Annales Mathematicae et Informaticae
47 (2017) pp. 211–218
http://ami.ektf.hu

Infinitary superperfect numbers

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Submitted June 13, 2017 — Accepted June 30, 2017

Abstract

We shall show that 9 is the only odd infinitary superperfect number.

Keywords: Odd perfect numbers, infinitary superperfect numbers, unitary divisors, infinitary divisors, the sum of divisors

MSC: 11A05, 11A25

1. Introduction

As usual, $\sigma(N)$ denotes the sum of divisors of a positive integer N. N is called perfect if $\sigma(N) = 2N$. It is a well-known unsolved problem to decide whether or not an odd perfect number exists. Interest to this problem has produced many analogous notions and problems concerning divisors of an integer. For example, Suryanarayana [15] called N to be superperfect if $\sigma(\sigma(N)) = 2N$. It is asked in this paper and still unsolved whether there are any odd superperfect numbers.

Some special classes of divisors have also been studied in several papers. One of them is the class of unitary divisors defined by Eckford Cohen [2]. A divisor d of n is called a unitary divisor if $\gcd(d, n/d) = 1$. Wall [16] introduced the notion of biunitary divisors. Letting $\gcd_1(a, b)$ denote the greatest common unitary divisor of a and b, a divisor d of a positive integer n is called a biunitary divisor if $\gcd_1(d, n/d) = 1$.

Graeme L. Cohen [3] generalized these notions and introduced the notion of k-ary divisors for any nonnegative integer k recursively. Any divisor of a positive integer n is called a 0-ary divisor of n and, for each nonnegative integer k, a divisor d of a positive integer n is called a $(k + 1)$-ary divisor if d and n/d does not have a common k-ary divisor other than 1. Clearly, a 1-ary divisor is a unitary divisor and a 2-ary divisor is a biunitary divisor. We note that a positive integer $d = \prod_i p_i^{f_i}$ with p_i distinct primes and $f_i \geq 0$ is a k-ary divisor of $n = \prod_i p_i^{e_i}$ if and only if $p_i^{f_i}$ is a k-ary divisor of $p_i^{e_i}$ for each i. G. L. Cohen [3, Theorem 1] showed that, if p^f is an $(e-1)$-ary divisor of p^e, then p^f is a k-ary divisor of p^e for any $k \geq e-1$ and called such a divisor to be an infinitary divisor. For any positive integer n, a divisor $d = \prod_i p_i^{f_i}$ of $n = \prod_i p_i^{e_i}$ is called an infinitary divisor if $p_i^{f_i}$ is an infinitary divisor of $p_i^{e_i}$ for each i, which is written as $d \mid_\infty n$.

According to E. Cohen [2], Wall [16] and G. L. Cohen [3] respectively, henceforth $\sigma^*(N), \sigma^{**}(N)$ and $\sigma_\infty(n)$ denote the sum of unitary, biunitary and infinitary divisors of N, respectively.

Replacing σ by σ^*, Subbarao and Warren [14] introduced the notion of a unitary perfect number. N is called unitary perfect if $\sigma^*(N) = 2N$. They proved that there are no odd unitary perfect numbers and $6, 60, 90, 87360$ are the first four unitary perfect numbers. Later the fifth unitary perfect number has been found by Wall [17], but no further instance has been found. Subbarao [13] conjectured that there are only finitely many unitary perfect numbers. Similarly, a positive integers N is called biunitary perfect if $\sigma^{**}(N) = 2N$. Wall [16] showed that $6, 60$ and 90, the first three unitary perfect numbers, are the only biunitary perfect numbers.

G. L. Cohen [3] introduced the notion of infinitary perfect numbers; a positive integer n is called infinitary perfect if $\sigma_\infty(n) = 2n$. Cohen [3, Theorem 16] showed that $6, 60$ and 90, exactly all of the biunitary perfect numbers, are also all of the infinitary perfect numbers not divisible by 8. Cohen gave 14 infinitary perfect numbers and Pedersen's database, which is now available at [8], contains 190 infinitary perfect numbers.

Combining the notion of superperfect numbers and the notion of unitary divisors, Sitaramaiah and Subbarao [10] studied unitary superperfect numbers, integers N satisfying $\sigma^*(\sigma^*(N)) = 2N$. They found all unitary superperfect numbers below 10^8. The first ones are $2, 9, 165, 238$. Thus, there are both even and odd ones. The author [18] showed that $9, 165$ are all the odd ones.

