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A note on the k-Narayana sequence

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Abstract

In the present article, we define the k-Narayana sequence of integer numbers. We study recurrence relations and some combinatorial properties of these numbers, and of the sum of their first n terms. These properties are derived from matrix methods. We also study some relations between the k-Narayana sequence and convolved k-Narayana sequence, and permanents and determinants of one type of Hessenberg matrix. Finally, we show how these sequences arise from a family of substitutions.

Keywords: The k-Narayana Sequence, Recurrences, Generating Function, Combinatorial Identities

MSC: 11B39, 11B83, 05A15.

1. Introduction

The Narayana sequence was introduced by the Indian mathematician Narayana in the 14th century, while studying the following problem of a herd of cows and calves: *A cow produces one calf every year. Beginning in its fourth year, each calf produces one calf at the beginning of each year. How many calves are there altogether after 20 years?* (cf. [1]).

This problem can be solved in the same way that Fibonacci solved its problem about rabbits (cf. [14]). If n is the year, then the Narayana problem can be modelled by the recurrence $b_{n+1} = b_n + b_{n-2}$, with $n \geqslant 2, b_0 = 0, b_1 = 1, b_2 = 1$ (cf. [1]). The first few terms are $0, 1, 1, 1, 2, 3, 4, 6, 9, 13, \dots$, (sequence A000930[1]). This sequence is called *Narayana sequence*.

In this paper, we introduce a generalization of the Narayana numbers. Specifically, for any nonzero integer number k the *k-Narayana sequence*, say $\{b_{k,n}\}_{k=0}^{\infty}$, is defined by the recurrence relation

$$b_{k,0} = 0, \quad b_{k,1} = 1, \quad b_{k,2} = k \quad \text{and} \quad b_{k,n} = kb_{k,n-1} + b_{k,n-3}.$$

The first few terms are

$$0,\ 1,\ k,\ k^2,\ k^3+1,\ k^4+2k,\ k^5+3k^2,\ k^6+4k^3+1,\ k^7+5k^4+3k,\ \dots$$

In particular:

$$\begin{aligned}
\{b_{1,n}\}_{n=0}^{\infty} &= \{0,1,1,1,2,3,4,6,9,13,19,28,41,\dots\}, \text{A000930, Narayana Seq.}\\
\{b_{2,n}\}_{n=0}^{\infty} &= \{0,1,2,4,9,20,44,97,214,472,1041,2296,\dots\}, \quad \text{A008998.}\\
\{b_{3,n}\}_{n=0}^{\infty} &= \{0,1,3,9,28,87,270,838,2601,8073,25057,\dots\}, \quad \text{A052541.}\\
\{b_{-1,n}\}_{n=0}^{\infty} &= \{0,1,-1,1,0,-1,2,-2,1,1,-3,4,-3,0,4,-7,\dots\}, \quad \text{A050935.}
\end{aligned}$$

Let $S_{k,n} = \sum_{i=1}^{n} b_{k,i}, \quad n \geq 1$, i.e., $S_{k,n}$ is the sum of the first n terms of the k-Narayana sequence. In this article we study the sequences $\{b_{k,n}\}$ and $\{S_{k,n}\}$. In Section 2.1, we give a combinatorial representation of $\{S_{k,n}\}$. Using the methods of [11], we find Binet-type formulae for $\{b_{k,n}\}$ and $\{S_{k,n}\}$ and their generating functions. We also study some identities involving these sequences, obtained from matrix methods. Similar researches have been made for tribonacci numbers [6, 7, 11], Padovan numbers [25], and generalized Fibonacci and Pell numbers [12, 10]. In Section 3 we obtain some relation determinants and permanents of certain Hessenberg matrices. In Section 4 we define the convolved k-Narayana sequences and we show some identities. In Section 5, we show how these sequences arises from a well known family of substitutions on an alphabet of three symbols.

2. Definitions and basic constructions

In this section, we define a new generating 3×3 matrix for the k-Narayana numbers. We also show the generating function and Binet formula for the k-Narayana sequence and some identities of the sum of the first n terms of the k-Narayana sequence.

For any integer number k, $(k \neq 0)$, we define the following matrix:

$$Q_k := \begin{bmatrix} k & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{2.1}$$

[1] Many integer sequences and their properties are expounded on *The On-Line Encyclopedia of Integer Sequences*[22].

