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Period of balancing sequence modulo powers of balancing and Pell numbers

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Abstract

The period of balancing numbers modulo m, denoted by $\pi(m)$, is the least positive integer n such that $(B_n, B_{n+1}) \equiv (0, 1) \pmod{m}$, where B_n is the n-th balancing number. While studying periodicity of balancing numbers, Panda and Rout found the results for $\pi(B_n)$ and $\pi(P_n)$, where P_n denotes the n-th Pell number. In this article we obtain the formulas of $\pi(B_n^{k+1})$ and $\pi(P_n^{k+1})$ for all $k \geq 1$.

Keywords: Balancing numbers; Lucas-balancing numbers; Periodicity; p-Adic order.

MSC: 11A05, 11B39, 11B50

1. Introduction

Let the sequence of balancing and Pell numbers be $\{B_n\}_{n\geq 1}$ and $\{P_n\}_{n\geq 1}$ respectively. These two sequences satisfy the recurrence relations $B_{n+1} = 6B_n - B_{n-1}$

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and $P_{n+1} = 2P_n + P_{n-1}$ with initial values $(B_0, B_1) = (0, 1) = (P_0, P_1)$ [1, 7]. Balancing numbers B_n and their associate Lucas-balancing numbers C_n are obtained from the Pell equation $C_n^2 - 8B_n^2 = 1$ [6]. The Lucas-balancing numbers satisfy the same recurrence relation as that of balancing numbers but with different initials $(C_0, C_1) = (1, 3)$ [6]. Some developments on balancing numbers and their related sequences can be found in [2, 3, 4, 12, 13].

Panda and Rout, in [8], defined the period of balancing numbers modulo m, denoted by $\pi(m)$, the least positive integer n satifying $(B_n, B_{n+1}) \equiv (0, 1) \pmod{m}$. They have derived the formulas $\pi(B_n) = 2n$ and $\pi(P_n) = n$ or $2n$ for the parity of n [8]. Later, Patel and Ray [9] studied the rank $r(m)$ and order $o(m)$ of the balancing sequence and established some connections between the period, rank and order. Among other relations, one important connection between them is that the product of rank and order is equal to period.

In [5], Marques obtained the order of appearance (rank) of Fibonacci numbers modulo powers of Fibonacci and Lucas numbers and derived the formula of $r(L_n^k)$ in some cases. Later, Pongsriiam extended the work of Marques and obtained the complete formula of $r(L_n^k)$ for all $n, k \geq 1$ [10]. Recently, Sanna [14] studied the p-adic valuation of Lucas sequences $u_n = au_{n-1} + bu_{n-2}$ with $u_0 = 0$ and $u_1 = 1$ for all $n \geq 2$. Among other results he has also established the following identity.

Theorem 1.1. *If p is a prime number such that $p \nmid b$, then*

$$\nu_p(u_n) = \begin{cases} \nu_p(n) + \nu_p(u_p) - 1, & \text{if } p|\Delta, p|n; \\ 0, & \text{if } p|\Delta, p \nmid n; \\ \nu_p(n) + \nu_p(u_{pr(p)}) - 1, & \text{if } p \nmid \Delta, r(p)|n, p|n; \\ \nu_p(u_{pr(p)}), & \text{if } p \nmid \Delta, r(p)|n, p \nmid n; \\ 0, & \text{if } p \nmid \Delta, r(p) \nmid n. \end{cases}$$

for each positive integer n, where $\Delta = a^2 + 4b$.

In the present study, we obtain the formula of $\pi(B_n^{k+1})$ and $\pi(P_n^{k+1})$ for every $k \geq 1$.

2. Preliminaries

The following results concerning about the periodicity of balancing numbers are found in [8].

Lemma 2.1. *If m divides n, then $r(m)$ divides $r(n)$.*

Lemma 2.2. *If m divides B_n if and only if $r(m)$ divides n.*

The following result is found in [6].

Lemma 2.3. *For every positive integers m and n, $B_{m+n} = B_m C_n + C_m B_n$.*

The following result is found in [7].

Lemma 2.4. *For $n \geq 0$, $C_n = 2Q_n^2 - (-1)^n$.*

The following two corollaries are direct consequences of theorem 1.1. The first one obtained for $(a,b) = (6,-1)$ and the second one for $(a,b) = (2,1)$.

Corollary 2.5. *For all prime p,*

$$\nu_p(B_n) = \begin{cases} \nu_p(n), & \text{if } p|\Delta, p|n, \\ \nu_p(n) + \nu_p(B_{r(p)}), & \text{if } p \nmid \Delta, r(p)|n, \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 2.6. *For all prime p,*

$$\nu_p(P_n) = \begin{cases} \nu_p(n), & \text{if } p|\Delta, p|n, \\ \nu_p(n) + \nu_p(P_{r(p)}), & \text{if } p \nmid \Delta, r(p)|n, \\ 0, & \text{otherwise.} \end{cases}$$

3. Main results

In order to prove the main results we need the following lemma.