Now we can call an integer N satisfying $\sigma_\infty(\sigma_\infty(N)) = 2N$ to be infinitary superperfect. We can see that 2 and 9 are infinitary superperfect, while 2 is also superperfect (in the ordinary sense) and 9 is also unitary superperfect. Below 2^{29}, we can find some integers n dividing $\sigma_\infty(\sigma_\infty(n))$ but we cannot find any other infinitary superperfect numbers.

Analogous to [18], we can show that following result.

Theorem 1.1. *9 is the only odd infinitary superperfect number.*

We can see that this immediately follows from the following result.

Theorem 1.2. *If N is an infinitary superperfect number with $\omega(\sigma_\infty(N)) \leq 2$, then $N = 2$ or $N = 9$.*

Indeed, if N is odd and $\sigma_\infty(\sigma_\infty(N)) = 2N$, then $\sigma_\infty(N)$ is a prime power or of the form $2^f q^{2^l}$ with $f, l \geq 0$ as shown in Section 3.

Table 1: All integers $N \leq 2^{29}$ for which $\sigma_\infty(\sigma_\infty(N)) = kN$

N	k	N	k
1	1	$428400 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 17$	3
2	2	$602208 = 2^5 \cdot 3^3 \cdot 17 \cdot 41$	6
$8 = 2^3$	3	$636480 = 2^6 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17$	4
$9 = 3^2$	2	$763776 = 2^7 \cdot 3^3 \cdot 13 \cdot 17$	10
$10 = 2 \cdot 5$	3	$856800 = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7$	6
$15 = 3 \cdot 5$	4	$1321920 = 2^6 \cdot 5^5 \cdot 5 \cdot 17$	7
$18 = 2 \cdot 3^2$	4	$1505520 = 2^4 \cdot 3^3 \cdot 5 \cdot 17 \cdot 41$	4
$24 = 2^3 \cdot 3$	5	$3011040 = 2^5 \cdot 3^3 \cdot 5 \cdot 17 \cdot 41$	8
$30 = 2 \cdot 3 \cdot 5$	5	$3084480 = 2^6 \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$	5
$60 = 2^2 \cdot 3 \cdot 5$	6	$21679488 = 2^7 \cdot 3^5 \cdot 17 \cdot 41$	7
$720 = 2^4 \cdot 3^2 \cdot 5$	3	$22276800 = 2^6 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$	6
$1020 = 2^2 \cdot 3 \cdot 5 \cdot 17$	4	$30844800 = 2^{10} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 17$	7
$4080 = 2^4 \cdot 3 \cdot 5 \cdot 17$	3	$31615920 = 2^4 \cdot 3^4 \cdot 5 \cdot 7 \cdot 17 \cdot 41$	4
$8925 = 3 \cdot 5^2 \cdot 7 \cdot 17$	4	$44553600 = 2^7 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$	12
$14688 = 2^5 \cdot 3^3 \cdot 17$	5	$50585472 = 2^7 \cdot 3^4 \cdot 7 \cdot 17 \cdot 41$	5
$14976 = 2^7 \cdot 3^2 \cdot 13$	5	$63231840 = 2^5 \cdot 3^4 \cdot 5 \cdot 7 \cdot 17 \cdot 41$	8
$16728 = 2^3 \cdot 3 \cdot 17 \cdot 41$	4	$126463680 = 2^6 \cdot 3^4 \cdot 5 \cdot 7 \cdot 17 \cdot 41$	6
$17850 = 2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 17$	8	$213721200 = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 257$	4
$35700 = 2^2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 17$	6	$230177280 = 2^9 \cdot 3 \cdot 5 \cdot 17 \cdot 41 \cdot 43$	9
$36720 = 2^4 \cdot 3^3 \cdot 5 \cdot 17$	6	$252927360 = 2^7 \cdot 3^4 \cdot 5 \cdot 7 \cdot 17 \cdot 41$	12
$37440 = 2^6 \cdot 3^2 \cdot 5 \cdot 13$	6	$307758528 = 2^6 \cdot 3^5 \cdot 7 \cdot 11 \cdot 257$	5
$66912 = 2^5 \cdot 3 \cdot 17 \cdot 41$	3	$345265920 = 2^8 \cdot 3^2 \cdot 5 \cdot 17 \cdot 41 \cdot 43$	3
$71400 = 2^3 \cdot 3 \cdot 5^2 \cdot 7 \cdot 17$	12	$427442400 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 257$	8
$285600 = 2^5 \cdot 3 \cdot 5^2 \cdot 7 \cdot 17$	9	$437898240 = 2^{10} \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 43$	5
$308448 = 2^5 \cdot 3^4 \cdot 7 \cdot 17$	5	$466794240 = 2^8 \cdot 3 \cdot 5 \cdot 11 \cdot 43 \cdot 257$	3
$381888 = 2^6 \cdot 3^3 \cdot 13 \cdot 17$	5	$512930880 = 2^6 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11 \cdot 257$	4

