gcd(m,n)}, & \text{if } \frac{m}{\gcd(m,n)} \equiv 1 \equiv \frac{n}{\gcd(m,n)} \pmod 2, \\ 2, & \text{otherwise;} \end{cases}$

3. $\gcd(L_m, M_n) = \begin{cases} \mu M_{\gcd(m,n)}, & \text{if } \frac{m}{\gcd(m,n)} + 1 \equiv 1 \equiv \frac{n}{\gcd(m,n)} \pmod 2, \\ 1 \text{ or } 2, & \text{otherwise,} \end{cases}$
   where $\mu = 1$ *or* $1/2$.

Proof. We omit the proof of the first statement, the easiest part, and start by proving the second one. The main tool is a Euclidean-like algorithm. Assume that $m = nq + r$, where q is an odd integer, and $0 \le r < 2n$. By (2.4) we have

$$M_m = \mu M_{nq} M_r \pm M_{nq-r}.$$

The terms of (M_n) is even, so μM_r is an integer. Let d be an integer which divides both M_m and M_n. Since q is odd, d divides M_{nq}, too. Thus $d \mid M_{nq-r}$ holds. On the other hand, if $d \mid M_n$ and $d \mid M_{nq-r}$, then similarly d divides M_m. Hence $\gcd(M_m, M_n) = \gcd(M_n, M_{nq-r})$.

Suppose now $m > n$ and $n \nmid m$. After the first Euclidean-like division by n, replace m by $nq - r$, and continue with this, while the subscript is larger than n. After the last step, $nq - r$ might be negative. It is obvious that after two steps m is decreased by $4n$. The last term of the sequence coming from these steps depends on the residue of the initial value of m modulo $4n$. Let $r_1 \equiv m \pmod n$, $r_2 \equiv m \pmod{4n}$, and $0 < r_1 < n$, $0 < r_2 < 4n$. In particular, for the last subscript r' we found

$$r' = \begin{cases} r_1, & \text{if } 0 < r_2 < n, \\ n - r_1, & \text{if } n < r_2 < 2n, \\ -r_1, & \text{if } 2n < r_2 < 3n, \\ r_1 - n, & \text{if } 3n < r_2 < 4n. \end{cases}$$

Obviously, $\gcd(n, r_1) = \gcd(n, r')$ and $0 < |r'| < n$, further if $d_1 \mid m$ and $d_1 \mid n$, then $d_1 \mid nq - r$. Moreover if d_1 divides both n and $nq - r$, then it must divide r and $m = nq + r$. This shows that $\gcd(m, n) = \gcd(nq - r, m)$. Thus $\gcd(m, n) = \gcd(r', n)$. Then apply this approach successively (replace the initial values of m by n, and n by $|r'|$, and continue), and finish when the remainder is zero. The last nonzero remainder is the gcd.

To complete the proof of the second case, suppose that $\gcd(m, n) = 1$. By the last division $n = 1$ follows, and denote the value of m by m_1. The parities of $m = nq + r$ and $nq - r$ coincide in each step. If both m and n are odd, then the values of $nq - r$, r' are odd, hence so is m_1. If m is even and n is odd, then r' is even, and then the next division-sequence begins with odd m and even n. By the last division (where $n = 1$) it follows that m_1 must be even. Similarly, if the initial value of m is odd and n is even, then m_1 is even, too.

Put $d_2 = \gcd(m, n)$. It occurs if we multiply all the terms in the last paragraph by d_2. If both m/d_2 and n/d_2 are odd, then the quotient in the last division (that is m_1) is odd, and by the algorithm and Lemma 2.1, we have $\gcd(M_m, M_n) = \gcd(M_{m_1 d_2}, M_{d_2}) = M_{d_2}$. If exactly one of m/d_2 and n/d_2 is even, then the last quotient (m_1) is even, and $\gcd(M_m, M_n) = \gcd(M_{m_1 d_2}, M_{d_2}) = 2$ follows by Lemma 2.2.

Now prove the third statement. The explicite formulae provide

$$2\mu L_{m+n} = L_n M_m + L_m M_n, \tag{2.8}$$

$$2\mu M_{m+n} = 12 L_n L_m + M_n M_m, \tag{2.9}$$

where $\mu = 2$ if both m and n are odd, and $\mu = 1$ otherwise.

First we show that $\gcd(L_k, M_k) = 2$ if $4 \mid k$, and $\gcd(L_k, M_k) = 1$ otherwise. It is clear for $k = 1, 2, 3, 4$. From (2.8) and (2.9) we obtain

$$L_{k+4} = \frac{1}{2}(L_k M_4 + L_4 M_k) = 7L_k + 2M_k,$$

$$M_{k+4} = \frac{1}{2}(12 L_k L_4 + M_k M_4) = 24 L_k + 7 M_k.$$

By the Euclidean algorithm we have

$$\begin{aligned}\gcd(L_{k+4}, M_{k+4}) &= \gcd(7L_k + 2M_k, 24L_k + 7M_k)\\ &= \gcd(7L_k + 2M_k, 3L_k + M_k)\\ &= \gcd(L_k, 3L_k + M_k) = \gcd(L_k, M_k).\end{aligned}$$

An induction implies the assertion for every k.

Now we show $\gcd(M_{kn}, L_n) = 1$ or 2, again by induction for k. We have just seen that it is true for $k = 1$. Now (2.9) implies

$$2\mu M_{kn+n} = 12 L_{kn} L_n + M_{kn} M_n.$$

Let d be an odd integer such that $d \mid M_{kn+n}$ and $d \mid L_n$. In this case $d \mid L_{kn}$, and we have shown that $\gcd(L_{kn}, M_{kn}) \le 2$, so d is relatively prime to M_{kn}. Thus $d \mid M_n$. Further $\gcd(L_n, M_n) \le 2$, and d is odd, so $d = 1$. If n is not divisible by 4, then L_n is odd, and $\gcd(M_{kn+n}, L_n)$ is necessarily 1. If $4 \mid n$, then M_{kn+n} is not divisible by 4, but L_{kn+n} is even, so $\gcd(M_{kn+n}, L_n) = 2$.

