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A combinatorial generalization of the gcd-sum function using a generalized Möbius function

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Abstract

Using the Souriau-Hsu-Möbius function with a natural parameter, a generalized Cesáro formula which is an extension of the classical gcd-sum formula is derived. The formula connects a combinatorial aspect of the generalized Möbius function with the number of integers whose prime factors have sufficiently high powers.

Keywords: gcd-sum function, Souriau-Hsu-Möbius function *MSC:* 11A25

1. Introduction

Let $\mathcal{A} := \{F : \mathbb{N} \to \mathbb{C}\}$ be the set of complex-valued arithmetic functions. For $F, G \in \mathcal{A}$, their addition and Dirichlet product (or convolution) are defined, respectively, by

$$(F+G)(n) = F(n) + G(n), \quad (F*G)(n) = \sum_{d \mid n} F(d)G(n/d).$$

It is well known, [7, Chapter 7] that $(\mathcal{A}, +, *)$ is commutative ring with identity I, where I(n) = 1 if n = 1 and I(n) = 0 when n > 1. The Souriau-Hsu-Möbius function ([9], [2]) is defined, for $\alpha \in \mathbb{C}$, by

$$\mu_{\alpha}(n) = \prod_{p|n} \binom{\alpha}{\nu_p(n)} (-1)^{\nu_p(n)},$$

where $n = \prod p^{\nu_p(n)}$ denotes the unique prime factorization of n. For some particular values of α , the corresponding Souriau-Hsu-Möbius functions represent certain well-known arithmetic functions, namely,

(i) when $\alpha = 0$, this corresponds to the convolution identity $\mu_0 = I$;

(ii) when $\alpha = 1$, this is the classical Möbius function $\mu_1 = \mu$;

(iii) when $\alpha = -1$, this is the inverse of the Möbius function $\mu_{-1} = \mu^{-1} =: u$, where u(n) = 1 $(n \in \mathbb{N})$ is the constant 1 function;

(iv) when $\alpha = -2$, this is the number of divisors function, $\mu_{-2} = d$, [2, p. 75]. Following [11], see also [6], for $\alpha \in \mathbb{C}$, $k \in \mathbb{Z}$, the (k, α) -Euler's totient is defined as an arithmetic function of the form

$$\varphi_{k,\alpha} := \zeta_k * \mu_\alpha \qquad \zeta_k(n) := n^k$$

When $k = \alpha = 1$, this function is the classical Euler's totient

$$\varphi_{1,1}(n) := \varphi(n) = \zeta_1 * \mu_1(n) = \sum_{d|n} d\mu\left(\frac{n}{d}\right) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

which counts the number of integers in $\{1, 2, ..., n\}$ that are relatively prime to n. Fixing $k = 1, \alpha = r \in \mathbb{N}$, the corresponding Euler's totient, referred to as *r*-Euler totient, is

$$\varphi_r(n) := \varphi_{1,r}(n) := \zeta_1 * \mu_r(n) = \sum_{d|n} d\mu_r\left(\frac{n}{d}\right)$$

As mentioned in [5, Example 6], the *r*-Euler totient has the following combinatorial meaning: an integer *a* is said to be r^{th} -degree prime to $n (\geq 2)$, briefly written as $(a, n)_r = 1$, if for each prime divisor *p* of *n*, there are integers $a_0, a_1, \ldots, a_{r-1}$ with $0 < a_i < p$ such that

$$a \equiv a_0 + a_1 p + \dots + a_{r-1} p^{r-1} \pmod{p^r}.$$

As a convention, we define

$$(a,1)_r := 1$$
 for any $a \in \mathbb{N}$.

When r = 1, the concept of being r^{th} -degree prime is merely that of being relatively prime.

In order to connect the concept of being r^{th} -degree prime with the *r*-Euler totient, we introduce another notion. A positive integer *n* is said to be *r*-powerful

if each of its prime factor appears with multiplicity at least r, i.e., $\nu_p(n) \ge r$ for each prime divisor p of n; as a convention, the integer 1 is adopted to be r-powerful for any $r \in \mathbb{N}$. Note that if n is r-powerful, then it is also s-powerful for all $s \in \mathbb{N}$ with $s \le r$. The following lemma shows that when n is r-powerful, the function $\varphi_r(n)$ counts the number of a's in the set $\{1, 2, \ldots, n\}$ such that $(a, n)_r = 1$; the proof given here is extracted from [11].

