Annales Mathematicae et Informaticae
47 (2017) pp. 129–139
http://ami.ektf.hu

Hyperbolic distance between hyperbolic lines

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Submitted June 2, 2016 — Accepted November 20, 2017

Abstract

We derive formulas for the hyperbolic distance between hyperbolic lines in the unit disk and in the upper half plane. We also build an algorithm in MATLAB/Octave to compute the hyperbolic distance.

Keywords: algorithms, hyperbolic geometry, hyperbolic distance, Poincaré model

MSC: 51M10

1. Introduction

The hyperbolic geometry was founded in the 19th century as an answer to the two millenniums old question about the parallel postulate. The hyperbolic geometry shows that the parallel postulate cannot be derived from the other four Euclid's postulates. The hyperbolic geometry has turned out to be a very useful tool in geometric function theory [9] and many applications including cosmology [1], Einstein's theory of general relativity [4, 8] and celestial mechanics [5].

The basic models of the hyperbolic geometry are the unit ball and the upper half space models. These models can be used to obtain geometry on any plane domain with at least 2 boundary points via the Riemann mapping theorem. Despite the hyperbolic geometry has many applications, some of the elementary properties has not been implemented to algorithms. In this article we consider one of these, namely the hyperbolic distance between two lines. We introduce an algorithm for the hyperbolic distance between two hyperbolic lines in the unit disk (Algorithm 1) and in the upper half plane (Algorithm 2).

2. Preliminary results

In this section we introduce notation and preliminary results. For basics of the hyperbolic geometry we refer reader to [2] and [3]. We denote the Euclidean n-space by $\mathbb{R}^n$, $n \geq 2$, and identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$.

For $x \in \mathbb{R}^n$ and $r > 0$ we denote Euclidean sphere with center x and radius r by $S^{n-1}(x,r) = \{y \in \mathbb{R}^n \colon |x-y| = r\}$.

When $a, b \in \mathbb{R}$, $a < b$, we denote open and closed intervals by $(a,b) = \{z \in \mathbb{R} \colon a < z < b\}$ and $[a,b] = \{z \in \mathbb{R} \colon a \leq z \leq b\}$. For half-open intervals we use notation $(a,b]$ and $[a,b)$. If $x, y \in \mathbb{R}^n$, $x \neq y$ and $n \geq 2$, we denote the closed Euclidean line segment by $[x,y] = \{z \in \mathbb{R}^n \colon z = x + t(y-x),\ t \in [0,1]\}$. If one or both of the end points are not included in the line segment, we use notation $(x,y]$, $[x,y)$ or (x,y).

We define the upper half space by

$$\mathbb{H}^n = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \colon x_n > 0\}$$

and the unit ball by

$$\mathbb{B}^n = \{x \in \mathbb{R}^n \colon |x| < 1\}.$$

Next we define the hyperbolic distance in these two domains. For $x, y \in \mathbb{H}^n$

$$\rho_{\mathbb{H}^n}(x,y) = \operatorname{arc\,cosh}\left(1 + \frac{|x-y|^2}{2x_n y_n}\right). \tag{2.1}$$

For $x, y \in \mathbb{B}^n$

$$\rho_{\mathbb{B}^n}(x,y) = 2 \operatorname{arc\,sinh} \frac{|x-y|}{\sqrt{1-|x|^2}\sqrt{1-|y|^2}}. \tag{2.2}$$

For noncollinear $a, b, c \in \mathbb{R}^n$ there exist a unique circle $C \subset \mathbb{R}^n$ containing the three points. We denote the center of the circle C by $\operatorname{center}(a,b,c)$. If $a, b, c \in \mathbb{C}$ then $\operatorname{center}(a,b,c)$ can be found by the following formula.

Lemma 2.1 ([6, Proposition 2.2])**.** *Let $C \subset \mathbb{C}$ be a circle and $a, b, c \in C$ be distinct. Then the center of C is*

$$\operatorname{center}(a,b,c) = \frac{(b-c)|a|^2 + (c-a)|b|^2 + (a-b)|c|^2}{(b-c)\bar{a} + (c-a)\bar{b} + (a-b)\bar{c}}.$$

Let $S \subset \mathbb{C}$ be a circular arc and C be the circle that contains S. We denote the center of C by $\operatorname{center}(S)$.