Our method does not seem to work to find all odd super perfect numbers since $\sigma(\sigma(N)) = 2N$ does not seem to imply that $\omega(\sigma(N)) \leq 2$. Even assuming that $\omega(\sigma(N)) \leq 2$, the property of σ that $\sigma(p^e)/p^e > 1 + 1/p$ prevents us from showing that $\sigma(\sigma(N)) < 2$. All that we know is the author's result in [19] that there are only finitely many odd superperfect numbers N with $\omega(\sigma(N)) \leq k$ for each k. For the biunitary analogues, the author [20] showed that 2 and 9 are the only integers N (even or odd!) such that $\sigma^{**}(\sigma^{**}(N)) = 2N$.

In Table 1, we give all integers $N \leq 2^{29}$ dividing $\sigma_\infty(\sigma_\infty(N))$. We found no other infinitary superperfect numbers other than 2 and 9, while we found several integers N dividing $\sigma_\infty(\sigma_\infty(N))$. From this table, we are led to conjecture that 2 is the only even infinitary superperfect number. On the other hand, it seems that for any integer $k \geq 3$, there exist infinitely many integers N for which $\sigma_\infty(\sigma_\infty(N)) = kN$.

2. Preliminary lemmas

In this section, we shall give several preliminary lemmas concerning the sum of infinitary divisors used to prove our main theorems.

We begin by introducing Theorem 8 of [3]: writing binary expansions of e, f as $e = \sum_{i \in I} 2^i$ and $f = \sum_{j \in J} 2^j$, p^f is an infinitary divisor of p^e if and only if J is a subset of I.

Hence, factoring $n = \prod_{i=1}^r p_i^{e_i}$ and writing a binary expansion of each e_i as $e_i = \sum_j y_{ij} 2^j$ with $y_{ij} \in \{0, 1\}$, we observe that, as is shown in [3][Theorem 13],

$$\sigma_\infty(n) = \prod_{i=1}^{r} \prod_{y_{ij}=1} \left(1 + p_i^{2^j}\right). \tag{2.1}$$

From this, we can easily deduce the following lemma.

Lemma 2.1. *Let $v_p(n)$ denote the exponent of a prime p in the factorization of the integer n and let $l(e)$ denote the number of 1's in the binary expansion of e. Then we have $v_2(\sigma_\infty(n)) \geq \sum_{p>2} l(v_p(n)) \geq \omega(n) - 1$. In particular, $\sigma_\infty(n)$ is odd if and only if n is a power of 2.*

Proof. For each prime factor p_i, write a binary expansion of each e_i as $e_i = \sum_j y_{ij} 2^j$ with $y_{ij} \in \{0, 1\}$. Hence, $l(e_i) = \sum_j y_{ij}$ holds for each i. Unless $p_i = 2$, $p_i^{2^j} + 1$ is even for any $j \geq 0$. By (2.1), each product $\sigma_\infty(p_i^{e_i}) = \prod_{y_{ij}=1} \left(1 + p_i^{2^j}\right)$ except $p_i = 2$ is divisible by 2 at least $l(e_i)$ times and $\sigma_\infty(n)$ at least $\sum_{p_i \neq 2} l(e_i)$ times. We can easily see that $\sum_{p_i \neq 2} l(e_i) \geq \omega(n) - 1$ since $l(m) > 0$ for any nonzero integer m. □

The following two lemmas follow almost immediately from Bang's result [1]. But we shall include direct proofs.

Lemma 2.2. *If p is a prime and $\sigma_\infty(p^e)$ is a prime power, then p is a Mersenne prime and $e = 1$ or $p = 2, e = 2^l$ and $\sigma_\infty(p^e)$ is a Fermat prime.*

Proof. If $e = 1$, then $p + 1$ must be a prime power. If p is odd, then $p + 1$ must be even and therefore a power of two. Hence, $p = 2$ or p is a Mersenne prime.