Lemma 1.1. Let $n, r \in \mathbb{N}$, and let $N_r(n)$ denote the number of integers $a \in \{1, 2, \ldots, n\}$ such that $(a, n)_r = 1$. We have :

- 1) The function $F(n) = N_r(n)$ if n is r-powerful and zero otherwise, is multiplicative.
- 2) If *n* is *r*-powerful, then $N_r(n) = \varphi_r(n) = n \prod_{p|n} (1 1/p)^r$, $N_r(1) = \varphi_r(1) := 1.$

Proof. 1) Let n be an r-powerful positive integer whose prime factorization is $n = p_1^{e_1} \cdots p_s^{e_s}$. By the Chinese remainder theorem, for any integers $\alpha_1, \ldots, \alpha_s$, there is a unique $a \pmod{n}$ such that

$$a \equiv \alpha_1 \pmod{p_1^{e_1}}, \ldots, a \equiv \alpha_s \pmod{p_s^{e_s}}.$$

Conversely, for any $a \pmod{n}$, there uniquely exist $\alpha_i \pmod{p_i^{e_i}}$ $(i = 1, \ldots, s)$ satisfying the above system of congruences. Thus,

$$(a,n)_r = 1 \iff (a, p_i^{e_i})_r = 1$$
 holds for every $i \in \{1, \dots, s\}$,

which shows at once that $N_r(n)$ is a multiplicative function of n.

2) Using part 1), it suffices to check that N_r and φ_r are equal on any prime power p^e with $e \ge r$. Recall that $N_r(p^e)$ is the number of $a \in \{1, 2, \ldots, p^e\}$ such that $(a, p^e)_r = 1$, i.e., such that there are integers $a_0, a_1, \ldots, a_{r-1}$ with $0 < a_i < p$ satisfying

$$a \equiv a_0 + a_1 p + \dots + a_{r-1} p^{r-1} \pmod{p^r}.$$

Thus, the number of such $a \pmod{p^r}$ is $(p-1)^r$, and so the total number of such $a \pmod{p^e}$ is $N_r(p^e) = p^{e-r}(p-1)^r$. On the other hand using $e \ge r$, we have

$$\varphi_r(p^e) = \sum_{d|p^e} d\mu_r(p^e/d) = p^0 \binom{r}{e} (-1)^e + p\binom{r}{e-1} (-1)^{e-1} + \dots + p^e\binom{r}{0} (-1)^0$$

= $p^{e-r}\binom{r}{r} (-1)^r + p^{e-r+1}\binom{r}{r-1} (-1)^{r-1} + \dots + p^e\binom{r}{0} (-1)^0$
= $p^{e-r} (p-1)^r = N_r(p^e).$

The classical gcd-sum function is an arithmetical function defined by

$$g(n) := \sum_{j=1}^{n} \gcd(j, n),$$
 (1.1)

and the classical gcd-sum formula states that

$$g(n) = \sum_{j=1}^{n} \gcd(j, n) = \sum_{d|n} d\varphi\left(\frac{n}{d}\right).$$
(1.2)

There have recently appeared quite a number of works related to the gcd-sum function and the gcd-sum formula. In [3], the gcd-sum function (1.1) is shown to be multiplicative, has a polynomial growth, and arises in the context of a lattice point counting problem, while the paper [1] studies the function $\sum_{j=1, \text{gcd}(j,n)|d} \text{gcd}(j,n)$, which is a generalization of the gcd-sum function (1.1).

In [8], the function $\sum_{j=1}^{n} \gcd(j, n)^{-1}$, which counts the orders of a generator of a cyclic group, is studied. In [4], an extended Cesáro formula

$$\sum_{j=1}^{n} f(\gcd(j,n)) = \sum_{d|n} f(d)\varphi\left(\frac{n}{d}\right) \quad (f \in \mathcal{A}),$$
(1.3)

which is another extension of (1.2), is investigated. Various properties of the gcdsum function (1.1) and its analogues are surveyed in [10]. Our objective here is to establish yet another generalization of the gcd-sum formula (1.2) by relating the *r*-Euler totient with the counting of *r*-powerful integers that are r^{th} -degree prime.

2. Generalized gcd-sum formula

Our generalized gcd-sum formula arises from replacing the usual Euler's totient on the right-hand side of (1.3) by the *r*-Euler totient, and using its combinatorial meaning to derive its corresponding generalized form. To do so, we need to extend the notion of r^{th} -degree primeness to that of *r*-gcd.

Definition. Let $r \in \mathbb{N}$, and let $n \in \mathbb{N}$ be *r*-powerful. For $j \in \mathbb{N}$, the integer *g* is the *r*-gcd of *j* and *n*, denoted by $g := (j, n)_r$, if $g = \operatorname{gcd}(j, n)$ satisfies two additional requirements

- 1. $\left(\frac{j}{g}, \frac{n}{g}\right)_r = 1$, and
- 2. n/g is r-powerful.