By induction on n, we show that

$$(Q_k)^n = \begin{bmatrix} k & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} b_{k,n+1} & b_{k,n-1} & b_{k,n} \\ b_{k,n} & b_{k,n-2} & b_{k,n-1} \\ b_{k,n-1} & b_{k,n-3} & b_{k,n-2} \end{bmatrix}, \quad n \geq 3. \tag{2.2}$$

Then Q_k is a generating matrix of the k-Narayana sequence.

Proposition 2.1. *For all integers m, n such that $0 < m < n$, we have the following relations*

1. $b_{k,n} = b_{k,m+1}b_{k,n-m} + b_{k,m-1}b_{k,n-m-1} + b_{k,m}b_{k,n-m-2}$.
2. $b_{k,n} = b_{k,m}b_{k,n-m+1} + b_{k,m-2}b_{k,n-m} + b_{k,m-1}b_{k,n-m-1}$.

Proof. It is clear that $Q_k^n = Q_k^m Q_k^{n-m}$. Then from Equation (2.2), we have

$$\begin{bmatrix} b_{k,n+1} & b_{k,n-1} & b_{k,n} \\ b_{k,n} & b_{k,n-2} & b_{k,n-1} \\ b_{k,n-1} & b_{k,n-3} & b_{k,n-2} \end{bmatrix} = \begin{bmatrix} b_{k,m+1} & b_{k,m-1} & b_{k,m} \\ b_{k,m} & b_{k,m-2} & b_{k,m-1} \\ b_{k,m-1} & b_{k,m-3} & b_{k,m-2} \end{bmatrix} \times \begin{bmatrix} b_{k,n-m+1} & b_{k,n-m-1} & b_{k,n-m} \\ b_{k,n-m} & b_{k,n-m-2} & b_{k,n-m-1} \\ b_{k,n-m-1} & b_{k,n-m-3} & b_{k,n-m-2} \end{bmatrix}. \tag{2.3}$$

Equating the (1,3)-th and (2,1)-th elements of the equation, we obtain the relations. □

Let $B_k(z)$ be the generating function of the k-Narayana numbers $b_{k,n}$. From standard methods we can obtain that

$$B_k(z) = \frac{z}{1 - kz - z^3}. \tag{2.4}$$

Moreover, from Equation (2.4) we obtain that the k-Narayana numbers are given by the following Binet's formula:

$$b_{k,n} = \frac{\alpha_k^{n+1}}{(\alpha_k - \beta_k)(\alpha_k - \gamma_k)} + \frac{\beta_k^{n+1}}{(\beta_k - \alpha_k)(\beta_k - \gamma_k)} + \frac{\gamma_k^{n+1}}{(\gamma_k - \alpha_k)(\gamma_k - \beta_k)}, \quad n \geq 0, \tag{2.5}$$

where $\alpha_k, \beta_k, \gamma_k$ are the zeros of characteristic equation of the k-Narayana numbers, $x^3 - kx^2 - 1 = 0$. Specifically,

$$\alpha_k = \frac{1}{3}\left(k + k^2 \sqrt[3]{\frac{2}{27 + 2k^3 + 3\sqrt{81 + 12k^3}}} + \sqrt[3]{\frac{27 + 2k^3 + 3\sqrt{81 + 12k^3}}{2}}\right),$$

$$\beta_k = \frac{1}{3}\left(k - \omega k^2 \sqrt[3]{\frac{2}{27 + 2k^3 + 3\sqrt{81 + 12k^3}}} + \omega^2 \sqrt[3]{\frac{27 + 2k^3 + 3\sqrt{81 + 12k^3}}{2}}\right),$$

$$\gamma_k = \frac{1}{3}\left(k + \omega^2 k^2 \sqrt[3]{\frac{2}{27 + 2k^3 + 3\sqrt{81 + 12k^3}}} - \omega \sqrt[3]{\frac{27 + 2k^3 + 3\sqrt{81 + 12k^3}}{2}}\right),$$

where $\omega = \frac{1+i\sqrt{3}}{2}$ is the primitive cube root of unity. If $k \geq 1$, the number α_k is a *Pisot number*, i.e., α_k is algebraic integer greater than 1, whereas its Galois conjugates (β_k and γ_k) have norm smaller than 1. The number α_1 is the fourth smallest Pisot number (cf. [2]).

