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On the right-continuity of infimal convolutions

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Abstract

We provide a sufficient condition for the right-continuity of the infimal convolution. First we introduce the generalized infimal convolution, and we show that this is not a proper generalization in the case of real-valued functions.

Following the main theorem, through some illustrating examples we prove that the result cannot be strengthened.

Keywords: infimal convolution, generalized infimal convolution, Darboux-property (intermediate value property), right-continuity

MSC: 26A03, 26A15, 44A35, 54C05, 54C30

1. Preliminaries

We use the following usual notions and notations in the whole paper.

Let
$$\underline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \qquad \text{and} \qquad \overline{\mathbb{R}} = \underline{\mathbb{R}} \cup \{\infty\}.$$
Note that $\overline{\mathbb{R}} = [-\infty, \infty]$ is called the extended real line.

In this paper a Darboux function is an extended real-valued function f of a real variable which has the intermediate value property, that is for any two real values a and b, and any y between $f(a)$ and $f(b)$, there is some c between a and b with $f(c) = y$.

A function $f : \mathbb{R} \to \overline{\mathbb{R}}$ is called right-continuous if for any $x_0 \in \mathbb{R}$ and for all neighborhood K of $f(x_0)$ there exists a positive real δ such that $f(x) \in K$ for all real x with $x_0 < x < x_0 + \delta$.

Now, we define $(f * g)$ and $(f \circledast g)$, the so called infimal convolution and generalized infimal convolution of f and g. (See in [5], [8] and [10])

Definition 1.1. Let $f : \mathbb{R} \to \underline{\mathbb{R}}$ be a function of $\mathbb{R}$ into $\underline{\mathbb{R}}$ and $g : V \subset \mathbb{R} \to \underline{\mathbb{R}}$ be a function of a subset V of $\mathbb{R}$ into $\underline{\mathbb{R}}$.

For any $x \in \mathbb{R}$, let

$$\Gamma(x) = \{(u, v) \in \mathbb{R} \times V : x \leq u + v\}$$

and define the functions $(f * g) : \mathbb{R} \to \overline{\mathbb{R}}$ and $(f \circledast g) : \mathbb{R} \to \overline{\mathbb{R}}$ of $\mathbb{R}$ into $\overline{\mathbb{R}}$ such that

$$(f * g)(x) = \inf_{v \in V} \big(f(x - v) + g(v)\big)$$

and

$$(f \circledast g)(x) = \inf\{f(u) + g(v) : (u, v) \in \Gamma(x)\}.$$

$(f * g)$ and $(f \circledast g)$ are called infimal convolution and generalized infimal convolution of f and g, respectively.

Remark 1.2. Note that we define the (generalized) infimal convolution only in the case, when the range of the factors f and g are subsets of $\underline{\mathbb{R}}$, and note also that if $V \neq \emptyset$, then the range of $(f \circledast g)$ and $(f * g)$ is also subset of $\underline{\mathbb{R}}$.

For all $x, y, z \in \underline{\mathbb{R}}$ we have $(-\infty) + x = x + (-\infty) = -\infty$ and $x \leq y \implies x + z \leq y + z$.

In this paper, the order of the domain and range of f and g is the usual order of (extended) real line, and the inf derives from this order also. The following theorem and corollary show that since the same ordering, the generalized infimal convolution is not a proper generalization contrary to the more abstract case, see [1] and [6].

Definition 1.3. Let $f : \mathbb{R} \to \underline{\mathbb{R}}$ and define $\hat{f} : \mathbb{R} \to \underline{\mathbb{R}}$ for all $u \in \mathbb{R}$ in the following way.

$$\hat{f}(u) = \inf_{a \geq u} f(a)$$

Proposition 1.4. *If $f : \mathbb{R} \to \underline{\mathbb{R}}$, then $\hat{f}$ is also, that is $\hat{f}(u) \neq \infty$ for all $u \in \mathbb{R}$. Moreover $\hat{f} \leq f$, $\hat{f}$ is increasing, and $\hat{f} = f$ if and only if f is increasing.*

The generalized infimal convolution could be expressed by classical one.