Lemma 3.1. *For $n \geq 2, k \geq 1$, $r(B_n^{k+1}) = nB_n^k$ and $r(P_n^{k+1}) = nP_n^k$.*

Proof. Since B_n divides B_n^{k+1}, $r(B_n) = n$ divides $r(B_n^{k+1})$ by Lemma 2.1. Further to prove $r(B_n^{k+1})$ divides nB_n^k analogously B_n^{k+1} divides $B_{nB_n^k}$ [Lemma 2.2], it is enough to exhibit $\nu_p(B_n^{k+1}) \leq \nu_p(B_{nB_n^k})$, for all prime p. Now for $p = 2$, using the Corollary 2.5, we have

$$\nu_2(B_{nB_n^k}) = \nu_2(nB_n^k) = \nu_2(n) + \nu_2(B_n^k) = \nu_2(B_n) + \nu_2(B_n^k) = \nu_2(B_n^{k+1}).$$

On the other hand for all odd primes, we have

$$\begin{aligned} \nu_p(B_{nB_n^k}) &= \nu_p(nB_n^k) + \nu_p(B_{r(p)}) \\ &= \nu_p(n) + \nu_p(B_{r(p)}) + \nu_p(B_n^k) \\ &= \nu_p(B_n) + \nu_p(B_n^k) \\ &= \nu_p(B_n^{k+1}). \end{aligned}$$

Now to see that $r(B_n^{k+1}) = nB_n^k/p^t$, for some $t \geq 0$. It suffices to show that B_n^{k+1} does not divides $B_{\frac{n}{2}B_n^k}$ for $p = 2$. That is $\nu_2\left(B_{\frac{nB_n^k}{2}}\right) < \nu_2(B_n^{k+1})$. So,

$$\begin{aligned} \nu_2\left(B_{nB_n^k/2}\right) &= \nu_2(nB_n^k/2) \\ &= \nu_2(n) + \nu_2(B_n^k) - \nu_2(2) \\ &= \nu_2(B_n) + \nu_2(B_n^k) - 1 \\ &< \nu_2(B_n^{k+1}). \end{aligned}$$

Again for all odd primes

$$\begin{aligned}\nu_p(B_{nB_n^k/p}) &= \nu_p(nB_n^k/p) + \nu_p(B_{r(p)}) \\ &= \nu_p(B_n) + \nu_p(B_n^k) - \nu_p(p) \\ &< \nu_p(B_n^{k+1}),\end{aligned}$$

which completes the first part of the lemma. The second part can be proved analogously. $\square$

Theorem 3.2. *For $n \geq 2$ and $k \geq 1$, $\pi(B_n^{k+1}) = \begin{cases} nB_n^k, & \text{if } n \equiv 0 \pmod 2; \\ 2nB_n^k, & \text{if } n \equiv 1 \pmod 2. \end{cases}$*

Proof. By virtue of Lemma 3.1, $B_{nB_n^k} \equiv 0 \pmod{B_n^{k+1}}$. In order to prove the theorem, it suffices to show the following cases, $B_{nB_n^k+1} \equiv 1 \pmod{B_n^{k+1}}$ for n is even only and for odd $B_{2nB_n^k+1} \equiv 1 \pmod{B_n^{k+1}}$. Using Lemma 2.3, we have

$$B_{2nB_n^k+1} = B_{2nB_n^k}C_1 + C_{2nB_n^k}B_1 \equiv C_{2nB_n^k} \pmod{B_n^{k+1}}. \tag{3.1}$$

Therefore using the identity $C_n^2 = 8B_n^2 + 1$ [6], we have

$$C_{2nB_n^k}^2 = 8B_{2nB_n^k}^2 + 1 \equiv 1 \pmod{B_n^{k+1}}.$$

Consequently, $C_{nB_n^k}^2 \equiv 1 \pmod{B_n^{k+1}}$, which implies

$$Q_{2nB_n^k}^2 \equiv 1 \pmod{B_n^{k+1}}. \tag{3.2}$$

By Lemma 2.4, we get

$$C_{2nB_n^k} = 2Q_{2nB_n^k}^2 - (-1)^{2nB_n^k} \equiv 1 \pmod{B_n^{k+1}},$$

which completes the second case. Again the use of Lemma 2.3 gives the identity $B_{nB_n^k+1} \equiv C_{nB_n^k} \pmod{B_n^{k+1}}$. In order to prove the first case, it suffices to show $C_{nB_n^k} \equiv 1 \pmod{B_n^{k+1}}$ for n is even only. Let $n = 2m$. As $C_{2n} = 16B_n^2 + 1$ [6], $C_{2mB_{2m}^k} = 16B_{mB_{2m}^k}^2 + 1$. We will now show that B_{2m}^{k+1} divides $B_{mB_{2m}^k}^2$, that is, $\nu_p(B_{2m}^{k+1}) \leq \nu_p(B_{mB_{2m}^k}^2)$ for all prime p.