We will show that if k is odd, then $\gcd(M_n, L_{kn}) = 1$ or 2. Clearly, it is true for $k = 1$. Suppose now that it holds for an odd k, and check it for $k+2$. It follows from (2.8) that

$$2\mu L_{kn+2n} = L_{kn} M_{2n} + M_{kn} L_{2n}.$$

Let be d an odd integer which divides both L_{kn+2n} and M_n. Then $d \mid M_{kn}$ holds since k is odd. But d is relatively prime to M_{2n}, so d must divide L_{kn}. We know that $\gcd(L_{kn}, M_{kn}) \le 2$, henceforward $d = 1$. If $4 \nmid n$, then odd k entails odd $L_{(k+2)n}$, and if $4 \mid n$, then $4 \nmid M_n$. Hence $\gcd(M_n, L_{kn+2n})$ is 1 or 2.

Assuming k is even, put $k = 2^l t$, where t is odd. Then M_n divides M_{tn}, and we have $L_{2tn} = \mu L_{tn} M_{tn}$, where μ is 1 or 1/2. So $M_{tn}/2 \mid L_{2tn}$, and by induction, $M_{tn}/2$ divides $L_{2^l tn}$. Subsequently, $\gcd(M_n, L_{kn})$ is M_n or $M_n/2$ for even k.

Thus the third statement is proven if one of n and m divides the other. For general m and n, suppose $m > n$, and let $m = nq + r$, where q is odd, $0 < r < 2n$. From (2.8), $2\mu L_{nq+r} = L_{nq} M_r + M_{nq} L_r$ follows. It is easy to see that for any odd d the conditions ($d \mid L_m$ and $d \mid M_n$), and ($d \mid M_n$ and $d \mid M_r$) are equivalent (for odd q use that M_n divides M_{nq} and $\gcd(M_{nq}, L_{nq})$ is 1 or 2). So it is enough to determine the greatest odd common divisior of M_n and M_r, for which we use the second part of this lemma.

Trivially, $\gcd(n, r) = \gcd(n, m)$. Denote this value by c. If m/c is even and n/c is odd, then (because q is odd) r/c is odd (say this is case A). By the lemma, $\gcd(M_n, M_r) = M_{\gcd(n,r)}$. If m/c is odd and n/c is even, then r/c is odd. If both m/c and n/c are odd, then r/c is even. In these two cases (we call them case B) $\gcd(M_n, M_r) = 2$ hold.

Clearly, M_n is not divisible by 8, moreover L_m and M_n are both divisible by 4 if and only if $4 \mid m$ and $n \equiv 2 \pmod 4$. In this case the exponent of 2 in $\gcd(n, m)$ is 1, m/c is even, and n/c is odd (this is case A), and $M_{\gcd(n,m)}$ is divisible by 4. It is easy to see that $\gcd(L_m, M_n) = M_{\gcd(n,m)}$. In the remaining situations of case A, $M_{\gcd(m,n)}$ is not divisible by 4. Thus $\gcd(L_m, M_n)$ is $M_{\gcd(n,m)}$ or one half of it. In case B, 4 does not divide L_m and M_n at the same time, so their gcd is 1 or 2.

If $m < n$, then $n = mp + r$. Now p is not necessarily odd, therefore we can suppose $0 < r < m$. Then from (2.9) we conclude $\gcd(L_m, M_n) = \gcd(L_m, M_r)$. To complete the proof we must use the previous case of this lemma. □

The next lemma gives lower and upper bounds on the terms of (L_n) and (M_n) by powers of dominant root α.

Lemma 2.5. *Suppose $n \ge 3$. We have*

$$\alpha^{n-0.944} < L_{2n} < \alpha^{n-0.943}, \quad \alpha^{n-0.181} < L_{2n+1} < \alpha^{n-0.180},$$
$$\alpha^{n} < M_{2n} < \alpha^{n+0.001}, \quad \alpha^{n+0.763} < M_{2n+1} < \alpha^{n+0.764}.$$

Further, independently from the parity of the subscript k,

$$\alpha^{k/2-0.944} < L_k < \alpha^{k/2-0.680} \quad \text{and} \quad \alpha^{k/2} < M_k < \alpha^{k/2+0.264}$$

hold.

Proof. Let n_0 be a positive integer, and assume $n \ge n_0$. The explicit formula (2.1) simplifies $L_{2n} = (\alpha^n - \beta^n)/(\alpha - \beta)$, which yields

$$L_{2n} \ge \frac{\alpha^n - \beta^{n_0}}{\alpha - \beta} = \alpha^n \frac{1 - (\frac{\beta}{\alpha})^{n_0} \alpha^{n_0 - n}}{\alpha - \beta} \ge \alpha^n \frac{1 - (\frac{\beta}{\alpha})^{n_0}}{\alpha - \beta}.$$

Supposing $n_0 \geq 3$, together with $0 < \beta/\alpha < 1$ it leads to

$$\frac{1-(\frac{\beta}{\alpha})^{n_0}}{\alpha-\beta} \geq \frac{1-(\frac{\beta}{\alpha})^{3}}{\alpha-\beta} = 0.28856\ldots > \alpha^{-0.944}.$$

Thus $L_{2n} > \alpha^{n-0.944}$. To get an upper bound is easier, since $\beta > 0$ implies

$$L_{2n} = \frac{\alpha^n - \beta^n}{\alpha-\beta} < \frac{\alpha^n}{\alpha-\beta} = \alpha^n \frac{1}{2\sqrt{3}} < \alpha^{n-0.943}.$$

For odd subscripts a similar treatment is available by

$$L_{2n+1} = \frac{1}{\alpha-\beta}\left[(\sqrt{3}+1)\alpha^n + (\sqrt{3}-1)\beta^n\right].$$

First we see

$$L_{2n+1} > \frac{1+\sqrt{3}}{2\sqrt{3}}\alpha^n > \alpha^{n-0.181}.$$

Now assume $n \geq n_0 \geq 3$. Consequently,

$$\begin{aligned} L_{2n+1} &\leq \frac{1}{\alpha-\beta}\left[(\sqrt{3}+1)\alpha^n + (\sqrt{3}-1)\beta^{n_0}\right] \\ &= \alpha^n \left[\frac{\sqrt{3}+1}{2\sqrt{3}} + \frac{\sqrt{3}-1}{2\sqrt{3}}\left(\frac{\beta}{\alpha}\right)^{n_0}\alpha^{n_0-n}\right] \\ &\leq \alpha^n \left[\frac{\sqrt{3}+1}{2\sqrt{3}} + \frac{\sqrt{3}-1}{2\sqrt{3}}\left(\frac{\beta}{\alpha}\right)^{3}\right] = \alpha^n \cdot 0.788753\ldots < \alpha^{n-0.180}. \end{aligned}$$

