Annales Mathematicae et Informaticae
45 *(2015) pp. 25–37*
http://ami.ektf.hu

Some inequalities for power series with nonnegative coefficients via a reverse of Jensen inequality

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Submitted April 28, 2015 — Accepted July 8, 2015

Abstract

Some inequalities for power series with nonnegative coefficients via a new reverse of Jensen inequality are given. Applications for some fundamental functions defined by power series are also provided.

Keywords: Power series, Jensen's inequality, Reverse of Jensen's inequality

MSC: 26D15; 26D10

1. Introduction

On utilizing some reverses of Jensen discrete inequality for convex functions, we obtained in [5] the following result for functions defined by power series with nonnegative coefficients:

Theorem 1.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $p \geq 1$, $0 < \alpha < R$ and $x > 0$ with $\alpha x^p, \alpha x^{p-1} < R$, then*

$$0 \leq \frac{f(\alpha x^p)}{f(\alpha)} - \left[\frac{f(\alpha x)}{f(\alpha)}\right]^p \leq p\left[\frac{f(\alpha x^p)}{f(\alpha)} - \frac{f\left(\alpha x^{p-1}\right)}{f(\alpha)}\frac{f(\alpha x)}{f(\alpha)}\right]. \tag{1.1}$$

Moreover, if $0 < x \le 1$, *then*

$$0 \le \frac{f(\alpha x^p)}{f(\alpha)} - \left[\frac{f(\alpha x)}{f(\alpha)}\right]^p \le p\left[\frac{f(\alpha x^p)}{f(\alpha)} - \frac{f(\alpha x^{p-1})}{f(\alpha)}\frac{f(\alpha x)}{f(\alpha)}\right] \tag{1.2}$$
$$\le \frac{1}{2}p\left(\frac{f(\alpha x^{2(p-1)})}{f(\alpha)} - \left[\frac{f(\alpha x^{p-1})}{f(\alpha)}\right]^2\right)^{1/2} \le \frac{1}{4}p$$

and

$$0 \le \frac{f(\alpha x^p)}{f(\alpha)} - \left[\frac{f(\alpha x)}{f(\alpha)}\right]^p \le p\left[\frac{f(\alpha x^p)}{f(\alpha)} - \frac{f(\alpha x^{p-1})}{f(\alpha)}\frac{f(\alpha x)}{f(\alpha)}\right] \tag{1.3}$$
$$\le \frac{1}{2}p\left(\frac{f(\alpha x^2)}{f(\alpha)} - \left[\frac{f(\alpha x)}{f(\alpha)}\right]^2\right)^{1/2} \le \frac{1}{4}p.$$

Corollary 1.2. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a power series with nonnegative coefficients and convergent on the open disk* $D(0,R)$ *with* $R > 0$ *or* $R = \infty$. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $u, v > 0$ *with* $v^p \le u^q < R$, *then*

$$\left[\frac{f(uv)}{f(u^q)}\right]^p \le \frac{f(v^p)}{f(u^q)} \le \frac{1}{4}p + \left[\frac{f(uv)}{f(u^q)}\right]^p \tag{1.4}$$

and

$$0 \le [f(v^p)]^{1/p}[f(u^q)]^{1/q} - f(uv) \le \frac{1}{4^{1/p}}p^{1/p}f(u^q). \tag{1.5}$$

Utilising a different approach in [6] we obtained the following results as well:

Theorem 1.3. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a power series with nonnegative coefficients and convergent on the open disk* $D(0,R)$ *with* $R > 0$ *or* $R = \infty$. *If* $p > 1$, $0 < \alpha < R$ *and* $0 < x \le 1$, *then*

$$0 \le \frac{f(\alpha x^p)}{f(\alpha)} - \left(\frac{f(\alpha x)}{f(\alpha)}\right)^p \le M_p\left(1 - \frac{f(\alpha x)}{f(\alpha)}\right)\frac{f(\alpha x)}{f(\alpha)} \le \frac{1}{4}M_p \tag{1.6}$$

and

$$0 \le \frac{f(\alpha x^p)}{f(\alpha)} - \left(\frac{f(\alpha x)}{f(\alpha)}\right)^p \le \frac{1}{4}\cdot\frac{1 - \left(\frac{f(\alpha x)}{f(\alpha)}\right)^{p-1}}{1 - \frac{f(\alpha x)}{f(\alpha)}} \le \frac{1}{4}M_p, \tag{1.7}$$

where

$$M_p := \begin{cases} 1 & \text{if } p \in (1,2], \\ \\ p-1 & \text{if } p \in (2,\infty). \end{cases}$$