When r = 1, the above definition of r-gcd is identical with the usual greatest common divisor. Let us look at some examples.

Example 1. Let $n = 2^3 \cdot 3^2 = 72$. The divisors of n are 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72. Consider $j \in \{1, 2, \dots, 72\}$. For r = 1, we have $(j, 72)_1 = 1$ when $j \in \{1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, 47, 49, 53, 55, 59, 61, 65, 67, 71\}$ $(j, 72)_1 = 2$ when $j \in \{2, 10, 14, 22, 26, 34, 38, 46, 50, 58, 62, 70\}$ $(j, 72)_1 = 3$ when $j \in \{3, 15, 21, 33, 39, 51, 57, 69\}$

 $(j, 72)_1 = 4$ when $j \in \{4, 20, 28, 44, 52, 68\}$ $(j, 72)_1 = 6$ when $j \in \{6, 30, 42, 66\}$ $(j, 72)_1 = 8$ when $j \in \{8, 16, 32, 40, 56, 64\}$ $(j, 72)_1 = 9$ when $j \in \{9, 27, 45, 63\}$ $(j, 72)_1 = 12$ when $j \in \{12, 60\}$ $(j, 72)_1 = 18$ when $j \in \{18, 54\}$ $(j, 72)_1 = 24$ when $j \in \{24, 48\}$ $(j, 72)_1 = 36$ when $j \in \{36\}$ $(j, 72)_1 = 72$ when $j \in \{72\}$. For r = 2, we have $(j, 72)_2 = 1$ when $j \in \{7, 23, 31, 35, 43, 59, 67, 71\}$ $(j, 72)_2 = 2$ when $j \in \{14, 46, 62, 70\}$ $(j, 72)_2 = 8$ when $j \in \{32, 40, 56, 64\}$ $(j, 72)_2 = 9$ when $j \in \{27, 63\}$ $(j, 72)_2 = 18$ when $j \in \{54\}$ $(j, 72)_2 = 72$ when $j \in \{72\},\$ and for the other values of j, the 2-gcd $(j, 72)_2$ are not defined.

For r = 3, we have $(j, 72)_3 = 9$ when $j \in \{63\}$, $(j, 72)_3 = 72$ when $j \in \{72\}$, and for the other values of j, the 3-gcd $(j, 72)_3$ are not defined.

For $r \ge 4$, we have $(j, 72)_3 = 72$ when $j \in \{72\}$, and the r-gcd $(j, 72)_r$ are not defined for the remaining j's.

Our next lemma connects the r-gcd with the r-Euler totient.

Lemma 2.1. Let $r, n \in \mathbb{N}$ with n being r-powerful, and let

$$A_{r,d}(n) := \{a \in \{1, 2, \dots, n\} ; (a, n)_r = d\}.$$

Then

$$|A_{r,d}(n)| = \varphi_r \left(n/d \right).$$

Proof. The result follows at once from the observation that

$$a \in \{1, 2, \dots, n\}$$
 and $(a, n)_r = d$,

if and only if $\frac{a}{d} \in \{1, 2, \dots, \frac{n}{d}\}, (\frac{a}{d}, \frac{n}{d})_r = 1, \frac{n}{d}$ is *r*-powerful, and the set of elements on the right-hand side has cardinality $\varphi_r(n/d)$.

We now state and prove our generalized Cesáro formula.

Theorem 2.2. Let $r, n \in \mathbb{N}$ with n being r-powerful. For an arithmetic function f, we have the following generalized Cesáro formula

$$\sum_{j=1}^{n} f((j,n)_r) = \sum_{d \mid r \mid n} f(d)\varphi_r\left(\frac{n}{d}\right).$$
(2.1)

where the symbol $d \mid_r n$ in the summation on the right-hand side indicates that the sum extends over all the divisors d for which n/d is r-powerful.