The bounds for the terms M_n can be shown by an analogous way. □

Lemma 2.6. *Suppose that a, b, z, and the fractions appearing below are integers. Then*

1. *if $3a \neq b$, then $\gcd(\frac{z+a}{2}, \frac{3z+b}{8}) \leq \left|\frac{3a-b}{2}\right|$,*
2. *if $2a \neq b$, then $\gcd(\frac{z+a}{2}, \frac{2z+b}{6}) \leq \left|\frac{2a-b}{2}\right|$,*
3. *if $a \neq b$, then $\gcd(\frac{z+a}{2}, \frac{z+b}{4}) \leq \left|\frac{a-b}{2}\right|$.*

Proof. The statements follow by a simple use of the Euclidean algorithm. □

Lemma 2.7. *Supposing $z \geq 4$, the following properties are valid.*

1. *If $z \equiv 1 \pmod 4$, then $M^2_{\frac{z-1}{2}} < 2L_z$, further $3L^2_{\frac{z-1}{2}} < 2L_z$.*
2. *If $z \equiv 3 \pmod 4$, then $M^2_{\frac{z-1}{2}} < 4L_z$.*
3. *If $z \equiv 2 \pmod 4$, then $M^2_{\frac{z-2}{2}} < 2L_z$.*

4. If $z \equiv 0 \pmod 4$, then $M^2_{\frac{z-2}{2}} < 4L_z$.

Proof. Use (2.5), (2.6), and

$$M_n = \begin{cases} L_{n-1} + L_{n+1}, & \text{if } n \text{ is even,} \\ 2(L_{n-1} + L_{n+1}), & \text{if } n \text{ is odd.} \end{cases} \tag{2.10}$$

Here (2.10) can be proven by induction. □

Lemma 2.8. *Suppose that a and b are positive real numbers and u_0 is a positive integer. Let $\kappa = \log_\alpha(a + \frac{b}{\alpha^{u_0}})$. If $u \geq u_0$, then*

$$a\alpha^u + b \leq \alpha^{u+\kappa}.$$

Proof. This is obvious by an easy calculation. □

3. Proof of Theorem 1.1

The conditions $1 \leq a < b < c$ entail $3 \leq x < y < z$. Obviously, $c \mid L_y - 1$ and $c \mid L_z - 1$. Thus $c \leq \gcd(L_y - 1, L_z - 1)$. Clearly, $L_z = bc + 1 < c^2$, which implies $\sqrt{L_z} < c$. Combining this with Lemma 2.5, we see

$$\alpha^{\frac{z}{4} - 0.472} = \alpha^{\frac{1}{2}\left(\frac{z}{2} - 0.944\right)} < \sqrt{L_z} < c < L_y < \alpha^{\frac{y}{2} - 0.680},$$

and then $z/4 - 0.472 < y/2 - 0.680$ yields $z < 2y - 0.832$. Hence $z \leq 2y - 1$.

Now we distinguish two cases.

Case I: $z \geq 117$.

The key point of this case is to estimate $G = \gcd(L_y - 1, L_z - 1)$. Assume that $i, j \in \{\pm 1, \pm 2\}$, and $\mu_i^*, \mu_j^* \in \{1, 1/2\}$. By Lemma 2.3,

$$\begin{aligned} G &= \gcd(\mu_i^* L_{\frac{y-i}{2}} M_{\frac{y+i}{2}}, \mu_j^* L_{\frac{z-j}{2}} M_{\frac{z+j}{2}}) \\ &\leq \gcd(L_{\frac{y-i}{2}} M_{\frac{y+i}{2}}, L_{\frac{z-j}{2}} M_{\frac{z+j}{2}}) \\ &\leq \gcd(L_{\frac{y-i}{2}}, L_{\frac{z-j}{2}}) \gcd(L_{\frac{y-i}{2}}, M_{\frac{z+j}{2}}) \gcd(M_{\frac{y+i}{2}}, L_{\frac{z-j}{2}}) \gcd(M_{\frac{y+i}{2}}, M_{\frac{z+j}{2}}). \end{aligned}$$

Let Q denote the last product. By Lemma 2.4

$$Q \leq L_{\gcd(\frac{y-i}{2}, \frac{z-j}{2})} M_{\gcd(\frac{y-i}{2}, \frac{z+j}{2})} M_{\gcd(\frac{y+i}{2}, \frac{z-j}{2})} M_{\gcd(\frac{y+i}{2}, \frac{z+j}{2})}$$

follows. We define d_1, d_2, d_3, d_4 according to the relations

$$\gcd\left(\frac{y-i}{2}, \frac{z-j}{2}\right) = \frac{z-j}{2d_1}, \quad \gcd\left(\frac{y-i}{2}, \frac{z+j}{2}\right) = \frac{z+j}{2d_2},$$
$$\gcd\left(\frac{y+i}{2}, \frac{z-j}{2}\right) = \frac{z-j}{2d_3}, \quad \gcd\left(\frac{y+i}{2}, \frac{z+j}{2}\right) = \frac{z+j}{2d_4}.$$

Let $d = \min\{d_1, d_2, d_3, d_4\}$.

First suppose $d \geq 5$. Now Lemma 2.5, together with $|i|, |j| \leq 2$ implies

$$\alpha^{\frac{z}{4}-0.472} < Q \leq L_{\frac{z-j}{2d}} M_{\frac{z+j}{2d}} M_{\frac{z-j}{2d}} M_{\frac{z+j}{2d}} \leq L_{\frac{z-j}{10}} M_{\frac{z+j}{10}} M_{\frac{z-j}{10}} M_{\frac{z+j}{10}}$$
$$< \alpha^{\frac{z+2}{20}-0.680} \left(\alpha^{\frac{z+2}{20}+0.264}\right)^3 = \alpha^{\frac{z+2}{5}+0.112}.$$

But $z/4 - 0.472 < (z+2)/5 + 0.112$ contradicting $z \geq 117$.