Corollary 1.4. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $u, v > 0$ with $v^p \le u^q < R$, then*

$$0 \le \frac{f(v^p)}{f(u^q)} - \left(\frac{f(uv)}{f(u^q)}\right)^p \le M_p \left(1 - \frac{f(uv)}{f(u^q)}\right) \frac{f(uv)}{f(u^q)} \le \frac{1}{4} M_p \tag{1.8}$$

and

$$0 \le \frac{f(v^p)}{f(u^q)} - \left(\frac{f(uv)}{f(u^q)}\right)^p \le \frac{1}{4} \cdot \frac{1 - \left(\frac{f(uv)}{f(u^q)}\right)^{p-1}}{1 - \frac{f(uv)}{f(u^q)}} \le \frac{1}{4} M_p. \tag{1.9}$$

For some similar exponential and logarithmic inequalities see [5] and [6] where further applications for some fundamental functions were provided .

For other recent results for power series with nonnegative coefficients, see [2, 8, 12, 13]. For more results on power series inequalities, see [2] and [8]–[11].

The most important power series with nonnegative coefficients that can be used to illustrate the above results are:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \ z \in \mathbb{C}, \ \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \ z \in D(0,1), \tag{1.10}$$

$$\ln \frac{1}{1-z} = \sum_{n=1}^{\infty} \frac{1}{n} z^n, \ z \in D(0,1), \ \cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}, \ z \in \mathbb{C},$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1}, \ z \in \mathbb{C}.$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \ z \in D(0,1), \tag{1.11}$$

$$\sin^{-1}(z) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)\,n!} z^{2n+1}, \ z \in D(0,1),$$

$$\tanh^{-1}(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \ z \in D(0,1),$$

$${}_2F_1(\alpha,\beta,\gamma,z) := \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \alpha,\beta,\gamma > 0$$
$$z \in D(0,1),$$

where Γ is *Gamma function.*

Motivated by the above results and utilizing a reverse of Jensen's inequality, we provide in this paper other inequalities for power series with nonnegative coefficients. Applications for some fundamental functions are given as well.

2. A reverse of Jensen's inequality

The following result holds:

Theorem 2.1. *Let $f : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, $\mathring{I}$ is the interior of I. If $x_i \in [m, M]$ and $w_i \geq 0$ $(i = 1, \dots, n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$, then we have the inequalities*

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} w_i f(x_i) - f\left(\sum_{i=1}^{n} w_i x_i\right) \\ &\leq 2 \max\left\{\frac{M - \sum_{i=1}^{n} w_i x_i}{M - m}, \frac{\sum_{i=1}^{n} w_i x_i - m}{M - m}\right\} \\ &\times \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m + M}{2}\right)\right]. \end{aligned} \tag{2.1}$$

Proof. We recall the following result obtained by the author in [4] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} & n \min_{i \in \{1,\dots,n\}} \{p_i\} \left[\frac{1}{n} \sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right)\right] \\ &\leq \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\right) \\ &\leq n \max_{i \in \{1,\dots,n\}} \{p_i\} \left[\frac{1}{n} \sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right)\right], \end{aligned} \tag{2.2}$$

where $f : C \to \mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i \in \{1,\dots,n\}} \subset C$ are vectors and $\{p_i\}_{i \in \{1,\dots,n\}}$ are nonnegative numbers with $P_n := \sum_{i=1}^{n} p_i > 0$.

For $n = 2$ we deduce from (2.2) that

$$\begin{aligned} & 2 \min\{t, 1 - t\} \left[\frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right)\right] \\ &\leq t f(x) + (1 - t) f(y) - f(tx + (1 - t) y) \\ &\leq 2 \max\{t, 1 - t\} \left[\frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right)\right] \end{aligned} \tag{2.3}$$

for any $x, y \in C$ and $t \in [0, 1]$.

