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Pell and Pell-Lucas numbers with only one distinct digit

Bernadette Faye[ab], Florian Luca[b]

[a]Ecole Doctorale de Mathematiques et d'Informatique
Université Cheikh Anta Diop de Dakar
BP 5005, Dakar Fann, Senegal
bernadette@aims-senegal.org

[b]School of Mathematics,
University of the Witwatersrand
Private Bag 3, Wits 2050, Johannesburg, South Africa
Florian.Luca@wits.ac.za

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Abstract

In this paper, we show that there are no Pell or Pell-Lucas numbers larger than 10 with only one distinct digit.

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MSC: 11B39, 11D61

1. Introduction

Let $\{P_n\}_{n\geq 0}$ be the sequence of Pell numbers given by $P_0 = 0$, $P_1 = 1$ and

$$P_{n+2} = 2P_{n+1} + P_n \quad \text{for all} \quad n \geq 0.$$

The Pell-Lucas sequence $(Q_n)_{n\geq 0}$ satisfies the same recurrence as the sequence of Pell numbers with initial condition $Q_0 = Q_1 = 2$. If $(\alpha, \beta) = (1 + \sqrt{2}, 1 - \sqrt{2})$ is the pair of roots of the characteristic equation $x^2 - 2x - 1 = 0$ of both the Pell and Pell-Lucas numbers, then the Binet formulas for their general terms are:

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad Q_n = \alpha^n + \beta^n \quad \text{for all} \quad n \geq 0.$$

Given an integer $g > 1$, a base g-repdigit is a number of the form

$$N = a\left(\frac{g^m - 1}{g - 1}\right) \quad \text{for some} \quad m \geq 1 \quad \text{and} \quad a \in \{1, \dots, g-1\}.$$

When $g = 10$ we refer to such numbers as *repdigits*. Here we use some elementary methods to study the presence of rep-digits in the sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$. This problem leads to a Diophantine equation of the form

$$U_n = V_m \quad \text{for some} \quad m, n \geq 0, \tag{1.1}$$

where $\{U_n\}_n$ and $\{V_m\}_m$ are two non degenerate linearly recurrent sequences with dominant roots. There is a lot of literature on how to solve such equations. See, for example, [1] [4], [6], [7], [8]. The theory of linear forms in logarithms à la Baker gives that, under reasonable conditions (say, the dominant roots of $\{U_n\}_{n\geq 0}$ and $\{V_m\}_{m\geq 0}$ are multiplicatively independent), equation (1.1) has only finitely many solutions which are effectively computable. In fact, a straightforward linear form in logarithms gives some very large bounds on $\max\{m, n\}$, which then are reduced in practice either by using the LLL algorithm or by using a procedure originally discovered by Baker and Davenport [1] and perfected by Dujella and Pethő [3].

In this paper, we do not use linear forms in logarithms, but show in an elementary way that 5 and 6 are respectively the largest Pell and Pell-Lucas numbers which has only one distinct digit in their decimal expansion. The method of the proofs is similar to the method from [5], paper in which the second author determined in an elementary way the largest repdigits in the Fibonacci and the Lucas sequences. We mention that the problem of determining the repdigits in the Fibonacci and Lucas sequence was revisited in [2], where the authors determined all the repdigits in all generalized Fibonacci sequences $\{F_n^{(k)}\}_{n\geq 0}$, where this sequence starts with $k-1$ consecutive 0's followed by a 1 and follows the recurrence $F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + \cdots + F_n^{(k)}$ for all $n \geq 0$. However, for this generalization, the method used in [2] involved linear forms in logarithms.

Our results are the following.

Theorem 1.1. *If*

$$P_n = a\left(\frac{10^m - 1}{9}\right) \quad \textit{for some} \quad a \in \{1, 2, \dots, 9\}, \tag{1.2}$$

then $n = 0, 1, 2, 3$.

Theorem 1.2. *If*

$$Q_n = a\left(\frac{10^m - 1}{9}\right) \quad \textit{for some} \quad a \in \{1, 2, \dots, 9\}, \tag{1.3}$$

then $n = 0, 1, 2$.