In the unit ball $\mathbb{B}^n$ for $z \in \mathbb{B}^n$ we can define a useful Möbius mapping T_z as in [10, 1.34]. For all $x \in \mathbb{B}^n$ define

$$T_z(x) = (p_z \circ q_z)(x), \tag{2.3}$$

where

$$q_z(x) = \frac{z}{|z|^2} + \left(\frac{1}{|z|^2} - 1\right)\left(x - \frac{z}{|z|^2}\right) \Big/ \left|x - \frac{z}{|z|^2}\right|^2$$

and

$$p_z(x) = x - 2x\frac{z^2}{|z|^2}.$$

Geometrically q_z is the inversion in sphere $S^{n-1}(z/|z^2|, 1/|z|^2 - 1)$ and p_z is the reflection in the $n-1$-dimensional hyperplane through 0, which is perpendicular to the line that contains 0 and z.

A useful property of the mapping T_z is the fact that it is a Möbius mapping and thus it preserves the hyperbolic distance in $\mathbb{B}^n$: For all $x, y \in \mathbb{B}^n$

$$\rho_{\mathbb{B}^n}(x, y) = \rho_{\mathbb{B}^n}(T_z(x), T_z(y)).$$

Since hyperbolic lines in the upper half space and the unit ball are arcs of Euclidean circles, we need repeatedly to find intersection points of two circles. Mathematically this is very straightforward and a solution is obtained by solving a pair of equations. Algorithmically this is also very simple, for example there are functions in MATLAB (function `circcirc`) and Octave (function `intersectCircles`). Our algorithms are independent of programming language and thus we introduce the formula for finding the intersection of two circles in the complex plane.

Let $C_1 = S^1(x, r)$ be circle with center $x \in \mathbb{C}$ and radius $r > 0$, and $C_2 = S^1(y, s)$ be circle with center $y \in \mathbb{C}$ and radius $s > 0$. If $r + s < |x - y|$ or $|x - y| + \min\{r, s\} < \max\{r, s\}$, then $C_1 \cap C_2 = \emptyset$.

We assume that $r + s < |x - y| < \max\{r, s\} - \min\{r, s\}$. Now $C_1 \cap C_2 \neq \emptyset$ and we derive a formula for the intersection points v. Let $v \in C_1 \cap C_2$ and choose a point z from the Euclidean line through x and y such that (x, v, z) and (y, z, v) form two right-angled triangle with the right-angle at z. If we denote $|v - z| = h$ and $|x - z| = t$, then $|y - z| = |x - y| - t$ and by the Pythagorean theorem

$$r^2 = h^2 + t^2 \quad \text{and} \quad s^2 = (|x - y| - t)^2 + h^2.$$

Now $h = \sqrt{r^2 - t^2}$ and

$$h^2 = r^2 - t^2 = s^2 - (|x - y| - t)^2,$$

which is equivalent to

$$t = \frac{r^2 - s^2 + |x - y|^2}{2|x - y|}.$$

We obtain

$$z = x + (y - x)\frac{t}{|x - y|} = x + (y - x)\frac{r^2 - s^2 + |x - y|^2}{2|x - y|^2}$$

and

$$v = z \pm i(x - y)\frac{h}{|x - y|} = z \pm i(x - y)\frac{\sqrt{4r^2|x - y|^2 - (r^2 - s^2 + |x - y|^2)^2}}{2|x - y|^2}.$$

Finally, we introduce an elementary lemma, which we need for our algorithm in the unit ball.

Lemma 2.2. *The function*

$$f(\alpha) = \frac{\frac{a}{\cos(\pi-\alpha)} - a}{1-(1-a\tan(\pi-\alpha))^2}$$

is decreasing on $(0,\pi)$ *and* $f(\alpha)\to 0$ *as* $\alpha\to\pi$.