Proposition 2.2. *Let $n > 2$ and the integers r, s, such that $0 \leqslant s < r$. Then the following equality holds*

$$\begin{aligned} b_{k,rn+s} = {} & \\ & (\alpha_k^r + \beta_k^r + \gamma_k^r) b_{k,r(n-1)+s} + ((\alpha_k\beta_k)^r + (\alpha_k\gamma_k)^r + (\beta_k\gamma_k)^r) b_{k,r(n-2)+s} \\ & + (\alpha_k\beta_k\gamma_k)^r b_{k,r(n-2)+s}, \end{aligned} \tag{2.6}$$

where α_k, β_k and γ_k are the roots of the characteristic equation of the k-Narayana numbers.

Proof. By induction on n, we can show that for any positive integer r, the numbers $(\alpha_k^r + \beta_k^r + \gamma_k^r)$, $(\alpha_k\beta_k)^r + (\alpha_k\gamma_k)^r + (\beta_k\gamma_k)^r$ and $(\alpha_k\beta_k\gamma_k)^r$ are always integers. Then, from Binet formula Equation (2.6) follows. □

Let

$$S_{k,n} = \sum_{i=1}^{n} b_{k,i}, \;\; n \geq 1, \quad \text{and} \quad S_{k,0} = 0.$$

By induction on n, we can proof the following identities:

- If $n \geq 3$, then $S_{k,n} = kS_{k,n-1} + S_{k,n-3} + 1$.
- If $n \geq 4$, then $S_{k,n} = (k+1)S_{k,n-1} - kS_{k,n-2} + S_{k,n-3} - S_{k,n-4}$.

Since the generating function of $\{b_{k,n}\}_n$ is $B_k(z)$, given in Equation (2.4), and using the Cauchy product of series. We obtain the generating function of $\{S_{k,n}\}_n$:

$$\sum_{i=0}^{\infty} S_{k,i} z^i = \frac{z}{(1-z)(1-kz-z^3)}. \tag{2.7}$$

Moreover, using similar techniques of [11], we obtain that the sum of the k-Narayana numbers are given by the following Binet-type formula:

$$\begin{aligned} S_{k,n} = {} & \\ & \frac{\alpha_k^{n+2}}{(\alpha_k - 1)(\alpha_k - \beta_k)(\alpha_k - \gamma_k)} + \frac{\beta_k^{n+2}}{(\beta_k - 1)(\beta_k - \alpha_k)(\beta_k - \gamma_k)} \\ & + \frac{\gamma_k^{n+2}}{(\gamma_k - 1)(\gamma_k - \alpha_k)(\gamma_k - \beta_k)}. \end{aligned} \tag{2.8}$$

On the other hand, using the results of [13] we obtain that

$$A_k^n = B_{k,n}, \tag{2.9}$$

where

$$A_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & k & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad B_{k,n} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ S_{k,n+1} & b_{k,n+2} & b_{k,n} & b_{k,n+1} \\ S_{k,n} & b_{k,n+1} & b_{k,n-1} & b_{k,n} \\ S_{k,n-1} & b_{k,n} & b_{k,n-2} & b_{k,n-1} \end{bmatrix}, \; n \geq 2.$$

Moreover, if $n, m \geq 3$, then

$$S_{k,n+m} = S_{k,n} + b_{k,n+1}S_{k,m+1} + b_{k,n-1}S_{k,m} + b_{k,n}S_{k,m-1}. \tag{2.10}$$

Define the diagonal matrix D_k and the matrix V_k as shown, respectively:

$$D_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha_k & 0 & 0 \\ 0 & 0 & \beta_k & 0 \\ 0 & 0 & 0 & \gamma_k \end{bmatrix}, \qquad V_k = \begin{bmatrix} k & 0 & 0 & 0 \\ -1 & \alpha_k^2 & \beta_k^2 & \gamma_k^2 \\ -1 & \alpha_k & \beta_k & \gamma_k \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

Note that $A_kV_k = V_kD_k$. Moreover, since the roots $\alpha_k, \beta_k, \gamma_k$ are different, it follows that $\det V_k \neq 0$, with $k \neq 0$. Then we can write $V_k^{-1}A_kV_k = D_k$, so the matrix A_k is similar to the matrix D_k. Hence $A_k^nV_k = V_kD_k^n$. By Equation (2.9), we have $B_{k,n}V_k = V_kD_k^n$. By equating the (3,1)-th element of the last equation, the result follows.

$$kS_{k,n} = b_{k,n+1} + b_{k,n} + b_{k,n-1} - 1, \quad n \geq 1.$$

2.1. Combinatorial representation of $S_{k,n}$

Let C_m be a $m \times m$ matrix defined as follows:

$$C_m(u_1, u_2, \ldots, u_m) = \begin{bmatrix} u_1 & u_2 & \ldots & u_{m-1} & u_m \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}.$$

This matrix, or some of its modifications, is called *companion matrix of the polynomial* $p(x) = x^n + u_1x^{n-1} + u_2x^{n-2} + \cdots + u_{m-1}x + u_m$, because its characteristic polynomial is $p(x)$.