In particular, taking f(n) = n, the generalized Cesáro formula becomes the generalized gcd-sum formula

$$\sum_{j=1}^{n} (j,n)_r = \sum_{d \mid_r n} d\varphi_r\left(\frac{n}{d}\right).$$
(2.2)

Proof. Writing $(j, n)_r = d$, using the above notation and Lemma 2.1, we get

$$\sum_{j=1}^{n} f((j,n)_r) = \sum_{j=1}^{n} \sum_{(j,n)_r=d} f(d) = \sum_{d \mid r \mid n} f(d) \left| A_{r,d}(n) \right| = \sum_{d \mid r \mid n} f(d) \varphi_r\left(\frac{n}{d}\right). \quad \Box$$

When r = 1, the generalized Cesáro formula is simply the classical Cesáro formula, and its representation via generalized Möbius function becomes a Dirichlet product of two arithmetic functions, viz.,

$$\sum_{j=1}^{n} f((j,n)_{1}) = \sum_{d \mid_{1} n} f(d)\varphi_{1}\left(\frac{n}{d}\right) = (f * \varphi)(n).$$

Example 2. Continuing from Example 1, let $n = 2^3 \cdot 3^2 = 72$. For r = 1, we have

$$\{d \in \{1, 2, \dots, 72\}; d \mid_1 72\} = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72\},\$$

and the values of $\varphi_1(72/d)$ with $d \mid_1 72$ are

$$\begin{aligned} \varphi_1(72) &= 24, \varphi_1(36) = 12, \varphi_1(24) = 8, \varphi_1(18) = 6, \varphi_1(12) = 4, \varphi_1(9) = 6, \\ \varphi_1(8) &= 4, \varphi_1(6) = 2, \varphi_1(4) = 2, \varphi_1(3) = 2, \varphi_1(2) = 1, \varphi_1(1) = 1. \end{aligned}$$

Using Example 1, the left-hand side of (2.1) is

$$\sum_{j=1}^{n} f((j,n)_{1}) = f(1) \times 24 + f(2) \times 12 + f(3) \times 8 + f(4) \times 6 + f(6) \times 4$$

+ $f(8) \times 6 + f(9) \times 4 + f(12) \times 2 + f(18) \times 2 + f(24) \times 2$
+ $f(36) \times 1 + f(72) \times 1$
= $\sum_{d \mid_{1} n} f(d)\varphi_{1}\left(\frac{n}{d}\right)$ (2.3)

which agrees with the theorem.

For r = 2, we have $\{d \in \{1, 2, \dots, 72\}; d \mid_2 72\} = \{1, 2, 8, 9, 18, 72\}$, and the values of $\varphi_2(72/d)$ with $d \mid_2 72$ are

$$\varphi_2(72) = 8, \varphi_2(36) = 4, \varphi_2(9) = 4, \varphi_2(8) = 2, \varphi_2(4) = 1, \varphi_2(1) = 1.$$

Using Example 1, the left-hand side of (2.1) is

$$\sum_{j=1}^{n} f((j,n)_2) = f(1) \times 8 + f(2) \times 4 + f(8) \times 4 + f(9) \times 2 + f(18) \times 1 + f(72) \times 1$$
$$= \sum_{d \mid > n} f(d)\varphi_2\left(\frac{n}{d}\right).$$
(2.4)

For r = 3, we have $\{d \in \{1, 2, ..., 72\}; d \mid_3 72\} = \{9, 72\}$, and the values of $\varphi_3(72/d)$ with $d \mid_3 72$ are $\varphi_3(8) = 1, \varphi_3(1) = 1$. Using Example 1, the left-hand side of (2.1) is

$$\sum_{j=1}^{n} f((j,n)_3) = f(9) \times 1 + f(72) \times 1 = \sum_{d \mid_3 n} f(d)\varphi_3\left(\frac{n}{d}\right).$$
(2.5)

For $r \ge 4$, we have $\{d \in \{1, 2, ..., 72\}; d \mid_r 72\} = \{72\}$, and the values of $\varphi_3(72/d)$ with $d \mid_r 72$ is $\varphi_r(1) = 1$. The left-hand side of (2.1) is

$$\sum_{j=1}^{n} f((j,n)_r) = f(72) \times 1 = \sum_{d \mid_r n} f(d)\varphi_r\left(\frac{n}{d}\right).$$
 (2.6)

The formulae such as (2.1)–(2.6) deal with a single r. We end this paper by remarking that such formulae can be absorbed into one single formula. For a positive integer n whose prime representation is $n = p_1^{\nu_1(n)} p_2^{\nu_2(n)} \cdots p_t^{\nu_t(n)}$, let

$$\nu(n) := \max \{\nu_i(n); 1 \le i \le t\}.$$

Corollary 2.3. For an arithmetic function f, we have the following generalized Cesáro formula

$$\sum_{r=1}^{\nu(n)} \sum_{j=1}^{n} f((j,n)_r) = \sum_{r=1}^{\nu(n)} \sum_{d \mid r} f(d)\varphi_r\left(\frac{n}{d}\right).$$

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