Proof. By differentiation we obtain

$$f'(\alpha) = -\frac{2a\sin\alpha + a\sin(2\alpha) + 2\cos(2\alpha) + 2}{2(1-\cos\alpha)(a\sin(\alpha)+2\cos\alpha)^2}$$

for $\alpha\in(0,\pi)$. We observe that $f'(\alpha)<0$ implying $f(\alpha)$ is decreasing, because $a\sin(2\alpha) = 2a\sin\alpha\cos\alpha$ and thus

$$2a\sin\alpha + a\sin(2\alpha) + 2\cos(2\alpha) + 2 = 2a\sin\alpha(1+\cos\alpha) + 2(1+\cos(2\alpha)) \geq 0$$

for $\alpha\in(0,\pi)$.

We denote $\beta = \pi-\alpha$ and calculate using l'Hospital's rule

$$\begin{aligned}\lim_{\alpha\to\pi} f(\alpha) = \lim_{\beta\to 0} f(\beta) &= \lim_{\beta\to 0}\frac{\dfrac{2a\sin\beta}{1+\cos(2\beta)}}{\dfrac{2a(1-a\tan\beta)}{(\cos\beta)^2}}\\ &= \lim_{\beta\to 0}\frac{2a\sin\beta(\cos\beta)^2}{(1+\cos(2\beta))(2a(1-a\tan\beta))} = 0\end{aligned}$$

and the assertion follows. □

3. The upper half plane

Let $a,b\in\mathbb{H}^2$ be two distinct points. If a, b and $\overline{a}$ are collinear, then $\mathrm{Re}(a)=\mathrm{Re}(b)$ and the hyperbolic line through a and b is the Euclidean ray

$$\{z\in\mathbb{H}^2 \colon z = (\mathrm{Re}(a),t),\ t>0\}. \tag{3.1}$$

If a, b and $\overline{a}$ are not collinear the hyperbolic line through a and b is the Euclidean semicircle

$$S^1(c,|a-c|)\cap\mathbb{H}^2, \quad c = \mathrm{center}(a,b,\overline{a}), \tag{3.2}$$

where the function center is defined in Lemma 2.1 and $c\in\partial\mathbb{H}^2$.

We derive next a formula for the hyperbolic distance between two hyperbolic lines.

Let $l_1, l_2 \subset \mathbb{H}^2$ be two distinct hyperbolic lines. If $l_1 \cap l_2 \neq \emptyset$ or both l_1 and l_2 are Euclidean rays as in (3.1), then $\rho_{\mathbb{H}^2}(l_1, l_2) = 0$. The latter one can be seen by selecting $x \in l_1$ and $y \in l_2$ with $\mathrm{Im}(x) = \mathrm{Im}(y) = t > 0$. Now (see Figure 1)

$$\rho_{\mathbb{H}^2}(l_1, l_2) \leq \rho_{\mathbb{H}^2}(x, y) = \operatorname{arc\,cosh}\left(1 + \frac{\mathrm{Re}(x-y)^2}{2t^2}\right) \to 0 \tag{3.3}$$

as $t \to 0$.

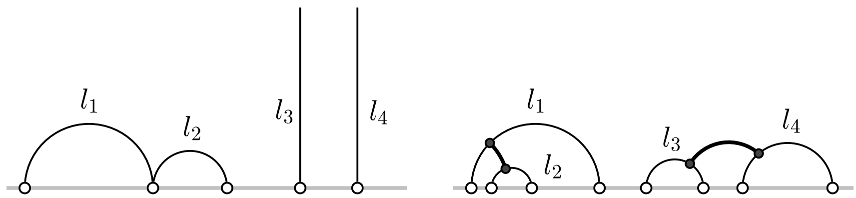

Figure 1: Left: Hyperbolic lines l_1, l_2, l_3 and l_4 with $\rho_{\mathbb{H}^2}(l_1, l_2) = 0 = \rho_{\mathbb{H}^2}(l_3, l_4)$ as in Proposition 3.1. Here $l_1 \cap l_2 = \emptyset$ but $\overline{l_1} \cap \overline{l_2} \neq \emptyset$ and l_3 and l_4 are as in (3.3). Right: Proposition 3.2 where the black points indicate the points p_i that give $\rho_{\mathbb{H}^2}(l_1, l_2)$ and $\rho_{\mathbb{H}^2}(l_3, l_4)$.