When $p = 2$, by virtue of Corollary 2.5, we have

$$\begin{aligned}\nu_2(B_{mB_{2m}^k}^2) &= 2 \cdot \nu_2(B_{mB_{2m}^k}) = 2 \cdot \nu_2(mB_{2m}^k) \\ &= 2 \cdot \nu_2(m) + 2 \cdot \nu_2(B_{2m}^k) \\ &\geq \nu_2(B_{2m}^{2k}) \\ &\geq \nu_2(B_{2m}^{k+1}).\end{aligned}$$

On the other hand, for any odd prime p, we obtain

$$\begin{aligned}\nu_p(B^2_{mB^k_{2m}}) &= 2\cdot\nu_p(B_{mB^k_{2m}})\\ &= 2[\nu_p(mB^k_{2m}) + \nu_p(B_{r(p)})]\\ &= \nu_p(B^{2k}_{2m}) + 2\cdot\nu_p(m) + 2\cdot\nu_p(B_{r(p)})\\ &\geq \nu_p(B^{k+1}_{2m}).\end{aligned}$$

This completes for n even. Now for n is odd, that is, for $n = 2m+1$, we need to show that

$$C_{(2m+1)B^k_{2m+1}} \not\equiv 1 \pmod{B^{k+1}_{2m+1}}.$$

From Eq. (3.2), we have

$$Q^2_{2(2m+1)B^k_{2m+1}} \equiv 1 \pmod{B^{k+1}_{2m+1}}.$$

Since $C_{(2m+1)B^k_{2m+1}} = 2Q^2_{(2m+1)B^k_{2m+1}} + 1$, it follows that

$$C_{(2m+1)B^k_{2m+1}} \not\equiv 1 \pmod{B^{k+1}_{2m+1}}.$$

This ends the proof. □

Theorem 3.3. *For $n \geq 2$ and $k \geq 1$,*

$$\pi(P^{k+1}_n) = \begin{cases} nP^k_n, & \text{if } n \equiv 0 \pmod 2;\\ 2nP^k_n, & \text{if } n \equiv 1 \pmod 2).\end{cases}$$

Proof. In order to derive the above result, we need to prove the following two cases, $B_{nP^k_n+1} \equiv 1 \pmod{P^{k+1}_n}$ for n is even only and $B_{2nP^k_n+1} \equiv 1 \pmod{P^{k+1}_n}$ for n is odd. For the proof of the first case, since $B_{nP^k_n+1} \equiv C_{nP^k_n} \pmod{P^{k+1}_n}$ by Lemma 2.3, it suffices to show that $C_{nP^k_n} \equiv 1 \pmod{P^{k+1}_n}$ for n even only.

Let $n = 2m$, we need to show that $C_{2mP^k_{2m}} \equiv 1 \pmod{P^{k+1}_{2m}}$. As $C_{2mP^k_{2m}} = 16B^2_{mP^k_{2m}} + 1$, it is enough to prove that P^{k+1}_{2m} divides $B^2_{mP^k_{2m}}$, that is, $\nu_p(P^{k+1}_{2m}) \leq \nu_p(B^2_{mP^k_{2m}})$ for all prime p. By virtue of Corollary 2.6, for even prime p,

$$\begin{aligned}\nu_2(B^2_{mP^k_{2m}}) &= 2\cdot\nu_2(B_{mP^k_{2m}}) = 2\cdot\nu_2(mP^k_{2m})\\ &= 2\cdot\nu_2(m) + 2\cdot\nu_2(P^k_{2m})\\ &\geq \nu_2(P^{k+1}_{2m}).\end{aligned}$$

Further for any odd prime p, we have

$$\begin{aligned}\nu_p(B^2_{mP^k_{2m}}) &= 2\cdot\nu_p(B_{mP^k_{2m}})\\ &= 2[\nu_p(mP^k_{2m}) + \nu_p(P_{r(p)})]\\ &= \nu_p(P^{2k}_{2m}) + 2\cdot\nu_p(m) + 2\cdot\nu_p(P_{r(p)})\end{aligned}$$

$$\geq \nu_p(P_{2m}^{k+1}).$$

This completes for n even.

Now for n is odd, we have to claim

$$C_{(2m+1)P_{2m+1}^k} \not\equiv 1 \pmod{P_{2m+1}^{k+1}}.$$

Since $C_{nP_n^k}^2 \equiv 1 \pmod{P_n^{k+1}}$, which implies

$$Q_{2nP_n^k}^2 \equiv 1 \pmod{P_n^{k+1}}. \tag{3.3}$$

From (3.3), we have

$$Q_{2(2m+1)P_{2m+1}^k}^2 \equiv 1 \pmod{P_{2m+1}^{k+1}}.$$

Further, using Lemma 2.4, we have $C_{(2m+1)P_{2m+1}^k} = 2Q_{(2m+1)P_{2m+1}^k}^2 + 1$ and the result follows. This ends the proof of first case. Furthermore,

$$B_{2nP_n^k+1} \equiv C_{2nP_n^k} = 16B_{nP_n^k}^2 + 1 \equiv 1 \pmod{P_n^{k+1}}.$$

This completes the proof of the second case. □

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