If $e = 2^l \geq 2$ is a power of two, then $\sigma_\infty(p^e) = p^{2^l} + 1$ must be a prime power, which is shown to be impossible by Lebesgue [6]. Hence, $p^{2^l} + 1$ must be prime. If $p > 2$, then $p^{2^l} + 1 > 2$ is even and therefore cannot be prime. If $p = 2$, then $\sigma_\infty(2^e) = 2^{2^l} + 1$ must be a Fermat prime.

If $l(e) > 0$, then $\sigma_\infty(p^e)$ has at least two factors $p^{2^k} + 1$ and $p^{2^l} + 1$ with $l > k$. If p is odd, then $p^{2^l} + 1$ cannot be prime power as above. If $p = 2$, then these two factors must give distinct Fermat primes. Hence, in both cases, $(p^{2^k} + 1)(p^{2^l} + 1)$ cannot be a prime power and neither can $\sigma_\infty(p^e)$. □

Lemma 2.3. *$\sigma_\infty(2^e)$ has at least $l(e)$ distinct prime factors. If p is an odd prime, then $\sigma_\infty(p^e)$ has at least $l(e) + 1$ distinct prime factors.*

Proof. Whether p is odd or two, $\sigma_\infty(p^e)$ is the product of $l(e)$ distinct numbers of the form $p^{2^l}+1$. If $k > l$, then $p^{2^k}+1 \equiv 2 \pmod{p^{2^l}+1}$ and therefore $p^{2^k}+1$ has a odd prime factor not dividing $p^{2^l}+1$. □

Finally, we shall introduce two technical lemmas needed in the proof.

Lemma 2.4. *If $p^2+1 = 2q^m$ with $m \geq 2$, then m must be a power of 2 and, for any given prime q, there exists at most one such m. If $p^{2^k}+1 = 2q^m$ with $k > 1$, then $m = 1$.*

Proof. Cohn [4] showed that $x^2+1 = 2y^n$ has no solution in positive integers x, y, n with $xy > 1$ and $n > 2$ other than $(x, y, n) = (239, 13, 4)$, quoting the result of Ljunggren [7] and the simpler proof of Steiner and Tzanakis [11] for $n = 4$ and rediscovering the result of Størmer [12, Théorème 8] for odd n.

Hence, if $p^2+1 = 2q^m$ with $m \geq 2$, then we must have $m = 2$ for any prime $q \neq 239$ and $m = 4$ for $q = 239$. This implies the former statement.

If $p^{2^k}+1 = 2q^m$ with $k > 1$, then $m = 2^l$ for some integer $l \geq 0$. Now the latter statement follows observing that $x^4+1 = 2y^2$, equivalent to $y^4 - x^4 = (y^2-1)^2$, has no solution other than $(1, 1)$ by Fermat's well-known right triangle theorem (see for example Theorem 2 in Chapter 4 of Mordell [9]). □

Lemma 2.5. *If p, q are odd primes satisfying $p^{2^k}+1 = 2q$ and $2^{2^{k+1}} \equiv 1 \pmod q$ with $k > 0$, then $(p, q) = (3, 5)$ and $k = 1$.*

Proof. Since q divides $2^{2^{k+1}}-1 = (2^{2^k}+1)(2^{2^k}-1)$, q must divide either of $2^{2^k}+1$ or $2^{2^k}-1$. In both cases, $q \leq 2^{2^k}+1$ and therefore, noting that $k > 0$,

$$2^{(2^k+1)(\log p/\log 2)} = p^{2^k+1} < 2q \leq 2(2^{2^k}+1) = 2^{2^k+1}+2 < 2^{2^k+2}. \tag{2.2}$$

Hence, we have $(2^k+1)(\log p/\log 2) < 2^k+2$ and $\log p/\log 2 < 1 + 1/(2^k+1)$, which leads to $k = 1, p = 3$ and $q = (3^2+1)/2 = 5$. □

3. Proofs of Theorems 1.1 and 1.2

We begin by noting that Theorem 1.1 follows from Theorem 1.2. Indeed, if N is odd and $\sigma_\infty(\sigma_\infty(N)) = 2N$, then Lemma 2.1 gives that $\omega(\sigma_\infty(N)) \leq 2$ and therefore Theorem 1.2 would yield Theorem 1.1.