We assume that at least one of l_1 and l_2 is a semicircle of type (3.2). Let us now assume that $\overline{l_1} \cap \overline{l_2} = \{d\} \subset \partial\mathbb{H}^2$ and $l_1 \cap l_2 = \emptyset$. To simplify notation, we may assume that $d = 0$. We first assume that both l_1 and l_2 are Euclidean semicircles of type (3.2). We denote $l_1 \subset S^1((-r, 0), r)$ for some $r > 0$ and $l_2 \subset S^1((\pm s, 0), s)$ for some $s > r$. We choose $x \in l_1$ to be $x = r(e^{\alpha i} - 1)$ and $y \in l_2$ to be $y = s(e^{\alpha i} - 1)$ or $y = s(e^{(\pi-\alpha)i} + 1)$ for some $\alpha \in (0, \pi/2)$. Now

$$|x - y| \leq r - r\cos\alpha + s - s\cos\alpha, \quad x_2 = r\sin\alpha, \quad y_2 = s\sin\alpha$$

and thus by (2.1) we obtain

$$\rho_{\mathbb{H}^2}(l_1, l_2) \leq \rho_{\mathbb{H}^2}(x, y) \leq \operatorname{arc\,cosh}\left(1 + \frac{(r+s)^2\cos^2\alpha}{2rs\sin^2\alpha}\right) \to 0$$

as $\alpha \to 0$.

At least one of l_1 and l_2 has to be a Euclidean semicircle and the case that the other one is a Euclidean ray can be considered similarly as above. Let l_1 and l_2 be as above with center$(l_2) = (s, 0)$. Denote the hyperbolic line that is the Euclidean ray by $l_2' = \{z \in \mathbb{H}^2 \colon \mathrm{Re}(z) = 0\}$. Then it is clear that

$$\rho_{\mathbb{H}^2}(l_1, l_2') \leq \rho_{\mathbb{H}^2}(l_1, l_2) \to 0, \quad \text{as } \alpha \to 0$$

and we conclude that for any two hyperbolic lines l_1 and l_2 in the case $\overline{l_1} \cap \overline{l_2} \neq \emptyset$ we have $\rho_{\mathbb{H}^2}(l_1, l_2) = 0$.

Proposition 3.1. *If l_1 and l_2 are hyperbolic lines in $\mathbb{H}^2$ with $\overline{l_1} \cap \overline{l_2} \neq \emptyset$, then $\rho_{\mathbb{H}^2}(l_1, l_2) = 0$.*

Next we assume, that $\overline{l_1} \cap \overline{l_2} = \emptyset$. Now by (3.3) at least one of the hyperbolic lines has to be a Euclidean semicircle.

We first assume that both l_1 and l_2 are Euclidean semicircles. We denote $l_1 \subset S^1((x,0),r)$ and $l_2 \subset S^1((y,0),s)$ for $x,y \in \mathbb{R}$, $x \neq y$, and $r,s > 0$ with $r > |x-y|$ and $s < r-|x-y|$. Let u be the radius and z the center of the Euclidean semicircle that is perpendicular to l_1 and l_2. By the Pythagorean theorem

$$u^2 = |x-z|^2 - r^2 = (|x-z| - |x-y|)^2 - s^2$$

and thus

$$|x-z| = \frac{r^2 - s^2 + |x-y|^2}{2|x-y|}, \quad u = \sqrt{|x-z|^2 - r^2}. \tag{3.4}$$

Now the circle

$$C_2 = \begin{cases} S^1((x-|x-z|,0),u), & \text{if } y < x, \\ S^1((x+|x-z|,0),u), & \text{if } x < y, \end{cases} \tag{3.5}$$

is perpendicular to both l_1 and l_2.

Proposition 3.2. *If l_1 and l_2 are hyperbolic lines of type* (3.2) *in* $\mathbb{H}^2$ *with* $\overline{l_1} \cap \overline{l_2} = \emptyset$ *and* center$(l_1) \neq$ center(l_2). *Then $\rho_{\mathbb{H}^2}(l_1,l_2) = \rho_{\mathbb{H}^2}(p_1,p_2)$, where $\{p_1\} = l_1 \cap C_2$ and $\{p_2\} = l_2 \cap C_2$. Here C_2 is as in* (3.5) *and* (3.4).