If we use the second inequality in (2.3) for the convex function $f : I \to \mathbb{R}$ where $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, we have for $t = \frac{M - \sum_{i=1}^{n} w_i x_i}{M - m}$ that

$$\frac{\left(M - \sum_{i=1}^{n} w_i x_i\right) f(m) + \left(\sum_{i=1}^{n} w_i x_i - m\right) f(M)}{M - m} \tag{2.4}$$

$$
\begin{aligned}
&- f\left(\frac{m\left(M-\sum_{i=1}^{n} w_i x_i\right)+M\left(\sum_{i=1}^{n} w_i x_i-m\right)}{M-m}\right) \\
&\leq 2 \max \left\{\frac{M-\sum_{i=1}^{n} w_i x_i}{M-m}, \frac{\sum_{i=1}^{n} w_i x_i-m}{M-m}\right\} \\
&\times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].
\end{aligned}
$$

By the convexity of f we have that

$$
\begin{aligned}
&\sum_{i=1}^{n} w_i f\left(x_i\right)-f\left(\sum_{i=1}^{n} w_i x_i\right) \\
&=\sum_{i=1}^{n} w_i f\left[\frac{m\left(M-x_i\right)+M\left(x_i-m\right)}{M-m}\right] \\
&- f\left(\sum_{i=1}^{n} w_i\left[\frac{m\left(M-x_i\right)+M\left(x_i-m\right)}{M-m}\right]\right) \\
&\leq \sum_{i=1}^{n} w_i \frac{\left(M-x_i\right) f(m)+\left(x_i-m\right) f(M)}{M-m} \\
&- f\left(\frac{m\left(M-\sum_{i=1}^{n} w_i x_i\right)+M\left(\sum_{i=1}^{n} w_i x_i-m\right)}{M-m}\right) \\
&=\frac{\left(M-\sum_{i=1}^{n} w_i x_i\right) f(m)+\left(\sum_{i=1}^{n} w_i x_i-m\right) f(M)}{M-m} \\
&- f\left(\frac{m\left(M-\sum_{i=1}^{n} w_i x_i\right)+M\left(\sum_{i=1}^{n} w_i x_i-m\right)}{M-m}\right).
\end{aligned} \tag{2.5}
$$

Utilizing the inequality (2.5) and (2.4) we deduce the desired inequality in (2.1). □

For some related integral versions, see [4].

Remark 2.2. Since, obviously,

$$
\frac{M-\sum_{i=1}^{n} w_i x_i}{M-m}, \frac{\sum_{i=1}^{n} w_i x_i-m}{M-m} \leq 1,
$$

then we obtain from the first inequality in (2.1) the simpler, however coarser inequality, namely

$$
0 \leq \sum_{i=1}^{n} w_i f\left(x_i\right)-f\left(\sum_{i=1}^{n} w_i x_i\right) \leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right], \tag{2.6}
$$

provided that $x_i \in [m, M]$ and $w_i \geq 0$ $(i=1,\ldots,n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$.

This inequality was obtained in 2008 by S. Simić in [14].

Example 2.3. a) If we write the inequality (2.1) for the convex function $f\colon [m,M]\subset[0,\infty)\to[0,\infty)$, $f(t)=t^p$, $p\geq 1$, then we have

$$\begin{aligned}0&\leq\sum_{i=1}^{n}w_ix_i^p-\left(\sum_{i=1}^{n}w_ix_i\right)^p \qquad (2.7)\\&\leq 2\max\left\{\frac{M-\sum_{i=1}^{n}w_ix_i}{M-m},\frac{\sum_{i=1}^{n}w_ix_i-m}{M-m}\right\}\left[\frac{m^p+M^p}{2}-\left(\frac{m+M}{2}\right)^p\right]\\&\leq 2\left[\frac{m^p+M^p}{2}-\left(\frac{m+M}{2}\right)^p\right],\end{aligned}$$

for any $x_i\in[m,M]$ and $w_i\geq 0$ $(i=1,\dots,n)$ with $W_n:=\sum_{i=1}^{n}w_i=1$.

b) If we apply the inequality (2.1) for the convex function $f\colon [m,M]\subset[0,\infty)\to[0,\infty)$, $f(t)=-\ln t$, then we have

$$\begin{aligned}0&\leq\ln\left(\sum_{i=1}^{n}w_ix_i\right)-\sum_{i=1}^{n}w_i\ln x_i \qquad (2.8)\\&\leq 2\max\left\{\frac{M-\sum_{i=1}^{n}w_ix_i}{M-m},\frac{\sum_{i=1}^{n}w_ix_i-m}{M-m}\right\}\ln\left(\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)\\&\leq\ln\left(\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^2\end{aligned}$$

for any $x_i\in[m,M]$ and $w_i\geq 0$ $(i=1,\dots,n)$ with $W_n:=\sum_{i=1}^{n}w_i=1$.