Theorem 1.5. *If $f : \mathbb{R} \to \underline{\mathbb{R}}$ and $g : V \subset \mathbb{R} \to \underline{\mathbb{R}}$, then for any $x \in \mathbb{R}$ we have*

$$(f \circledast g)(x) = (\hat{f} * g)(x).$$

Proof. By definition
$$(f \circledast g)(x) \leq f(u) + g(v)$$
for all $(u, v) \in \Gamma(x)$, therefore
$$(f \circledast g)(x) \leq \inf_{u \geq x-v} \big(f(u) + g(v)\big)$$
for all $v \in V$, and it follows that
$$(f \circledast g)(x) \leq \inf_{v \in V} \Big(\inf_{u \geq x-v} \big(f(u) + g(v)\big)\Big).$$
On the other hand
$$\inf_{v \in V} \Big(\inf_{u \geq x-v} \big(f(u) + g(v)\big)\Big) \leq \inf_{u \geq x-v} \big(f(u) + g(v)\big)$$
for all $v \in V$, and
$$\inf_{u \geq x-v} \big(f(u) + g(v)\big) \leq f(u) + g(v)$$
for all $(u, v) \in \Gamma(x)$, therefore
$$\inf_{v \in V} \Big(\inf_{u \geq x-v} \big(f(u) + g(v)\big)\Big) \leq (f \circledast g)(x).$$
Now we have that
$$(f \circledast g)(x) = \inf_{v \in V} \Big(\inf_{u \geq x-v} \big(f(u) + g(v)\big)\Big),$$
and by the required definitions
$$(\hat{f} * g)(x) = \inf_{v \in V} \big(\hat{f}(x-v) + g(v)\big) = \inf_{v \in V} \Big(\inf_{u \geq x-v} \big(f(u)\big) + g(v)\Big).$$
If we notice that $\inf(A + a) = \inf A + a$ for all $A \subset \underline{\mathbb{R}}$ and $a \in \underline{\mathbb{R}}$, and write $\{f(u) : u \geq x - v\}$ and $g(v)$ in place of A and a respectively, then the proof is complete. □

According to Proposition 1.4 and Theorem 1.5 we obtain:

Corollary 1.6. *If $f : \mathbb{R} \to \underline{\mathbb{R}}$ is increasing and $g : V \subset \mathbb{R} \to \underline{\mathbb{R}}$, then for any $x \in \mathbb{R}$ we have*
$$(f \circledast g)(x) = (f * g)(x).$$

The infimal convolution could not always be expressed by generalized one. Namely, there is not exist $f : \mathbb{R} \to \underline{\mathbb{R}}$ and $g : V \subset \mathbb{R} \to \underline{\mathbb{R}}$ such that $(f \circledast g) = (\sin * \sin)$, because
$$(\sin * \sin)(0) = \inf_{v \in \mathbb{R}} \big(\sin(0 - v) + \sin(v)\big) = 0,$$
and
$$(\sin * \sin)\left(\frac{\pi}{2}\right) = \inf_{v \in \mathbb{R}} \left(\sin\left(\frac{\pi}{2} - v\right) + \sin(v)\right) = \inf_{v \in \mathbb{R}} \big(\cos(v) + \sin(v)\big) = -\sqrt{2},$$
but by Proposition 2.1 $(f \circledast g)$ is increasing.

2. Main Theorem

The following proposition has been proved in [10]. Its proof is included here for the reader's convenience.

Proposition 2.1. *If $f : \mathbb{R} \to \underline{\mathbb{R}}$ and $g : V \subset \mathbb{R} \to \underline{\mathbb{R}}$, then $(f \circledast g)$ is increasing.*

Proof. Suppose that $x, y \in \mathbb{R}$ such that $x \leq y$. Then, by the corresponding definitions, it is clear that $\Gamma(y) \subset \Gamma(x)$ and thus $(f \circledast g)(x) \leq (f \circledast g)(y)$. □