Next we deal with the case $x = y$. Let l_1 and l_2 be hyperbolic lines of type (3.2) in $\mathbb{H}^2$ with $\overline{l_1} \cap \overline{l_2} = \emptyset$ and $l_1 \subset S^1((x,0),r)$ and $l_2 \subset S^1((x,0),s)$ for $x \in \mathbb{R}$ and $r,s > 0$. Now (see Figure 2)

$$\rho_{\mathbb{H}^2}(l_1,l_2) = \rho_{\mathbb{H}^2}((x,r),(x,s)). \tag{3.6}$$

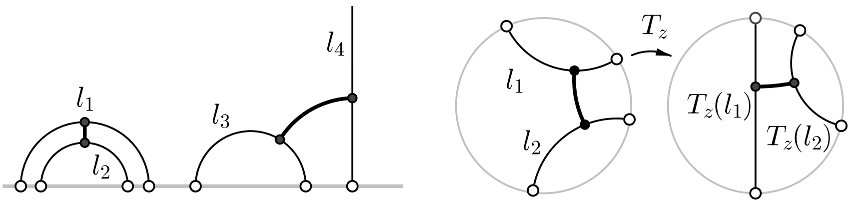

Figure 2: Left: Formula (3.6) for l_1 and l_2. Proposition 3.3 for l_3 and l_4. The black points indicate the points p_i that give $\rho_{\mathbb{H}^2}(l_1,l_2)$ and $\rho_{\mathbb{H}^2}(l_3,l_4)$. Right: Function T_z can be used to map l_1 to a Euclidean line segment. Proposition 4.1 gives the hyperbolic distance $\rho_{\mathbb{B}^2}(T_z(l_1),T_z(l_2)) = \rho_{\mathbb{B}^2}(l_1,l_2)$. The black points indicate the points p_i that give $\rho_{\mathbb{B}^2}(l_1,l_2)$ and $\rho_{\mathbb{B}^2}(T_z(l_1),T_z(l_2))$.

We then assume that l_1 is a Euclidean ray and l_2 is a Euclidean semicircle. We denote $l_1 = \{z \in \mathbb{H}^2 \colon z = (x,t),\, t > 0\}$ and $l_2 \subset S^1((y,0),r)$ for $x,y \in \mathbb{R}$ and $0 < r < |x-y|$. By the Pythagorean theorem we obtain that the circle

$$C_3 = S^1((x,0), \sqrt{|x-y|^2 - r^2}) \tag{3.7}$$

is perpendicular to l_1 and l_2.

Proposition 3.3. *Let l_1 and l_2 be hyperbolic lines in $\mathbb{H}^2$ with $\overline{l_1} \cap \overline{l_2} = \emptyset$. If l_1 is of type* (3.1) *and l_2 is of type of type* (3.2)*, then $\rho_{\mathbb{H}^2}(l_1, l_2) = \rho_{\mathbb{H}^2}(p_1, p_2)$, where $\{p_1\} = l_1 \cap C_3$ and $\{p_2\} = l_2 \cap C_3$. Here C_3 is as in* (3.7)*.*

Putting (3.6) and the results of Propositions 3.1, 3.2 and 3.3 together gives us Algortihm 1, the algorithm for the hyperbolic distance between hyperbolic lines in $\mathbb{H}^2$.

Data: points $a, b, c, d \in \mathbb{H}^2$ with $a \neq b$ and $c \neq d$
Result: $\rho_{\mathbb{H}^2}(l_1, l_2)$ for the hyperbolic line l_1 through a and b, and the hyperbolic line l_2 through c and d

/* Case A */
if *l_1 and l_2 are Euclidean rays* **then**
    return 0
/* Case B */
else if *l_1 and l_2 are a Euclidean ray and a semicircle* **then**
    if $\overline{l_1} \cap \overline{l_1} \neq \emptyset$ **then**
        return 0
    else
        calculate $\rho_{\mathbb{H}^2}(l_1, l_2)$ using Proposition 3.3
    end
/* Case C: l_1 and l_2 are semicircles */
else
    if $\overline{l_1} \cap \overline{l_1} \neq \emptyset$ **then**
        return 0
    else if center(l_1) == center(l_2) **then**
        calculate $\rho_{\mathbb{H}^2}(l_1, l_2)$ using (3.6)
    else
        calculate $\rho_{\mathbb{H}^2}(l_1, l_2)$ using Proposition 3.2
    end
end

Algorithm 1: Algorithm for hyperbolic distance between hyperbolic lines in $\mathbb{H}^2$.