In order to prove Theorem 1.2, we shall first show that if $\sigma_\infty(N)$ is odd or a prime power, then N must be 2. If $\sigma_\infty(N)$ is a prime power, then Lemma 2.2 immediately yields that $N = 2^e$ or $\sigma_\infty(N)$ is a power of 2, where the latter case cannot occur since $\sigma_\infty(\sigma_\infty(N))$ must be odd in the latter case while we must have $\sigma_\infty(\sigma_\infty(N)) = 2N$. If $\sigma_\infty(N)$ is odd, then N must be a power of 2 by Lemma 2.1.

Thus, we see that if $\sigma_\infty(N)$ is odd or a prime power, then $N = 2^e$ must be a power of 2. Now we can easily see that $\sigma_\infty(\sigma_\infty(N)) = 2N = 2^{e+1}$ must also be a power of 2. Hence, for each prime-power factor $q_i^{f_i}$ of $\sigma_\infty(N)$, $\sigma_\infty(q_i^{f_i})$ is also a power of 2. By Lemma 2.2, each $f_i = 1$ and q_i is a Mersenne prime. Hence, we see that $\sigma_\infty(N) = \sigma_\infty(2^e)$ must be a product of Mersenne primes. Let r be an integer such that $2^{2^r} \mid_\infty N$. Then $2^{2^r} + 1$ must also be a product of Mersenne primes. By the first supplementary law, only $r = 0$ is appropriate and therefore $e = 0$. Thus, we conclude that if $\sigma_\infty(N)$ is odd or a prime power, then $N = 2$.

Henceforth, we are interested in the case $\sigma_\infty(N) = 2^f q^{2^l}$ with $f > 0$ and $l \geq 0$. Factor $N = \prod_i p_i^{e_i}$. Our proof proceeds as follows: (I) if $l = 0$, then there exists exactly one prime factor p_i of N such that q divides $\sigma_\infty(p_i^{e_i})$, (IA) if $l = 0$ and $f = 1$, then $N = 9$, (IBa) it is impossible that $l = 0, f > 1$ and $p_i \mid q + 1$, (IBb) it is impossible that $l = 0, f > 1$ and p_i does not divide $q + 1$, (II) if $l > 0$, then there exists at most one prime factor p_i of $q^{2^l} + 1$ such that $p_i^{2^k} + 1 = 2q$, (IIa) it is impossible that $q^{2^l} + 1$ has no such prime factor, (IIb) it is impossible that $q^{2^l} + 1$ has one such prime factor p_i.

First we shall settle the case $l = 0$, that is, $\sigma_\infty(N) = 2^f q$. Since q divides N exactly once, there exists exactly one index i such that q divides $\sigma_\infty(p_i^{e_i})$.

For any index j other than i, we must have $\sigma_\infty(p_j^{e_j}) = 2^{k_j}$ and therefore, by Lemma 2.2, we have $e_j = 1$ and $p_j = 2^{k_j} - 1$ for some intger k_j. Clearly, p_j must divide $2N = \sigma_\infty(\sigma_\infty(N)) = \sigma_\infty(2^f)(q + 1)$ and the first supplementary law yields that $p_j \mid (q + 1)$ unless $p_j = 3$.

If $f = 1$, then $N = 2^k p^e$ for an odd prime p by Lemma 2.1 and $2N = \sigma_\infty(2q) = 3(q+1)$. Hence, $p = 3$ and $\sigma_\infty(2^k 3^e) = 2q$. But, we observe that $k = 0$ and $N = 3^e$ since $\sigma_\infty(3^e) > 2$ is even. By Lemma 2.1, we have $e = 2^u$ and $3^e + 1 = 2q = 2(2 \times 3^{e-1} - 1) = 4 \times 3^{e-1} - 2$. Hence, $3^{e-1} = 3$, that is, $N = 9$ and $q = 5$. This gives an infinitary superperfect number 9.

Now we consider the case $f > 1$. If $2^{2^m} \mid_\infty 2^f$ with $m > 0$ and p divides $2^{2^m} + 1$, then p must be congruent to 1 (mod 4) and therefore must be p_i. By Lemma 2.2, we must have $2^{2^m} + 1 = p_i$. Hence, $f = 2^m$ and $\sigma_\infty(2^f) = p_i$ or $f = 2^m + 1$ and $\sigma_\infty(2^f) = 3p_i$.