2. Proof of Theorem 1.1

We start by listing the periods of $\{P_n\}_{n\geq 0}$ modulo 16, 5, 3 and 7 since they will be useful later

$$\begin{aligned}
&0,\ 1,\ 2,\ 5,\ 12,\ 13,\ 6,\ 9,\ 8,\ 9,\ 10,\ 13,\ 4,\ 5,\ 14,\ 1,\ 0,\ 1 \pmod{16}\\
&0,\ 1,\ 2,\ 0,\ 2,\ 4,\ 0,\ 4,\ 3,\ 0,\ 3,\ 1,\ 0,\ 1 \pmod{5}\\
&0,\ 1,\ 2,\ 2,\ 0,\ 2,\ 1,\ 1,\ 0,\ 1 \pmod{3}\\
&0,\ 1,\ 2,\ 5,\ 5,\ 1,\ 0,\ 1 \pmod{7}.
\end{aligned} \tag{2.1}$$

We also compute P_n for $n \in [1, 20]$ and conclude that the only solutions in this interval correspond to $n = 1, 2, 3$. From now, we suppose that $n \geq 21$. Hence,

$$P_n \geq P_{21} = 38613965 > 10^7.$$

Thus, $m \geq 7$. Now we distinguish several cases according to the value of a. We first treat the case when $a = 5$.

Case $a = 5$.

Since $m \geq 7$, reducing equation (1.2) modulo 16 we get

$$P_n = 5\left(\frac{10^m - 1}{9}\right) \equiv 3 \pmod{16}.$$

A quick look at the first line in (2.1) shows that there is no n such that $P_n \equiv 3 \pmod{16}$.

From now on, $a \neq 5$. Before dealing with the remaining cases, let us show that m is odd. Indeed, assume that m is even. Then, $2 \mid m$, therefore

$$11 \mid \frac{10^2 - 1}{9} \mid \frac{10^m - 1}{9} \mid P_n.$$

Since, $11 \mid P_n$, it follows that $12 \mid n$. Hence,

$$2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 = 13860 = P_{12} \mid P_n = a \cdot \frac{10^m - 1}{9},$$

and the last divisibility is not possible since $a(10^m - 1)/9$ cannot be a multiple of 10. Thus, m is odd.

We are now ready to deal with the remaining cases.

Case $a = 1$.

Reducing equation (1.2) modulo 16, we get $P_n \equiv 7 \pmod{16}$. A quick look at the first line of (2.1) shows that there is no n such that $P_n \equiv 7 \pmod{16}$. Thus, this case is impossible.

Case $a = 2$.

Reducing equation (1.2) modulo 16, we get

$$P_n = 2\left(\frac{10^m - 1}{9}\right) \equiv 14 \pmod{16}.$$

A quick look at the first line of (2.1) gives $n \equiv 14 \pmod{16}$. Reducing also equation (1.2) modulo 5, we get $P_n \equiv 2 \pmod 5$, and now line two of (2.1) gives $n \equiv 2, 4 \pmod{12}$. Since also $n \equiv 14 \pmod{16}$, we get that $n \equiv 14 \pmod{48}$. Thus, $n \equiv 6 \pmod 8$, and now row three of (2.1) shows that $P_n \equiv 1 \pmod 3$. Thus,

$$2\left(\frac{10^m - 1}{9}\right) \equiv 1 \pmod 3.$$

The left hand side above is $2(10^{m-1} + 10^{m-2} + \cdots + 10 + 1) \equiv 2m \pmod 3$, so we get $2m \equiv 1 \pmod 3$, so $m \equiv 2 \pmod 3$, and since m is odd we get $m \equiv 5 \pmod 6$. Using also the fact that $n \equiv 2 \pmod 6$, we get from the last row of (2.1) that $P_n \equiv 2 \pmod 7$. Thus,

$$2\left(\frac{10^m - 1}{9}\right) \equiv 2 \pmod 7,$$

leading to $10^m - 1 \equiv 9 \pmod 7$, so $10^{m-1} \equiv 1 \pmod 7$. This gives $6 \mid m - 1$, or $m \equiv 1 \pmod 6$, contradicting the previous conclusion that $m \equiv 5 \pmod 6$.