Now let $d = 4$, that is one of d_1, d_2, d_3, d_4 equals 4. Assume that $\eta_1, \eta_2 \in \{\pm 1\}$. Then $|\eta_1 j|, |\eta_2 i| \leq 2$, and we can assume $z + \eta_1 j \geq y + \eta_2 i$. Contrary, if it does not hold, then by the definition of d the inequality $5/4(z-2) \leq y + 2$ is true, which together with $z > y$ implies $5z \leq 4y + 18 < 5y + 18$. So $z < 18$, which is not the case. Now we have only two possibilities:

$$\frac{z + \eta_1 j}{8} = \frac{y + \eta_2 i}{2} \quad \text{or} \quad \frac{z + \eta_1 j}{8} = \frac{y + \eta_2 i}{6}.$$

In the first case we have $z = 4y + (4\eta_2 i - \eta_1 j) \geq 4y - 10$, and by $z \leq 2y - 1$ we get $4y - 10 \leq 2y - 1$, which implies $y \leq 4$, and then $z \leq 7$, a contradiction.

In the second case let $\eta_1', \eta_2' \in \{\pm 1\}$, such that $(\eta_1', \eta_2') \neq (\eta_1, \eta_2)$. Clearly,

$$y = \frac{3z + 3\eta_1 j - 4\eta_2 i}{4}, \quad \text{and} \quad \frac{y + \eta_2' i}{2} = \frac{3z + 3\eta_1 j + 4(\eta_2' - \eta_2) i}{8}.$$

Put $t = 4(\eta_2' - \eta_2)$. Thus $t = 0$ or ± 8. Applying the first assertion of Lemma 2.6 with $a = \eta_1' j$ and $b = 3\eta_1 j + ti$, it gives

$$\gcd\left(\frac{z + \eta_1' j}{2}, \frac{y + \eta_2' i}{2}\right) = \gcd\left(\frac{z + \eta_1' j}{2}, \frac{3z + 3\eta_1 j + ti}{8}\right) \leq \left|\frac{3\eta_1' j - 3\eta_1 j - ti}{2}\right|,$$

which does not exceed 14. This conclusion is correct if $3a - b \neq 0$, that is if $3\eta_1' - 3\eta_1 j - ti \neq 0$. If $3a - b = 0$, then $3 \mid t$, and then $t = 0$. Thus η_1' must be equal to η_1, so $(\eta_1', \eta_2') = (\eta_1, \eta_2)$, which has been excluded. Subsequently, three of the four factors of Q is at most M_{14} ($M_n \geq L_n$ for any index n) and the fourth factor is $L_{\frac{z \pm j}{8}}$ or $M_{\frac{z \pm j}{8}}$, none of them exceeding $M_{\frac{z+2}{8}}$. So

$$Q \leq M_{14}^3 M_{\frac{z+2}{8}} = 10084^3 M_{\frac{z+2}{8}},$$

and then, by Lemma 2.5, we have

$$\alpha^{\frac{z}{4}-0.472} < Q < \alpha^{21.003} \alpha^{\frac{z+2}{16}+0.264}.$$

Now we conclude $z < 116.7$, and it is a contradiction with $z \geq 117$.

Suppose $d = 3$. We have the two possibilities

$$\frac{z + \eta_1 j}{6} = \frac{y + \eta_2 i}{2} \quad \text{and} \quad \frac{z + \eta_1 j}{6} = \frac{y + \eta_2 i}{4}.$$

In the first case $2y-1 \geq z = 3(y+\eta_2 i) - \eta_1 j \geq 3y-8$ implies $y \leq 7$, and then $z \leq 13$, which is impossible.

In the second case we repeat the treatment of case $d=4$, the variables $\eta_1^{'}$ and $\eta_2^{'}$ satisfy the same conditions. Now $y = (2z + 2\eta_1 j - 3\eta_2 i)/3$ provides

$$\frac{y+\eta_2^{'} i}{2} = \frac{2z+2\eta_1 j - 3\eta_2 i + 3\eta_2^{'} i}{6} = \frac{2z+2\eta_1 j + 3(\eta_2^{'} - \eta_2)i}{6}.$$

Let be $t = 3(\eta_2^{'} - \eta_2)$ with value 0 or ± 6. Use the second assertion of Lemma 2.6 with $a = \eta_1^{'} j$, $b = 2\eta_1 j + ti$. If $2a - b \neq 0$ then

$$\gcd\left(\frac{z+\eta_1^{'} j}{2}, \frac{y+\eta_2^{'} i}{2}\right) = \gcd\left(\frac{z+\eta_1^{'} j}{2}, \frac{2z+2\eta_1 j + ti}{6}\right) \leq \left|\frac{2\eta_1^{'} j - 2\eta_1 j - ti}{2}\right|,$$

which is less then or equal to 10. If $2a-b=0$, that is if $2\eta_1^{'} j - 2\eta_1 j - ti = 0$, then $3 \mid t$ and $j \nmid t$ show $3 \mid \eta_1^{'} - \eta_1$, which can hold only if $\eta_1^{'} = \eta_1$. But in this case t must be zero, too. So $(\eta_1^{'}, \eta_2^{'}) = (\eta_1, \eta_2)$, which is not allowed. We have

$$\alpha^{\frac{z}{4} - 0.472} < Q \leq M_{10}^3 M_{\frac{z+2}{6}} < 724^3 \alpha^{\frac{z+2}{12} + 0.264}$$

by using Lemma 2.5. This implies $z < 96$, again a contradiction.