4. The unit disk

In this section we find the hyperbolic distance between two hyperbolic lines in the unit disk. For all $x \in \mathbb{B}^2$ we denote $x^* = x/|x|^2$.

Let $a, b \in \mathbb{B}^2$ be two distinct points, $a \neq 0$. If points a, b and a^* are collinear, then the hyperbolic line through a and b is the Euclidean line segment

$$\{z \in \mathbb{B}^2 \colon z = ta/|a|,\ t \in (-1, 1)\}. \tag{4.1}$$

If points a, b and a^* are not collinear, then the hyperbolic line through a and b is the circular arc

$$S^1(c, |a - c|) \cap \mathbb{B}^2, \quad c = \text{center}(a, b, a^*). \tag{4.2}$$

By mapping T_z defined in (2.3) we can map any hyperbolic line of type (4.2) to type (4.1) and preserve hyperbolic distances, see Figure 2. The selection of z that does the trick is

$$z = c\left(1 - \frac{|a - c|}{|c|}\right), \quad c = \text{center}(a, b, a^*). \tag{4.3}$$

Let $l_1, l_2 \subset \mathbb{B}^2$ by hyperbolic lines. By the discussion above, we may assume that l_1 is of type (4.1) and after rotation about the origin we may choose $l_1 = (-i, i)$.

All the hyperbolic lines l_3 perpendicular to l_1 have $\text{Re}(\text{center}(l_3)) = 0$. Let us denote the end points of l_2 by a_2 and b_2, and the Euclidean line through points a_2 and b_2 by L. The hyperbolic lines perpendicular to l_2 satisfy $\text{center}(l_2) \in L$. Since the shortest hyperbolic segment joining l_1 and l_2 is perpendicular to both l_1 and l_2, we want hyperbolic line l_3 with $\text{center}(l_3) \in L \cap \{z \in \mathbb{B}^2 \colon \text{Re}(z) = 0\}$.

The last thing we need to do, is to find the radius of the circle C_3 that contains l_3. Since C_3 is perpendicular to the unit circle we obtain by the Pythagorean theorem

$$C_3 = S^1(\text{center}(l_3), r_3)), \quad r_3 = \sqrt{\text{center}(l_3)^2 - 1}. \tag{4.4}$$

We have obtained the following proposition.

Proposition 4.1. *Let l_1 and l_2 be hyperbolic lines in $\mathbb{B}^2$ with $\overline{l_1} \cap \overline{l_2} = \emptyset$. If l_1 is of type* (4.1) *and l_2 is of type of type* (4.2), *then $\rho_{\mathbb{B}^2}(l_1, l_2) = \rho_{\mathbb{B}^2}(p_1, p_2)$, where $\{p_1\} = l_1 \cap C_3$ and $\{p_2\} = l_2 \cap C_3$. Here C_3 is as in* (4.4).

Finally, we need to consider the case $\overline{l_1} \cap \overline{l_2} \neq \emptyset$. If $l_1 \cap l_2 \neq \emptyset$, then clearly $\rho_{\mathbb{B}^2}(l_1, l_2) = 0$. We assume that $l_1 \cap l_2 = \emptyset$. As above, we may assume that $l_1 = (-i, i)$. Now we can consider

$$l_2 = \{z \in \mathbb{B}^2 \colon z = a - i + ae^{i\alpha}\}$$

for $a > 0$. We choose $y = a + i + ae^{i\alpha}$ for small enough α and $x \in l_1$ such that $x \in L'$, where L' is a Euclidean line through y and $a - i$. Now $y \to -i$ and $x \to -i$ as $\alpha \to \pi$. Since $|x| = 1 - a\tan(\pi - \alpha)$ and $|x - (a - i)| = a/\cos(\pi - \alpha)$ we can estimate

$$\begin{aligned} \rho_{\mathbb{B}^2}(l_1, l_2) \le \rho_{\mathbb{B}^2}(x, y) &= 2 \operatorname{arc\,sinh} \frac{|x - y|}{1 - |x|^2} \\ &= 2 \operatorname{arc\,sinh} \frac{\frac{a}{\cos(\pi - \alpha)} - a}{1 - (1 - a\tan(\pi - \alpha))^2} \to 0 \end{aligned}$$

as $\alpha \to 0$, where the limit follows from Lemma 2.2. We conclude that $\rho_{\mathbb{B}^2}(l_1, l_2) = 0$ whenever $\overline{l_1} \cap \overline{l_2} \neq \emptyset$.