Chen and Louck ([5]) showed the following result about the matrix power of $C_m(u_1, u_2, \ldots, u_m)$.

Theorem 2.3. *The (i,j)-th entry $c_{ij}^{(n)}(u_1,\ldots,u_m)$ in the matrix $C_m^n(u_1,\ldots,u_m)$ is given by the following formula:*

$$c_{ij}^{(n)}(u_1,\ldots,u_m)=\sum_{(t_1,t_2,\ldots,t_m)}\frac{t_j+t_{j+1}+\cdots+t_m}{t_1+t_2+\cdots+t_m}\times\binom{t_1+\cdots+t_m}{t_1,\ldots,t_m}u_1^{t_1}\cdots u_m^{t_m}, \tag{2.11}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+mt_m=n-i+j$, the coefficient in (2.11) is defined to be 1 if $n=i-j$, and

$$\binom{n}{n_1,\ldots,n_m}=\frac{n!}{n_1!\ \cdots\ n_m!}$$

is the multinomial coefficient.

Let R_k and $W_{k,n}$ be the following 4×4 matrices

$$R_k=\begin{bmatrix}k+1 & -k & 1 & -1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix},\quad W_{k,n}=\begin{bmatrix}S_{k,n+1} & f_{k,n} & b_{k,n} & -S_{k,n}\\ S_{k,n} & f_{k,n-1} & b_{k,n-1} & -S_{k,n-1}\\ S_{k,n-1} & f_{k,n-2} & b_{k,n-2} & -S_{k,n-2}\\ S_{k,n-2} & f_{k,n-3} & b_{k,n-3} & -S_{k,n-3}\end{bmatrix},$$

where $f_{k,n}=kf_{k,n-1}+f_{k,n-3}-k$ with $f_{k,-1}=1, f_{k,0}=0, f_{k,1}=-k, f_{k,2}=1-k-k^2$.

Proposition 2.4. *If $n\geq 2$, then $R_k^n=W_{k,n}$.*

Proof. We have

$$R_kW_{k,n-1}=\begin{bmatrix}k+1 & -k & 1 & -1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}S_{k,n} & f_{k,n-1} & b_{k,n-1} & -S_{k,n-1}\\ S_{k,n-1} & f_{k,n-2} & b_{k,n-2} & -S_{k,n-2}\\ S_{k,n-2} & f_{k,n-3} & b_{k,n-3} & -S_{k,n-3}\\ S_{k,n-3} & f_{k,n-4} & b_{k,n-4} & -S_{k,n-4}\end{bmatrix}$$
$$=\begin{bmatrix}S_{k,n+1} & f_{k,n} & b_{k,n} & -S_{k,n}\\ S_{k,n} & f_{k,n-1} & b_{k,n-1} & -S_{k,n-1}\\ S_{k,n-1} & f_{k,n-2} & b_{k,n-2} & -S_{k,n-2}\\ S_{k,n-2} & f_{k,n-3} & b_{k,n-3} & -S_{k,n-3}\end{bmatrix}=W_{k,n}.$$

Then $W_{k,n}=R_k^{n-1}W_{k,1}$. Finally, by direct computation follows $W_{k,1}=R_k$. Hence $R_k^n=W_{k,n}$. □

Note that the characteristic polynomial of the matrix R_k is $p_{R_k}(x)=p_{Q_k}(x)(x-1)=x^4-(k+1)x^3+kx^2-x+1$, where $p_{Q_k}(x)$ is the characteristic polynomial of the matrix Q_k. So the roots of R_k are $\alpha_k,\beta_k,\gamma_k,1$.

Corollary 2.5. *Let $S_{k,n}$ be the sums of the k-Narayana numbers. Then*

$$S_{k,n}=\sum_{(t_1,t_2,t_3,t_4)}\binom{t_1+t_2+t_3+t_4}{t_1,t_2,t_3,t_4}(-1)^{t_2+t_4}(k+1)^{t_1}k^{t_2},$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + 3t_3 + 4t_4 = n - 1$.