Now suppose $d = 2$. The only possibility is

$$\frac{z+\eta_1 j}{4} = \frac{y+\eta_2 i}{2}.$$

($\eta_1^{'}$ and $\eta_2^{'}$ are the same as in the previous cases.) It leads to $y = (z + \eta_1 j - 2\eta_2 i)/2$, and then to

$$\frac{y+\eta_2^{'} i}{2} = \frac{z+\eta_1 j - 2\eta_2 i + 2\eta_2^{'} i}{4} = \frac{z+\eta_1 j + ti}{4},$$

where $t = 2(\eta_2^{'} - \eta_2) \in \{0, \pm 4\}$. Let $a = \eta_1^{'} j$, $b = \eta_1 j + ti$. If $a \neq b$, then by the third assertion of Lemma 2.6 we have

$$\gcd\left(\frac{z+\eta_1^{'} j}{2}, \frac{y+\eta_2^{'} i}{2}\right) = \gcd\left(\frac{z+\eta_1^{'} j}{2}, \frac{z+\eta_1 j + ti}{4}\right) \leq \left|\frac{\eta_1^{'} j - \eta_1 j - ti}{2}\right| \leq 6.$$

Thus

$$\alpha^{\frac{z}{4} - 0.472} < Q \leq M_6^3 M_{\frac{z+2}{4}} < \alpha^{9.003} \alpha^{\frac{z+2}{8} + 0.264},$$

and we arrived at a contradiction via $z < 80$. If $a - b = 0$, then $(\eta_1^{'} - \eta_1)j = ti$. Now, if $j = \pm 1$, then (because t is divisible by 4) $4 \mid \eta_1^{'} - \eta_1$ must hold. This occurs only if $\eta_1^{'} = \eta_1$, hence $t = 0$, so $\eta_2^{'} = \eta_2$, which has been excluded. Thus we may suppose $j = \pm 2$ and $\eta_1^{'} \neq \eta_1$. In this case $\eta_1^{'} - \eta_1 = \pm 2$, and $i = \pm 1$. The factors of Q belong to $(-\eta_1, \eta_2)$ and $(\eta_1, -\eta_2)$ can be estimated by M_6. If $(\eta_1, \eta_2) = (1, 1)$, then this factor is $\gcd(M_{\frac{y+i}{2}}, M_{\frac{z+j}{2}})$, which is 2 via $(z+j)/4 = (y+i)/2$ and

Lemma 2.4. If $(\eta_1,\eta_2)=(1,-1)$, then similarly $\gcd(L_{\frac{y-i}{2}},M_{\frac{z+j}{2}})\le 2$. In this two cases we have

$$\alpha^{\frac{z}{4}-0.472}<Q\le 2M_6^2M_{\frac{z+2}{4}}<\alpha^{6.527}\alpha^{\frac{z+2}{8}+0.264},$$

and then $z\le 60$, a contradiction.

Let $(\eta_1,\eta_2)=(-1,-1)$ or $(-1,1)$. From $(z+\eta_1 j)/4=(y+\eta_2 i)/2$ and $|j|=2$, $|i|=1$ it is easy to see that $(z-\eta_1 j)/2=2(y-\eta_2 i)/2$ or $(z-\eta_1 j)/2=2(y-\eta_2 i)/2\pm 4$. If the first case holds, then $\gcd((z-\eta_1 j)/2,(y-\eta_2 i)/2)=(z-\eta_1 j)/4$. Further if $(\eta_1,\eta_2)=(-1,-1)$, then the factor of Q belonging to $(-\eta_1,-\eta_2)$ is $\gcd(M_{\frac{y+i}{2}},M_{\frac{z+j}{2}})=2$ (by Lemma 2.4). If $(\eta_1,\eta_2)=(-1,1)$, then the factor $\gcd(L_{\frac{y-i}{2}},M_{\frac{z+j}{2}})=1$ or 2. If $(z-\eta_1 j)/2=2(y-\eta_2 i)/2\pm 4$ holds, it can be seen by the Euclidean algorithm that $\gcd((z-\eta_1 j)/2,(y-\eta_2 i)/2)\le 4$, and the factor of Q is at most $M_4=14$. So in these cases we conclude

$$\alpha^{\frac{z}{4}-0.472}<Q\le M_4M_6^2M_{\frac{z+2}{4}}<\alpha^{8.005}\alpha^{\frac{z+2}{8}+0.264},$$

and this implies $z<72$.

Assume $d=1$. Now

$$\frac{z+\eta_1 j}{2}=\frac{y+\eta_2 i}{2},$$

where $\eta_1,\eta_2=\pm 1$, and it reduces to $z\pm j=y\pm i$ with $i,j\in\{\pm 1,\pm 2\}$ According to Lemma 2.3 the values depend of the residue y and z modulo 4. Altogether, it means that we need to verify 16 cases.

1. $y\equiv z\equiv 1 \pmod 4$. Clearly, now $i=j=1$, so $z\pm 1=y\pm 1$. The condition $y\equiv z \pmod 4$ leads immediately to $y=z$, a contradiction.

2. $y\equiv 1,\ z\equiv 2 \pmod 4$. Now $i=1,\ j=2$. Thus $z\pm 2=y\pm 1$, and then $z=y\pm 3$ or $z=y\pm 1$. Considering them modulo 4, the only possibility is $z=y+1$. By Lemma 2.3, we conclude

$$L_y-1=L_{\frac{y-1}{2}}M_{\frac{y+1}{2}}=L_{\frac{z-2}{2}}M_{\frac{z}{2}},\quad\text{and}\quad L_z-1=L_{\frac{z-2}{2}}M_{\frac{z+2}{2}}.$$

The common factor $L_{\frac{z-2}{2}}$ together with $\gcd(M_{\frac{z}{2}},M_{\frac{z+2}{2}})=2$ and by Lemma 2.5 provides a contradiction again, since

$$\alpha^{\frac{z}{4}-0.472}<\gcd(L_y-1,L_z-1)=2L_{\frac{z-2}{2}}<\alpha^{0.527}\alpha^{\frac{z-2}{4}-0.680}=\alpha^{\frac{z}{4}-0.653}.$$

3. $y\equiv 1,\ z\equiv 3 \pmod 4$. Here $i=1,\ j=-1$, and the only possibility is $z=y+2$. It follows that

$$L_y-1=L_{\frac{y-1}{2}}M_{\frac{y+1}{2}}=L_{\frac{z-3}{2}}M_{\frac{z-1}{2}},\quad L_z-1=L_{\frac{z+1}{2}}M_{\frac{z-1}{2}},$$

where $\gcd(L_{\frac{z+1}{2}},L_{\frac{z-3}{2}})=1$. Now

$$c|\gcd(L_y-1,L_z-1)=M_{\frac{z-1}{2}}=c_1c>c_1\sqrt{L_z}$$

holds with an appropriate integer c_1. By Lemma 2.7, $M_{\frac{z-1}{2}} < 2\sqrt{L_z}$. So we have $c_1\sqrt{L_z} < M_{\frac{z-1}{2}} < 2\sqrt{L_z}$, which implies $c_1 < 2$, i.e. $c_1 = 1$. Thus $c = M_{\frac{z-1}{2}}$, and we can see from the factorization of $L_y - 1$ and $L_z - 1$ that $a = L_{\frac{z-3}{2}}$, $b = L_{\frac{z+1}{2}}$. Lemma 2.5 shows

$$\alpha^{\frac{x}{2}-0.680} > L_x = ab + 1 = L_{\frac{z-3}{2}} L_{\frac{z+1}{2}} + 1 > L_{\frac{z-3}{2}} L_{\frac{z+1}{2}} > \alpha^{\frac{z-3}{4}-0.944}\alpha^{\frac{z+1}{4}-0.944}.$$