Combining Proposition 4.1 with the above discussion we obtain the following algorithm.

Data: points $a, b, c, d \in \mathbb{B}^2$ with $a \neq b$ and $c \neq d$
Result: $\rho_{\mathbb{B}^2}(l_1, l_2)$ for the hyperbolic line l_1 through a and b, and the hyperbolic line l_2 through c and d
if *l_1 and l_2 are circular arcs* **then**
  use function T_z defined in (2.3) for z as in (4.3) to transform l_1 into a Euclidean line segment
end
if $\overline{l_1} \cap \overline{l_2} \neq \emptyset$ **then**
  return 0
else
  calculate $\rho_{\mathbb{B}^2}(l_1, l_2)$ Proposition 4.1
end

Algorithm 2: Algorithm for hyperbolic distance between hyperbolic lines in $\mathbb{B}^2$.

5. Testing the algorithms

We compared Algorithms 1 and 2 with other solutions to the problem using random points. We implemented the algorithms in MATLAB/Octave and tested the performance. We made additionally visual testing for strategically chosen points and random points for Algorithms 1 and 2. Next we shortly introduce other methods.

The easiest way to find the minimum distance between two hyperbolic lines is to generate m points for each line and find the shortest hyperbolic distance between the points pairwise. We call this method the linear search (LS). In the LS algorithm the points on the hyperbolic line are equally spaced in the Euclidean distance.

An other way is to represent the each hyperbolic line with a real variable and minimise the hyperbolic distance with respect to the variables. For example, if a hyperbolic line is an arc of a circle we can write it as $x + re^{it}$ for t in a suitable interval. If both hyperbolic lines are circular arcs we can minimise

$$\rho(x + re^{it}, x' + r'e^{is})$$

with respect to variables t and s. In MATLAB/Octave we may use function `fminsearch`. We call this algorithm the minimum search (MS). The starting point for minimisation was selected to be the midpoints of the domains for t and s.

We tested the algorithms with 1000 random quadruples of points and compared the running times. For the linear search we also varied the number of points m with values 50, 250 and 500. Additionally we checked which of the three algorithms gave the lowest value. For the LM algorithm we estimated the error by comparing the value to Algorithm 1 or Algorithm 2 depending on the domain. The MS algorithm uses minimisation, which gives the points that are used to compute the hyperbolic distance in the domain. If the minimisation points were not in the original domain, then the minimisation did not work and we did not include the result to our test. We kept track how often this happened and reported the success rate.

For every set of random point Algorithm 1 or Algorithm 2 gave the minimum value of the 3 algorithms. However, in some cases also the MS algorithm gave the same value.

	LS, $m = 50$	LS, $m = 250$	LS, $m = 500$	MS	Alg. 1 / 2
$\mathbb{H}^2$	1.1 (0.08)	12.3 (0.01)	58.0 (0.009)	49.1 (53%)	1.4
$\mathbb{B}^2$	1.6 (0.02)	16.3 (0.004)	77.5 (0.002)	56.4 (22%)	2.6

Table 1: Average evaluation time (in ms) for LS, MS and Algorithms 1 and 2. For LS algorithm error is given in parentheses and MS algorithm success rate is given in parentheses.

From Table 1 we can see that the success rate for MS algorithm is poor. The algorithm gives good results when it works, but it is much slower compared to Algorithms 1 and 2. Table 1 also shows that LS algorithm works, but the quality is poor (10^{-2}) with $m = 50$. Choosing $m = 500$ gives better quality, but the evaluation time becomes longer than for the other algorithms. We may conclude that Algorithms 1 and 2 outperform LS and MS algorithms.

Finally, we note that Algorithms 1 and 2 do not work in higher dimensions ($n \geq 3$) in general and it remains an open problem how the generalisation should be implemented.

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