Proof. In Theorem 2.3, we consider the $(2,1)$-th entry, with $n = 4, u_1 = k + 1, u_2 = -k, u_3 = 1$ and $u_4 = -1$. Then the proof follows from Proposition 2.4 by considering the matrices R_k and $W_{k,n}$. □

For example,

$$\begin{aligned} S_{2,5} = &\sum_{t_1+2t_2+3t_3+4t_4=4} \binom{t_1+t_2+t_3+t_4}{t_1, t_2, t_3, t_4} (-1)^{t_2+t_4} 3^{t_1} 2^{t_2} \\ &= \binom{4}{4,0,0,0} 3^4 - \binom{3}{2,1,0,0} 3^2 \cdot 2 + \binom{2}{1,0,1,0} 3 \\ &\qquad - \binom{1}{0,0,0,1} + \binom{2}{0,2,0,0} 2^2 = 36. \end{aligned}$$

3. Hessenberg matrices and the k-Narayana sequence

An *upper Hessenberg matrix*, A_n, is a $n \times n$ matrix, where $a_{i,j} = 0$ whenever $i > j + 1$ and $a_{j+1,j} \neq 0$ for some j. That is, all entries bellow the superdiagonal are 0 but the matrix is not upper triangular:

$$A_n = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n-1} & a_{2,n} \\ 0 & a_{3,2} & a_{3,3} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & \cdots & a_{n,n-1} & a_{n,n} \end{bmatrix}. \tag{3.1}$$

We consider two types of upper Hessenberg matrix whose determinants and permanents are the k-Narayana numbers. The following result about upper Hessenberg matrices, proved in [8], will be used.

Theorem 3.1. *Let* a_1, $p_{i,j}, (i \leqslant j)$ *be arbitrary elements of a commutative ring* R, *and let the sequence* $a_1, a_2, \ldots$ *be defined by:*

$$a_{n+1} = \sum_{i=1}^{n} p_{i,n} a_i, \quad (n = 1, 2, \ldots).$$

If

$$A_n = \begin{bmatrix} p_{1,1} & p_{1,2} & p_{1,3} & \cdots & p_{1,n-1} & p_{1,n} \\ -1 & p_{2,2} & p_{2,3} & \cdots & p_{2,n-1} & p_{2,n} \\ 0 & -1 & p_{3,3} & \cdots & p_{3,n-1} & p_{3,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{n-1,n-1} & p_{n-1,n} \\ 0 & 0 & 0 & \cdots & -1 & p_{n,n} \end{bmatrix}.$$

Then $a_{n+1} = a_1 \det A_n$, *for* $n \geq 1$.

Let $L_{k,n}$ be a n-square matrix as follow

$$L_{k,n} = \begin{bmatrix} k & 0 & 1 & & & & 0 \\ -1 & k & 0 & 1 & & & \\ & -1 & k & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & -1 & k & 0 & 1 \\ & & & & -1 & k & 0 \\ 0 & & & & & -1 & k \end{bmatrix}.$$

Then from the above Theorem, it is clear that

$$\det L_{k,n} = b_{k,n+1}, \quad \text{for} \quad n \geq 1. \tag{3.2}$$

Theorem 3.2 (Trudi's formula [17]). *Let m be a positive integer. Then*

$$\det \begin{bmatrix} a_1 & a_2 & \cdots & & a_m \\ a_0 & a_1 & \cdots & & \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & a_1 & a_2 \\ 0 & 0 & \cdots & a_0 & a_1 \end{bmatrix}$$
$$= \sum_{(t_1,t_2,\ldots,t_m)} \binom{t_1 + \cdots + t_m}{t_1, \cdots, t_m} (-a_0)^{m-t_1-\cdots-t_m} a_1^{t_1} a_2^{t_2} \cdots a_m^{t_m}, \tag{3.3}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + mt_m = m$.

From Trudi's formula and Equation (3.2), we have

$$b_{k,n+1} = \sum_{t_1+3t_3=n} \binom{t_1+t_3}{t_1, t_3} k^{t_1}.$$

For example,

$$b_{2,7} = \sum_{t_1+3t_3=6} \binom{t_1+t_3}{t_1, t_3} 2^{t_1} = \binom{2}{0,2} + \binom{4}{3,1} 2^3 + \binom{6}{6,0} 2^6 = 97.$$

The permanent of a matrix is defined in a similar manner to the determinant but all the sign used in the Laplace expansion of minors are positive. The permanent of a n-square matrix is defined by

$$\text{per}A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation extends over all permutations σ of the symmetric group S_n ([16]). Let $A = [a_{ij}]$ be a $m \times n$ real matrix with row vectors $r_1, r_2, \ldots, r_m$. We say A is *contractible on column* k if column k contains exactly two nonzero entries, in a similar manner we define *contractible on row*. Suppose A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A by replacing row i with $a_{jk}r_i + a_{ik}r_j$, and deleting row j and column k. The matrix $A_{ij:k}$ is called the *contraction of A on column k relative to rows i and j*. If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = [A^T_{ij:k}]^T$ is called the *contraction of A on row k relative to columns i and j*.