Clearly, $x > z - 3.416$, and then $x \geq z - 3$. In our case $x < y = z - 2$ holds, so $x = z - 3$. This implies $L_{z-3} - 1 = L_x - 1 = L_{\frac{z-3}{2}} L_{\frac{z+1}{2}}$, which entails $L_{\frac{z-3}{2}} \mid L_{z-3} - 1$. Combining it with $L_{\frac{z-3}{2}} \mid L_{z-3}$, we have $L_{\frac{z-3}{2}} = 1$, and z is too small.

4. $y \equiv 1,\ z \equiv 0 \pmod 4$. In this case $z = y + 3$, and

$$L_y - 1 = L_{\frac{y-1}{2}} M_{\frac{y+1}{2}} = L_{\frac{z-4}{2}} M_{\frac{z-2}{2}}, \quad L_z - 1 = \frac{1}{2} L_{\frac{z+2}{2}} M_{\frac{z-2}{2}}.$$

The distance of the subscripts of the appropriate terms of (L_n) is 3, so $\gcd(L_{\frac{z-4}{2}}, \frac{1}{2}L_{\frac{z+2}{2}}) \leq \gcd(L_{\frac{z-4}{2}}, L_{\frac{z+2}{2}}) = 1$ or 3. So $\gcd(L_y - 1, L_z - 1) \mid 3M_{\frac{z-2}{2}}$. Therefore there exist a positive integer c_1 such that

$$c \mid \gcd(L_y - 1, L_z - 1) \mid 3M_{\frac{z-2}{2}} = c_1 c > c_1\sqrt{L_z}.$$

Lemma 2.7 implies $M_{\frac{z-2}{2}} < 2\sqrt{L_z}$, and so $6\sqrt{L_z} > 3M_{\frac{z-2}{2}} > c_1\sqrt{L_z}$ hold. Thus $c_1 < 6$. Since $L_{\frac{z+2}{2}}$ is odd, $M_{\frac{z-2}{2}}$ does not divide $L_z - 1$. So we have $\gcd(L_y - 1, L_z - 1) = \lambda M_{\frac{z-2}{2}}/2$, where $\lambda = 1$ or 3.

When $\lambda = 1$, c divides $M_{\frac{z-2}{2}}/2 = 3M_{\frac{z-2}{2}}/6$, which implies $c_1 \geq 6$, a contradiction.

Assuming $\lambda = 3$, it yields $c \mid 3M_{\frac{z-2}{2}}/2$. Thus either $c = 3M_{\frac{z-2}{2}}/2$ ($c_1 = 2$) or $c = 3M_{\frac{z-2}{2}}/4$ ($c_1 = 4$) holds. We can exclude the second case, because $(z-2)/2$ is odd, and so $M_{\frac{z-2}{2}}$ is not divisible by 4. In the first case $b = L_{\frac{z+2}{2}}/3$ and $a = 2L_{\frac{z-4}{2}}/3$ follow from

$$bc = L_z - 1 = \frac{1}{2} M_{\frac{z-2}{2}} L_{\frac{z+2}{2}} \quad \text{and} \quad ac = L_y - 1 = M_{\frac{z-2}{2}} L_{\frac{z-4}{2}},$$

respectively.

Using the fact that $L_{2k-2}L_{2k+1} + 1 = L_{2k-1}L_{2k}$ holds for every positive integer k (this comes from the explicit formula (2.1)), we can write

$$L_x = ab + 1 = \frac{2}{9} L_{\frac{z-4}{2}} L_{\frac{z+2}{2}} + 1 = \frac{2}{9}(L_{\frac{z-2}{2}} L_{\frac{z}{2}} - 1) + 1 = \frac{2}{9} L_{\frac{z-2}{2}} L_{\frac{z}{2}} + \frac{7}{9}.$$

By Lemma 2.5 we obtain

$$\alpha^{\frac{x}{2}-0.680} > L_x = \frac{2}{9} L_{\frac{z-2}{2}} L_{\frac{z}{2}} + \frac{7}{9} > \frac{2}{9} L_{\frac{z-2}{2}} L_{\frac{z}{2}} > \alpha^{-1.143}\alpha^{\frac{z-2}{4}-0.681}\alpha^{\frac{z}{4}-0.944}$$

(since $(z-2)/2$ is odd). It implies $x > z - 5.176$, so $x \geq z - 5$ holds.

We will reach the contradiction by showing $ab+1 < L_{z-5}$. Knowing that z is even, $L_{z-5} > \alpha^{\frac{z-5}{2}-0.681} = \alpha^{\frac{z}{2}-3.181}$ follows from Lemma 2.5. Since

$$L_{\frac{z-2}{2}} L_{\frac{z}{2}} > \alpha^{\frac{z-4}{2}-0.681} \alpha^{\frac{z}{4}-0.944} = \alpha^{\frac{z}{2}-2.125}$$

and $z \geq 16$, the exponent of α is at least 5.875. Applying Lemma 2.8 with $u_0 = 5$, we have $\kappa = \log_\alpha((2+7\alpha^{-5})/9) < -1.138$, and then

$$ab+1 = \frac{2}{9} L_{\frac{z-2}{2}} L_{\frac{z}{2}} + \frac{7}{9} < \alpha^{-1.138} \alpha^{\frac{z-2}{4}-0.68} \alpha^{\frac{z}{4}-0.943} = \alpha^{\frac{z}{2}-3.261}.$$

From these inequalities

$$L_{z-5} > \alpha^{\frac{z}{2}-3.181} > \alpha^{\frac{z}{2}-3.261} > ab+1$$

follows, and the proof of this part is complete.