Case $a = 3$.

In this case, we have that $3 \mid P_n$, therefore $4 \mid n$ by the third line of (2.1). Further,

$$P_n = 3\left(\frac{10^m - 1}{9}\right) \equiv 5 \pmod{16}.$$

The first line of (2.1) shows that $n \equiv 3, 13 \pmod{16}$, contradicting the fact that $4 \mid n$. Thus, this case in impossible.

Case $a = 4$.

We have $4 \mid P_n$, which implies that $4 \mid n$. Reducing equation (1.2) modulo 5 we get that $P_n \equiv 4 \pmod 5$. Row two of (2.1) shows that $n \equiv 5, 7 \pmod{12}$, which contradicts the fact that $4 \mid n$. Thus, this case is impossible.

Case $a = 6$.

Here, we have that $3 \mid P_n$, therefore $4 \mid n$. Hence,

$$12 \mid P_n = 6\left(\frac{10^m - 1}{9}\right),$$

which is impossible.

Case $a = 7$**.**

In this case, we have that $7 \mid P_n$, therefore $6 \mid n$ by row four of (2.1). Hence,

$$70 = P_6 \mid P_n = 7\left(\frac{10^m - 1}{9}\right),$$

which is impossible.

Case $a = 8$.

We have that $8 \mid P_n$, so $8 \mid n$. Hence,

$$8 \cdot 3 \cdot 17 = 408 = P_8 \mid P_n = 8\left(\frac{10^m - 1}{9}\right),$$

implying $17 \mid 10^m - 1$. This last divisibility condition implies that $16 \mid m$, contradicting the fact that m is odd.

Case $a = 9$**.**

We have $9 \mid P_n$, thus $12 \mid n$. Hence,

$$13860 = P_{12} \mid P_n = 10^m - 1,$$

a contradiction.

This completes the proof of Theorem 1.1.

3. The proof of Theorem 1.2

We list the periods of $\{Q_n\}_{n \geq 0}$ modulo 8, 5 and 3 getting

$$\begin{aligned} &2,\ 2,\ 6,\ 6,\ 2,\ 2 \pmod 8\\ &2,\ 2,\ 1,\ 4,\ 4,\ 2,\ 3,\ 3,\ 4,\ 1,\ 1,\ 3,\ 2,\ 2 \pmod 5 \qquad (3.1)\\ &2,\ 2,\ 0,\ 2,\ 1,\ 1,\ 0,\ 1,\ 2,\ 2 \pmod 3 \end{aligned}$$

(3.2)

We next compute the first values of Q_n for $n \in [1, 20]$ and we see that there is no solution $n > 3$ in this range. Hence, from now on,

$$Q_n > Q_{21} = 109216786 > 10^8,$$

so $m \geq 9$. Further, since Q_n is always even and the quotient $(10^m - 1)/9$ is always odd, it follows that $a \in \{2, 4, 6, 8\}$. Further, from row one of (3.1) we see that Q_n is never divisible by 4. Thus, $a \in \{2, 6\}$.

Case $a = 2$**.**

Reducing equation (1.3) modulo 8, we get that

$$Q_n = 2\left(\frac{10^m - 1}{9}\right) \equiv 6 \pmod 8.$$

Row one of (3.1) shows that $n \equiv 2, 3 \pmod 4$. Reducing equation (1.3) modulo 5 we get that $Q_n \equiv 2 \pmod 5$, and now row two of (3.1) gives that $n \equiv 0, 1, 5 \pmod{12}$, so in particular $n \equiv 0, 1 \pmod 4$. Thus, we get a contradiction.

Case $a = 6$.

First $3 \mid n$, so by row three of (3.1), we have that $n \equiv 2, 6 \pmod 8$. Next reducing (1.3) modulo 8 we get

$$Q_n = 6\left(\frac{10^m - 1}{9}\right) \equiv 2 \pmod 8.$$

and by the first row of (3.1) we get $n \equiv 0, 1 \pmod 4$. Thus, this case is impossible.

This completes the proof of Theorem 1.2.

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