Brualdi and Gibson [3] proved the following result about the permanent of a matrix.

Lemma 3.3. *Let A be a nonnegative integral matrix of order $n > 1$ and let B be a contraction of A. Then*

$$\text{per}A = \text{per}B.$$

There are a lot of relations between determinants or permanents of matrices and number sequences. For example, Yilmaz and Bozkurt [25] obtained some relations between Padovan sequence and permanents of one type of Hessenberg matrix. Kiliç [11] obtained some relations between the tribonacci sequence and permanents of one type of Hessenberg matrix. Öcal et al. [18] studied some determinantal and permanental representations of k-generalized Fibonacci and Lucas numbers. Janjić [8] considered a particular upper Hessenberg matrix and showed its relations with a generalization of the Fibonacci numbers. In [15], Li obtained three new Fibonacci-Hessenberg matrices and studied its relations with Pell and Perrin sequence. More examples can be found in [4, 9, 20, 21, 24].

Define the n-square Hessenberg matrix $J_k(n)$ as follows:

$$J_k(n) = \begin{bmatrix} k^2 & 1 & k & & & & 0 \\ 1 & k & 0 & 1 & & & \\ & 1 & k & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & k & 0 & 1 \\ & & & & 1 & k & 0 \\ 0 & & & & & 1 & k \end{bmatrix}. \tag{3.4}$$

Theorem 3.4. *Let $J_k(n)$ be a n-square matrix as in (3.4), then*

$$\text{per}J_k(n) = b_{k,n+2}, \tag{3.5}$$

where $b_{k,n}$ is the n-th k-Narayana number.

Proof. Let $J_{k,r}(n)$ be the r-th contraction of $J_k(n)$, by construction is a $n-r \times n-r$ matrix. By definition of the matrix $J_k(n)$, it can be contracted on column 1, then

$$J_{k,1}(n) = \begin{bmatrix} k^3+1 & k & k^2 & & & & 0 \\ 1 & k & 0 & 1 & & & \\ & 1 & k & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & k & 0 & 1 \\ & & & & 1 & k & 0 \\ 0 & & & & & 1 & k \end{bmatrix}.$$

After contracting $J_{k,1}(n)$ on the first column we have

$$J_{k,2}(n) = \begin{bmatrix} k^4+2k & k^2 & k^3+1 & & & & 0 \\ 1 & k & 0 & 1 & & & \\ & 1 & k & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & k & 0 & 1 \\ & & & & 1 & k & 0 \\ 0 & & & & & 1 & k \end{bmatrix}.$$

According to this procedure, the r-th contraction is

$$J_{k,r}(n) = \begin{bmatrix} b_{k,r+3} & b_{k,r+1} & b_{k,r+2} & & & & 0 \\ 1 & k & 0 & 1 & & & \\ & 1 & k & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & k & 0 & 1 \\ & & & & 1 & k & 0 \\ 0 & & & & & 1 & k \end{bmatrix}.$$

Hence, the $(n-3)$-th contraction is

$$J_{k,n-3}(n) = \begin{bmatrix} b_{k,n} & b_{k,n-2} & b_{k,n-1} \\ 1 & k & 0 \\ 0 & 1 & k \end{bmatrix},$$

which, by contraction of $J_{k,n-3}(n)$ on column 1,

$$J_{k,n-2}(n) = \begin{bmatrix} b_{k,n+1} & b_{k,n-1} \\ 1 & k \end{bmatrix}.$$

Then from Lemma 3.3,

$$\mathrm{per} J_k(n) = \mathrm{per} J_{k,n-2}(n) = k b_{k,n+1} + b_{k,n-1} = b_{k,n+2}.$$ □

4. The convolved k-Narayana numbers

The *convolved k-Narayana numbers* $b_{k,j}^{(r)}$ are defined by

$$B_k^{(r)}(z) = (1 - kz - z^3)^{-r} = \sum_{j=0}^{\infty} b_{k,j+1}^{(r)} z^j, \ r \in \mathbb{Z}^+.$$