Theorem 2.2. *If $f : \mathbb{R} \to \underline{\mathbb{R}}$ is a Darboux function and $g : V \subset \mathbb{R} \to \underline{\mathbb{R}}$, then $(f \circledast g)$ is right-continuous.*

Proof. Since the domain of $(f \circledast g)$ is the real line, we need only show that, for any $x \in \mathbb{R}$, we have

$$(f \circledast g)(x) = \lim_{t \to x+} (f \circledast g)(t).$$

Since Proposition 2.1 $(f \circledast g)$ is increasing, therefore by [7] Theorem 4.29. we have

$$(f \circledast g)(x) \leq \inf_{t > x} (f \circledast g)(t) = \lim_{t \to x+} (f \circledast g)(t).$$

To prove the corresponding equality, define

$$y = \inf_{t > x} (f \circledast g)(t),$$

and assume to the contrary that $(f \circledast g)(x) < y$. Then, $y \neq -\infty$.

Let $(u, v) \in \Gamma(x)$ such that

$$f(u) + g(v) < y.$$

There exists such (u, v) because y is not a lower bound of $\{f(u) + g(v) : (u, v) \in \Gamma(x)\}$.

Hence by using that $(f \circledast g)(u + v) \leq f(u) + g(v)$, we can infer that

$$(f \circledast g)(u + v) < y.$$

Thus, by the definition of y we necessarily have

$$u + v \leq x.$$

Moreover, since $(u, v) \in \Gamma(x)$, we also have $x \leq u + v$, and thus $x = u + v$.

Also by the corresponding definitions, we can note that

$$y \leq (f \circledast g)(x + 1) = (f \circledast g)(u + 1 + v) \leq f(u + 1) + g(v).$$

Hence, since $y \neq -\infty$, we can see that $g(v) \neq -\infty$.

Now, from the inequalities

$$f(u) + g(v) < y \leq f(u + 1) + g(v),$$

we can infer that

$$f(u) < y - g(v) \leq f(u+1).$$

Thus, by the Darboux property of f, there exists $s \in]u, u+1[$, such that

$$f(u) < f(s) < y - g(v),$$

and thus

$$f(s) + g(v) < y.$$

Moreover, since $x = u + v < s + v$, we can see that

$$y \leq (f \circledast g)(s+v) \leq f(s) + g(v) < y.$$

This contradiction proves the required equality. □

Corollary 2.3. *If $f : \mathbb{R} \to \underline{\mathbb{R}}$ and $g : V \subset \mathbb{R} \to \underline{\mathbb{R}}$ such that f is a continuous increasing function, then $(f * g)$ is right-continuous.*

Note that since f is increasing, hence it is a Darboux function if and only if it is continuous.

3. Some Illustrating examples

The following two examples show that $(f \circledast g)$ may be discontinuous even if $f, g : \mathbb{R} \to \underline{\mathbb{R}}$ are increasing continuous real-valued functions.

Example 3.1. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^3$. Since f is increasing

$$(f \circledast f)(x) = (f * f)(x) = \inf_{v \in \mathbb{R}} \left(x^3 - 3x^2 v + 3xv^2 \right).$$

Let $p(v) = x^3 - 3x^2 v + 3xv^2$.

If $x < 0$, then

$$\lim_{v \to \pm\infty} p(v) = -\infty$$

therefore $(f \circledast f)(x) = -\infty$.

If $x > 0$, then p is differentiable at everywhere, and $p'(v) = -3x^2 + 6xv = 3x(2v - x)$. Now we have that $p'(v) < 0$, $p'(v) = 0$ and $p'(v) > 0$ if $v < \frac{x}{2}$, $v = \frac{x}{2}$ and $v > \frac{x}{2}$, respectively. Therefore

$$(f \circledast f)(x) = \inf_{v \in \mathbb{R}} p(v) = \min_{v \in \mathbb{R}} p(v) = p\left(\frac{x}{2}\right) = \frac{x^3}{4}.$$

By Theorem 2.2, $(f \circledast f)$ is right-continuous. Hence one has

$$(f \circledast f)(0) = \lim_{x \to 0+} (f \circledast f)(x) = 0.$$

In the above example, the convolution has an infinite jump discontinuity. Since the increasingness, an arbitrary convolution has at most 2 infinite jump, and at most countable many finite jump, and has not any essential discontinuity. (See [7] Theorem 4.30.)