5. $\boldsymbol{y \equiv 2, \quad z \equiv 1 \pmod 4}$. Now $z = y+3$, further

$$L_y - 1 = L_{\frac{y-2}{2}} M_{\frac{y+2}{2}} = L_{\frac{z-5}{2}} M_{\frac{z-1}{2}}, \quad L_z - 1 = L_{\frac{z-1}{2}} M_{\frac{z+1}{2}}.$$

It is easy to see from Lemma 2.4 that $\gcd(L_{\frac{z-5}{2}}, L_{\frac{z-1}{2}}) = 1$, $\gcd(M_{\frac{z+1}{2}}, M_{\frac{z-1}{2}}) = 2$, $\gcd(L_{\frac{z-5}{2}}, M_{\frac{z+1}{2}}) \leq M_3 = 10$, $\gcd(M_{\frac{z-1}{2}}, L_{\frac{z-1}{2}}) \leq 2$. Consequently,

$$\alpha^{\frac{z}{4}-0.472} < \gcd(L_y - 1, L_z - 1) \leq 40 < \alpha^{2.802},$$

and then $z < 14$, a contradiction again.

6. $\boldsymbol{y \equiv z \equiv 2 \pmod 4}$. In this case $i = j = 2$. Then $z = y+4$ follows. The identities

$$L_y - 1 = L_{\frac{y-2}{2}} M_{\frac{y+2}{2}} = L_{\frac{z-6}{2}} M_{\frac{z-2}{2}}, \quad L_z - 1 = L_{\frac{z-2}{2}} M_{\frac{z+2}{2}}$$

and $\gcd(L_{\frac{z-6}{2}}, L_{\frac{z-2}{2}}) = 1$, $\gcd(M_{\frac{z-2}{2}}, M_{\frac{z+2}{2}}) = 2$ (because both terms cannot be divisible by 4), $\gcd(L_{\frac{z-6}{2}}, M_{\frac{z+2}{2}}) \leq M_4 = 14$, $\gcd(M_{\frac{z-2}{2}}, L_{\frac{z-2}{2}}) \leq 2$ (see Lemma 2.4) induce

$$\alpha^{\frac{z}{4}-0.472} < \gcd(L_y - 1, L_z - 1) \leq 56 < \alpha^{3.057},$$

which gives $z < 15$.

7. $\boldsymbol{y \equiv 2, \quad z \equiv 3 \pmod 4}$. Here $z = y+1$, moreover we have

$$L_y - 1 = L_{\frac{y-2}{2}} M_{\frac{y+2}{2}} = L_{\frac{z-3}{2}} M_{\frac{z+1}{2}}, \quad L_z - 1 = L_{\frac{z+1}{2}} M_{\frac{z-1}{2}}.$$

Again by Lemma 2.4,

$$\gcd(L_{\frac{z-3}{2}}, L_{\frac{z+1}{2}}) = 1, \quad \gcd(M_{\frac{z+1}{2}}, M_{\frac{z-1}{2}}) = 2,$$
$$\gcd(L_{\frac{z-3}{2}}, M_{\frac{z-1}{2}}) \leq 2, \quad \gcd(M_{\frac{z+1}{2}}, L_{\frac{z+1}{2}}) \leq 2.$$

Thus

$$\alpha^{\frac{z}{4}-0.472} < \gcd(L_y-1, L_z-1) \le 8 < \alpha^{1.579}$$

follows, which implies $z < 9$.

8. $\boldsymbol{y \equiv 2,\ \ z \equiv 0\ \ (\mathrm{mod}\ 4)}$. Now $i = 2$, $j = -2$, and $y \pm 2 = j \mp 2$ cannot hold modulo 4.

9. $\boldsymbol{y \equiv 3,\ \ z \equiv 1\ \ (\mathrm{mod}\ 4)}$. In this case the only possibility is $z = y + 2$. Obviously,

$$L_y - 1 = L_{\frac{y+1}{2}} M_{\frac{y-1}{2}} = L_{\frac{z-1}{2}} M_{\frac{z-3}{2}}, \quad L_z - 1 = L_{\frac{z-1}{2}} M_{\frac{z+1}{2}}$$

hold. Beside the common factor, we get $\gcd(M_{\frac{z-3}{2}}, M_{\frac{z+1}{2}}) = 2$ (because the subscripts are odd). Hence $\gcd(L_y - 1, L_z - 1) = 2L_{\frac{z-1}{2}}$, further we see

$$c \mid \gcd(L_y-1, L_z-1) = 2L_{\frac{z-1}{2}} = c_1 c > c_1\sqrt{L_z}$$

with an appropriate c_1. By the second assertion of case (1) in Lemma 2.7, $\sqrt{L_z} > \sqrt{3/2}L_{\frac{z-1}{2}}$, subsequently

$$2L_{\frac{z-1}{2}} > c_1\sqrt{L_z} > c_1\sqrt{\frac{3}{2}}L_{\frac{z-1}{2}}$$

holds, providing $c_1 < \frac{2\sqrt{2}}{\sqrt{3}} < 2$. So only $c_1 = 1$ is possible. Thus $c = 2L_{\frac{z-1}{2}}$, and from the factorizations

$$ac = L_y - 1 = L_{\frac{z-1}{2}} M_{\frac{z-3}{2}}, \quad bc = L_z - 1 = L_{\frac{z-1}{2}} M_{\frac{z+1}{2}}$$

we obtain

$$a = \frac{1}{2}M_{\frac{z-3}{2}} \quad \text{and} \quad b = \frac{1}{2}M_{\frac{z+1}{2}}.$$

Finally, we show that $c < b$. (2.10) yields $M_{2k+1} = 2L_{2k} + 2L_{2k+2} > 4L_{2k}$. Now $(z-1)/2$ is even, so $2L_{\frac{z-1}{2}} < \frac{1}{2}M_{\frac{z+1}{2}}$. Thus $c < b$, contradicting the condition $a < b < c$.