Note that

$$b_{k,m+1}^{(r)} = \sum_{j_1+j_2+\cdots+j_r=m} b_{k,j_1+1} b_{k,j_2+1} \cdots b_{k,j_r+1}. \tag{4.1}$$

The generating functions of the convolved k-Narayana numbers for $k = 2$ and $r = 2, 3, 4$ are

$$\begin{aligned}
B_2^{(2)}(z) &= \frac{1}{(1-2z-z^3)^2} = 1 + 4z + 12z^2 + 34z^3 + 92z^4 + 240z^5 + 611z^6 + \cdots \\
B_2^{(3)}(z) &= \frac{1}{(1-2z-z^3)^3} = 1 + 6z + 24z^2 + 83z^3 + 264z^4 + 792z^5 + 2278z^6 + \cdots \\
B_2^{(4)}(z) &= \frac{1}{(1-2z-z^3)^4} = 1 + 8z + 40z^2 + 164z^3 + 600z^4 + 2032z^5 + \cdots .
\end{aligned}$$

Let A and C be matrices of order $n \times n$ and $m \times m$, respectively, and B be an $n \times m$ matrix. Since

$$\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \det A \det C,$$

the principal minor $M^{(k)}(i)$ of $L_{k,n}$ is equal to $b_{k,i} b_{k,n-i+1}$. It follows that the principal minor $M^{(k)}(i_1, i_2, \ldots, i_l)$ of the matrix $L_{k,n}$ is obtained by deleting rows and columns with indices $1 \leqslant i_1 < i_2 < \cdots < i_l \leqslant n$:

$$M^{(k)}(i_1, i_2, \ldots, i_l) = b_{k,i_1} b_{k,i_2-i_1} \cdots b_{k,i_l-i_{l-1}} b_{k,n-i_l+1}. \tag{4.2}$$

Then from (4.2) we have the following theorem.

Theorem 4.1. *Let $S_{n-l}^{(k)}, (l = 0, 1, 2, \ldots, n-1)$ be the sum of all principal minors of $L_{k,n}$ of order $n - l$. Then*

$$S_{n-l}^{(k)} = \sum_{j_1+j_2+\cdots+j_{l+1}=n-l} b_{k,j_1+1} b_{k,j_2+1} \cdots b_{k,j_{l+1}+1} = b_{k,n-l+1}^{(l+1)}. \tag{4.3}$$

For example,

$$S_4^{(2)} = 2 \cdot \det \begin{bmatrix} 2 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix} + \det \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix} +$$

$$\det\begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix} + \det\begin{bmatrix} 2 & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = 92 = b_{2,5}^{(2)}.$$

Since the coefficients of the characteristic polynomial of a matrix are, up to the sign, sums of principal minors of the matrix, then we have the following.

Corollary 4.2. *The convolved k-Narayana number $b_{k,n-l+1}^{(l+1)}$ is equal, up to the sign, to the coefficient of x^l in the characteristic polynomial $p_n(x)$ of $L_{k,n}$.*

For example, the characteristic polynomial of the matrix $L_{2,5}$ is $x^5 - 10x^4 + 40x^3 - 83x^2 + 92x - 44$. So, it is clear that the coefficient of x is $b_{2,5}^{(2)} = 92$.

5. Sequences and substitutions

In this section, we show that the k-Narayana sequence is related to a substitution on an alphabet of 3 symbols.

A *substitution* or a *morphism* on a finite alphabet $\mathcal{A} = \{1, \ldots, r\}$ is a map ζ from $\mathcal{A}$ to the set of finite words in $\mathcal{A}$, i.e., $\mathcal{A}^* = \cup_{i\geq 0}\mathcal{A}^i$. The map ζ is extended to $\mathcal{A}^*$ by concatenation, i.e., $\zeta(\emptyset) = \emptyset$ and $\zeta(UV) = \zeta(U)\zeta(V)$, for all U, $V \in \mathcal{A}^*$. Let $A^{\mathbb{N}}$ denote the set of one-sided infinite sequences in $\mathcal{A}$. The map ζ, is extended to $\mathcal{A}^{\mathbb{N}}$ in the obvious way. We call $u \in \mathcal{A}^{\mathbb{N}}$ a *fixed point* of ζ if $\zeta(u) = u$ and *periodic* if there exists $l > 0$ so that it is fixed for ζ^l. To these fixed or periodic points we can associate dynamical systems, which have been studied extensively, see for instance [19].