In the following more complicated example $(f \circledast g)$ is also real-valued, and $(f \circledast g)$ has infinitely many finite jump. For any $x \in \mathbb{R}$ let $[x]$ and $\{x\}$ denote the integer and fractional part of x.

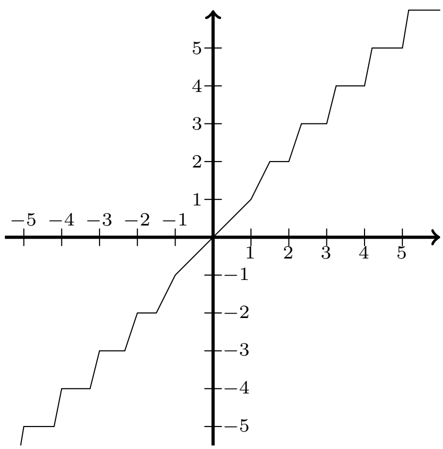
Example 3.2. Let $\alpha : \mathbb{R} \to \mathbb{R}$ such that

$$\alpha(x) = \begin{cases} 0, \text{ if} & x < 0, \\ x, \text{ if} & 0 \leq x \leq 1, \\ 1, \text{ if} & 1 < x. \end{cases}$$

It is easy to see that α is an increasing continuous function. Moreover let $g : \mathbb{R}_0^+ \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ such that

$$g(x) = \sum_{n=0}^{\infty} \alpha((n+1)(x-n)) \qquad \text{and} \qquad f(x) = \begin{cases} g(x), & \text{if } 0 \leq x, \\ -g(-x), & \text{if } x < 0. \end{cases}$$

This figure shows the graph of f.



For all $x \geq 0$ if $n \in \mathbb{Z}$ with $n \geq [x]+1 > x$ we have that $x - n < 0$ and $n+1 > 0$, hence $(n+1)(x-n) < 0$, therefore $\alpha((n+1)(x-n)) = 0$. It follows that

$$g(x) = \sum_{n=0}^{[x]} \alpha((n+1)(x-n)) \tag{3.1}$$

for all $x \geq 0$, and if $a \geq 0$, then

$$g(x) = \sum_{n=0}^{[a]} \alpha((n+1)(x-n))$$

on $[0,a]$.

It means, that definition of g is correct and g is really a real-valued function. It also means that the function series is uniformly convergent on $[0,a]$ for all $a \geq 0$, hence the continuity of $\alpha((n+1)(x-n))$ follows the continuity of g on $[0,a]$ for all $a \geq 0$ and hence g is continuous. Note that by (3.1) we can also easily see the increasingness of g.

Moreover, if $n \in \mathbb{Z}$ with $0 \leq n \leq [x]-1$, then since $(n+1)(x-n) \geq 1$, hence $\alpha((n+1)(x-n)) = 1$. It follows by (3.1) that

$$g(x) = \sum_{n=0}^{[x]-1} 1 + \alpha(([x]+1)(x-[x])) = [x] + \alpha(([x]+1)\{x\}). \tag{3.2}$$

$[x]+1 \geq 1$ and $\{x\} \geq 0$ imply that $([x]+1)\{x\} \geq \{x\}$ and it follows by the increasingness of α, that $\alpha(([x]+1)\{x\}) \geq \alpha(\{x\}) = \{x\}$. Now, since (3.2) we have that

$$g(x) \geq [x] + \{x\} = x \tag{3.3}$$

for all $x \geq 0$.

If $n \in \mathbb{Z}_0^+$, then by (3.2)

$$g(n) = [n] + \alpha(([n]+1)\{n\}) = n, \tag{3.4}$$

and

$$g(x+n) - g(x) = [x+n] + \alpha(([x+n]+1)\{x+n\}) - [x] - \alpha(([x]+1)\{x\}) \geq n \tag{3.5}$$

for all $x \geq 0$.