If p_i divides $q + 1$, then $e_i \geq 2$. By Lemma 2.3, we must have $e_i = 2^v$ and $p_i^{e_i} + 1 = 2q$. Since $p_i = \sigma_\infty(2^f)$, $p_i^{e_i - 1}$ divides $q + 1$ and therefore $2(q+1) = p_i^{e_i} + 3$, Hence, $p_i^{e_i} \equiv -3 \pmod{p_i^{e_i - 1}}$, which is impossible since $p_i > 3$ now.

If p_i does not divide $q + 1$, then $e_i = 1$ and $2^{k_i} q = p_i + 1 = 2^{2^m} + 2$. Hence, $k_i = 1$ and $q = 2^{2^m - 1} + 1$. Now $m = 1$ with $q = 3$ is the only m such that q is prime. Hence, we have $p_i = 2q - 1 = 5$, $\sigma_\infty(2^f) = 5$ or 15 and $N = \sigma_\infty(2^f)(q + 1)/2 = 10$ or 30, neither of which is infinitary superperfect. Thus, the case $\sigma_\infty(N) = 2^f q$ with $f > 1$ has turned out to be impossible and $N = 3^2$ is the only infinitary superperfect number with $\sigma_\infty(N) = 2q$.

Now the remaining case is $\sigma_\infty(N) = 2^f q^g$ with $g > 1$. We can take a positive integer l such that $q^{2^l} \mid_\infty q^g$. If p is odd and divides $\sigma_\infty(q^{2^l}) = q^{2^l} + 1$, then p divides $\sigma_\infty(\sigma_\infty(N)) = 2N$ and therefore p divides N. If $p^{2^k} \mid_\infty N$, then $p^{2^k} + 1$ divides $\sigma_\infty(N) = 2^f q^{2^l}$ and therefore we can write $p^{2^k} + 1 = 2q^t$. We note that $p \equiv 1 \pmod 4$ since p is odd and divides $q^{2^l} + 1$ with $l > 0$. Hence, we see that a) if $k = 0$, then $p + 1 = 2q^t$, b) if $k = 1$, then $p^2 + 1 = 2q$ or $2q^{2^u}$ by Lemma 2.4 and

c) if $k > 1$, then $p^{2^k} + 1 = 2q$ by Lemma 2.4.

Clearly, there exists at most one prime factor p_i of N such that $p_i^{2^k} + 1 = 2q$ for some integer $k > 0$. Moreover, by Lemma 2.4, there exists at most one prime factor p_j of N such that $p_j^2 + 1 = 2q^{2^u}$ for some integer $u > 0$. Letting i and j denote the indices of such primes respectively if these exist, $q^{2^l} + 1$ can be written in the form

$$q^{2^l} + 1 = 2p_i^{s_i} p_j^{s_j} (2q^{t_1} - 1)(2q^{t_2} - 1)..., \tag{3.1}$$

where $s_i, s_j \geq 0$ may be zero.

If $s_i \neq 0$, then we have $2p_i^{s_i} p_j^{s_j} \equiv \pm 1 \pmod{q}$ and therefore, observing that $p_i^{2^{k+1}} \equiv p_j^4 \equiv 1 \pmod{q}$, we have $2^{2^{k+1}} \equiv 1 \pmod{q}$. By Lemma 2.5, we must have $p_i = 3, e_i = 2$ and $q = 5$ and p_j cannot exist. Since $p_i = 3$ divides $q^{2^l} + 1$, we must have $l = 0$, contrary to the assumption $l > 0$.

If $s_i = 0$, then we must have $2p_j^{s_j} \equiv \pm 1 \pmod{q}$. If s_j is even, then $2p_j^{s_j} \equiv 2(-1)^{s_j/2} \equiv \pm 2 \pmod{q}$ cannot be $\pm 1 \pmod{q}$. Hence, s_j must be odd and $2p_j \equiv \pm 1 \pmod{q}$. Since $p_j^4 \equiv 1 \pmod{q}$, we have $2^4 \equiv 1 \pmod{q}$ and $q \equiv 1 \pmod{4}$. Equivalently, we have $q = 5$ and therefore $p_j^2 + 1 = 2 \times 5^{2^k}$ with $k > 0$. By Lemma 2.5, we must have $p_j = 7$ and $k = 1$. However, this is impossible since 7 divides neither $\sigma_\infty(5^2) = 2 \times 13$ nor $\sigma_\infty(2^f)$ by the first supplementary law. Now our proof is complete.

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