We write $l_i(U)$ for the number of occurrences of the symbol i in the word U and denote the column-vector $\mathbf{l}(U) = (l_1(U), \ldots, l_r(U))^t$. The *incidence matrix* of the substitution ζ is defined as the matrix $M_\zeta = M = (m_{ij})$ whose entry $m_{ij} = l_i(\zeta(j))$, for $1 \leq i, j \leq k$. Note that $M_\zeta(\mathbf{l}(U)) = \mathbf{l}(\zeta(U))$, for all $U \in \mathcal{A}^*$.

We consider the following substitution:

$$\zeta_k = \begin{cases} 1 \to 1^k 2, \\ 2 \to 3, \\ 3 \to 1; \end{cases}$$

where 1^k is the word $\overbrace{1 \cdots 1}^{k}$, and $k \geq 1$.

The substitution ζ_k has an unique fixed point in $\{1, 2, 3\}^{\mathbb{N}}$, and the words $\zeta_k^n(1)$ are prefixes of this fixed point. The fixed point of ζ_1 (sequence A105083) starts with the symbols:

$$1231121231231123112123112123123112123123112311212312311231 \cdots .$$

Some of the dynamical and geometrical properties associated to these sequences, have been studied in [23].

Let $a_{k,n}$ be the number of symbols of the word $\zeta_k^n(1)$. So $a_{k,0} = 1$, $a_{k,1} = k+1$ and $a_{k,2} = k^2 + k + 1$. For $n \geq 3$, we have

$$\begin{aligned}\zeta^n(1) &= \zeta^{n-1}(1^k 2) = (\zeta^{n-1}(1))^k \zeta^{n-1}(2)\\ &= (\zeta^{n-1}(1))^k \zeta^{n-2}(3) = (\zeta^{n-1}(1))^k \zeta^{n-3}(1).\end{aligned}$$

Hence

$$a_{k,n} = k a_{k,n-1} + a_{k,n-3}, \quad \text{for } n \geq 3.$$

The fist few terms of $\{a_{k,n}\}_{n=0}^{\infty}$ are

$$1,\ k+1,\ k^2+k+1,\ k^3+k^2+k+1,\ k^4+k^3+k^2+2k+1,\ k^5+k^4+k^3+3k^2+2k+1, \ldots$$

In particular:

$$\begin{aligned}&\{a_{1,n}\}_{n=0}^{\infty} = \{1,2,3,4,6,9,13,19,28,41,60,\ldots\}, \text{i.e., } a_{1,n} = b_{1,n+3} \text{ for all } n \geq 1.\\ &\{a_{2,n}\}_{n=0}^{\infty} = \{1,3,7,15,33,73,161,355,783,1727,3809,\ldots\}, \quad \text{A193641}.\\ &\{a_{3,n}\}_{n=0}^{\infty} = \{1,4,13,40,124,385,1195,3709,11512,35731,\ldots\}, \quad \text{A098183}.\end{aligned}$$

By the definition of the matrix associated to the substitution, we have M_{ζ_k} is equal to the matrix Q_k, defined in Section 2. By the recurrence of the substitution we have that the entry (i,j)-th of the matrix $M_{\zeta_k}^n$, corresponds to the number of occurrences of the symbol i in the word $\zeta_k^n(j)$, i.e., $l_i(\zeta_k^n(j))$. Since $a_{k,n}$ is the length of the word $\zeta_k^n(1)$, we have

$$\begin{aligned}a_{k,n} &= l_1(\zeta_k^n(1)) + l_2(\zeta_k^n(1)) + l_3(\zeta_k^n(1)) && (5.1)\\ &= (Q_k)_{1,1}^n + (Q_k)_{1,2}^n + (Q_k)_{1,3}^n && (5.2)\\ &= b_{k,n+1} + b_{k,n} + b_{k,n-1}. && (5.3)\end{aligned}$$

This identity shows the relation between the k-Narayana sequence and the sequence $\{a_{k,n}\}_{n=0}^{\infty}$.

If we consider the substitutions

$$\zeta_{k,i} = \begin{cases} 1 \to 1^{k-i}21^i, \\ 2 \to 3, \\ 3 \to 1; \end{cases}$$

with $0 \leq i \leq k-1$. Obviously the length of the words $\zeta_k^n(1)$ and $\zeta_{k,i}^n(1)$ coincide.

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