Since the increasingness of g and (3.4) we also have that

$$g(x) \leq g([x]+1) = [x]+1 \leq x+1, \tag{3.6}$$

for all $x \geq 0$.

To prove the increasingness of f, by using the increasingness of g let $x_1 < x_2 < 0$, and see the following

$$f(x_1) = -g(-x_1) \leq -g(-x_2) = f(x_2) \leq 0 = g(0) = f(0).$$

It follows by Corollary 1.6 that

$$(f \circledast f)(x) = (f * f)(x).$$

By (3.3) and (3.6) we have that

$$x \leq f(x) \leq x+1, \qquad \text{for all } x \geq 0, \tag{3.7}$$

and

$$x-1 \leq f(x) \leq x, \qquad \text{for all } x < 0. \tag{3.8}$$

Now, we prove that $(f \circledast f)(x) = [x]$ for all $x \geq 0$.

At first, let $x \in \mathbb{Z}_0^+$. In this case one has

$$f(x-0) + f(0) = g(x) = x,$$

therefore $(f \circledast f)(x) \leq x$.

To prove the converse inequality we show that $f(x-v)+f(v) \geq x$, for all $v \in \mathbb{R}$.
If $0 \leq v \leq x$, then $f(x-v) + f(v) \geq (x-v) + v = x$ by (3.7).
If $x < v$, then $f(x-v) + f(v) = -g(v-x) + g(v) = g((v-x)+x) - g(v-x) \geq x$ by (3.5).
If $v < 0$, then $f(x-v) + f(v) = g(x-v) - g(-v) \geq x$ by (3.5).

At second, let $x \in \mathbb{R}_0^+ \setminus \mathbb{Z}$. In this case one has

$$(f \circledast f)(x) \geq (f \circledast f)([x]) = [x],$$

by Proposition 2.1.

To prove the converse inequality we show that $f(x-v)+f(v) = [x]$ with $v \in \mathbb{Z}$ such that $v \geq [x] + 1/(1 - \{x\})$.

Really, for such v we have that

$$(v - [x])(1 - \{x\}) \geq 1,$$

that is

$$\alpha(([v-x]+1)\{v-x\}) = 1.$$

It means by (3.2) that

$$g(v-x) = [v-x] + 1 = v - [x],$$

therefore

$$f(x-v) + f(v) = -g(v-x) + g(v) = v - (v - [x]) = [x].$$

Finally, we show that $(f \circledast f)$ is real-valued. For this, it is enough to show that if $x \in \mathbb{R}$, then by (3.7) and (3.8)

$$x - 2 = (x-v-1) + (v-1) \leq f(x-v) + f(v) \leq (x-v+1) + (v+1) = x+2,$$

for all $v \in \mathbb{R}$, therefore

$$x - 2 \leq (f \circledast f)(x) \leq x + 2.$$

Finally, the following example shows that the Darboux property in Theorem 2.2 is essential.

Example 3.3. Let $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = 1_{\mathbb{R}^+} = \begin{cases} 0, \text{ if} & x \leq 0 \\ 1, \text{ if} & 0 < x. \end{cases}$$

For all real x and v we have that $f(x-v), f(v) \in \{0,1\}$, therefore $f(x-v) + f(v) \in \{0,1,2\}$, hence $(f * f)(x) \in \{0,1,2\}$.

With $v = 0$ we have that $f(x-0) + f(0) = f(x)$, hence $(f * f)(x) \leq f(x)$ for all $x \in \mathbb{R}$.

To prove the converse inequality, it is enough to show that $1 \leq (f * f)(x)$ for all $x > 0$. If $0 < x$, then for all $v \in \mathbb{R}$ $x - v > 0$ or $v > 0$, that is $f(x-v) = 1$ or $f(v) = 1$, hence $1 \leq f(x-v) + f(v)$. It follows that $1 \leq (f * f)(x)$ for all $x > 0$.

Now, we have that

$$(f * f) = f,$$

which is not right-continuous.

Note that by Corollary 1.6, the increasingness of f follows that

$$(f \circledast f)(x) = (f * f)(x).$$

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