10. $\boldsymbol{y \equiv 3,\ \ z \equiv 2\ \ (\mathrm{mod}\ 4)}$. We find $z = y + 3$, and

$$L_y - 1 = L_{\frac{y+1}{2}} M_{\frac{y-1}{2}} = L_{\frac{z-2}{2}} M_{\frac{z-4}{2}}, \quad L_z - 1 = L_{\frac{z-2}{2}} M_{\frac{z+2}{2}}.$$

By Lemma 2.4, $\gcd(M_{\frac{z-4}{2}}, M_{\frac{z+2}{2}}) = 2$ follows (not $M_3 = 10$, because if the subscripts are divisible by 3, dividing them by 3 exactly one of the integers will be odd). Now

$$\alpha^{\frac{z}{4}-0.472} < \gcd(L_y-1, L_z-1) = 2L_{\frac{z-2}{2}} < \alpha^{0.527}\alpha^{\frac{z-2}{4}-0.680}$$

leads to a contradiction.

11. $\boldsymbol{y \equiv z \equiv 3 \pmod 4}$. In this case, $i = j = -1$ implies $y = z$, which is a contradiction.

12. $\boldsymbol{y \equiv 3,\quad z \equiv 0 \pmod 4}$. Here $z = y + 1$, further

$$L_y - 1 = L_{\frac{y+1}{2}} M_{\frac{y-1}{2}} = L_{\frac{z}{2}} M_{\frac{z-2}{2}}, \quad L_z - 1 = \frac{1}{2} L_{\frac{z+2}{2}} M_{\frac{z-2}{2}}$$

hold. Lemma 2.4 provides $\gcd(L_{\frac{z}{2}}, L_{\frac{z+2}{2}}) = 1$, and we obtain $\gcd(L_y - 1, L_z - 1) = \frac{1}{2} M_{\frac{z-2}{2}}$ (because $L_{\frac{z+2}{2}}$ is odd). Hence

$$c| \gcd(L_y - 1, L_z - 1) = \frac{1}{2} M_{\frac{z-2}{2}} = c_1 c > c_1 \sqrt{L_z}.$$

By Lemma 2.7 we have $M_{\frac{z-2}{2}} < 2\sqrt{L_z}$. Thus $M_{\frac{z-2}{2}} > 2c_1 \sqrt{L_z} > c_1 M_{\frac{z-2}{2}}$, which implies $c_1 < 1$, an impossibility.

13. $\boldsymbol{y \equiv 0,\quad z \equiv 1 \pmod 4}$. In this case $z = y + 1$, moreover

$$L_y - 1 = \frac{1}{2} L_{\frac{y+2}{2}} M_{\frac{y-2}{2}} = \frac{1}{2} L_{\frac{z+1}{2}} M_{\frac{z-3}{2}}, \quad L_z - 1 = L_{\frac{z-1}{2}} M_{\frac{z+1}{2}}.$$

By Lemma 2.4, we obtain $\gcd(L_{\frac{z+1}{2}}, L_{\frac{z-1}{2}}) = 1$, $\gcd(M_{\frac{z-3}{2}}, M_{\frac{z+1}{2}}) = 2$, $\gcd(L_{\frac{z+1}{2}}, M_{\frac{z+1}{2}}) \le 2$, $\gcd(M_{\frac{z-3}{2}}, L_{\frac{z-1}{2}}) \le 2$. Then

$$\alpha^{\frac{z}{4} - 0.472} < \gcd(L_y - 1, L_z - 1) \le 8 < \alpha^{1.579}$$

implies $z < 9$.

14. $\boldsymbol{y \equiv 0,\quad z \equiv 2 \pmod 4}$. Now, by Lemma 2.3, $i = -2$, $j = 2$, and $y \mp 2 = z \pm 2$ follow, which is not possible.

15. $\boldsymbol{y \equiv 0,\quad z \equiv 3 \pmod 4}$. In this case $z = y + 3$, and

$$L_y - 1 = \frac{1}{2} L_{\frac{y+2}{2}} M_{\frac{y-2}{2}} = \frac{1}{2} L_{\frac{z-1}{2}} M_{\frac{z-5}{2}}, \quad L_z - 1 = L_{\frac{z+1}{2}} M_{\frac{z-1}{2}}.$$

Via Lemma 2.4 we see $\gcd(L_{\frac{z-1}{2}}, L_{\frac{z+1}{2}}) = 1$, $\gcd(M_{\frac{z-5}{2}}, M_{\frac{z-1}{2}}) = 2$, $\gcd(L_{\frac{z-1}{2}}, M_{\frac{z-1}{2}}) = 1$, (because $\frac{z-1}{2}$, and so $L_{\frac{z-1}{2}}$ is odd), $\gcd(M_{\frac{z-5}{2}}, L_{\frac{z+1}{2}}) \le M_3 = 10$. These lead to a contradiction via

$$\alpha^{\frac{z}{4} - 0.472} < \gcd(L_y - 1, L_z - 1) \le 20 < \alpha^{2.275}.$$

16. $\boldsymbol{y \equiv z \equiv 0 \pmod 4}$. In the last case the only possibility is $z = y + 4$. We have

$$L_y - 1 = \frac{1}{2} L_{\frac{y+2}{2}} M_{\frac{y-2}{2}} = \frac{1}{2} L_{\frac{z-2}{2}} M_{\frac{z-6}{2}}, \quad L_z - 1 = \frac{1}{2} L_{\frac{z+2}{2}} M_{\frac{z-2}{2}}.$$

By Lemma 2.4, we get

$$\gcd(L_{\frac{z-2}{2}}, L_{\frac{z+2}{2}}) = 1,$$

$$\gcd(M_{\frac{z-6}{2}}, M_{\frac{z-2}{2}}) = 2,$$
$$\gcd(L_{\frac{z-2}{2}}, M_{\frac{z-2}{2}}) = 1 \text{ (because } (z-2)/2 \text{ is odd)},$$
$$\gcd(M_{\frac{z-6}{2}}, L_{\frac{z+2}{2}}) \leq M_4 = 14.$$

Then we obtain $z < 10$ from

$$\alpha^{\frac{z}{4}-0.472} < \gcd(L_y - 1, L_z - 1) \leq 14 < \alpha^{2.004}.$$

Case II: $z \leq 116$. The proof of Theorem 1 will be complete, if we check the finitely many cases $3 \leq x < y < z \leq 116$. It has been done by a computer verification based on the following observation. The equations (1.2) imply

$$(L_x - 1)(L_y - 1) = a^2bc = a^2(L_z - 1).$$

Thus

$$\sqrt{\frac{(L_x - 1)(L_y - 1)}{L_z - 1}} \tag{3.1}$$

must be an integer. Checking the given range we found that (3.1) is never an integer.

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