This inequality is equivalent to

$$\begin{aligned}1&\leq\frac{\sum_{i=1}^{n}w_ix_i}{\prod_{i=1}^{n}x_i^{w_i}}\leq\left(\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^{2\max\left\{\frac{M-\sum_{i=1}^{n}w_ix_i}{M-m},\frac{\sum_{i=1}^{n}w_ix_i-m}{M-m}\right\}} \qquad (2.9)\\&\leq\frac{(m+M)^2}{4mM}\end{aligned}$$

for any $x_i\in[m,M]$ and $w_i\geq 0$ $(i=1,\dots,n)$ with $W_n:=\sum_{i=1}^{n}w_i=1$.

We can state the following result connected to *Hölder's inequality*:

Proposition 2.4. *If $x_i\geq 0$, $y_i>0$ for $i\in\{1,\dots,n\}$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and such that*

$$0\leq k\leq\frac{x_i}{y_i^{q-1}}\leq K \text{ for } i\in\{1,\dots,n\}, \qquad (2.10)$$

then we have

$$0\leq\frac{\sum_{i=1}^{n}x_i^p}{\sum_{i=1}^{n}y_i^q}-\left(\frac{\sum_{i=1}^{n}x_iy_i}{\sum_{i=1}^{n}y_i^q}\right)^p \qquad (2.11)$$

$$\leq 2\max\left\{\frac{K-\frac{\sum_{i=1}^{n}x_iy_i}{\sum_{i=1}^{n}y_i^q}}{K-k},\frac{\frac{\sum_{i=1}^{n}x_iy_i}{\sum_{i=1}^{n}y_i^q}-k}{K-k}\right\}\left[\frac{k^p+K^p}{2}-\left(\frac{k+K}{2}\right)^p\right]$$
$$\leq 2\left[\frac{k^p+K^p}{2}-\left(\frac{k+K}{2}\right)^p\right].$$

Proof. The inequalities (2.11) follow from (2.7) by choosing

$$z_i=\frac{x_i}{y_i^{q-1}} \text{ and } w_i=\frac{y_i^q}{\sum_{j=1}^{n}y_j^q}, i\in\{1,\dots,n\}.$$

The details are omitted. □

Remark 2.5. Let $p>1, \frac{1}{p}+\frac{1}{q}=1$. Assume that

$$0\leq k\leq\frac{a_i}{b_i^{q-1}}\leq K,\ \text{for } i\in\{1,\dots,n\}. \tag{2.12}$$

If $p_i>0$ for $i\in\{1,\dots,n\}$, then for $x_i:=p_i^{1/p}a_i$ and $y_i:=p_i^{1/q}b_i$ we have

$$\frac{x_i}{y_i^{q-1}}=\frac{p_i^{1/p}a_i}{\left(p_i^{1/q}b_i\right)^{q-1}}=\frac{p_i^{1/p}a_i}{p_i^{(q-1)/q}b_i^{q-1}}=\frac{p_i^{1/p}a_i}{p_i^{1/p}b_i^{q-1}}=\frac{a_i}{b_i^{q-1}}\in[k,K]$$

for $i\in\{1,\dots,n\}$.

If we write the inequality (2.11) for these choices, we get the weighted inequalities

$$0\leq\frac{\sum_{i=1}^{n}p_ia_i^p}{\sum_{i=1}^{n}p_ib_i^q}-\left(\frac{\sum_{i=1}^{n}p_ia_ib_i}{\sum_{i=1}^{n}p_ib_i^q}\right)^p \tag{2.13}$$
$$\leq 2\max\left\{\frac{K-\frac{\sum_{i=1}^{n}p_ia_ib_i}{\sum_{i=1}^{n}p_ib_i^q}}{K-k},\frac{\frac{\sum_{i=1}^{n}p_ia_ib_i}{\sum_{i=1}^{n}p_ib_i^q}-k}{K-k}\right\}\left[\frac{k^p+K^p}{2}-\left(\frac{k+K}{2}\right)^p\right]$$
$$\leq 2\left[\frac{k^p+K^p}{2}-\left(\frac{k+K}{2}\right)^p\right].$$

From this inequality we have:

$$\left(\frac{\sum_{i=1}^{n}p_ia_ib_i}{\sum_{i=1}^{n}p_ib_i^q}\right)^p\leq\frac{\sum_{i=1}^{n}p_ia_i^p}{\sum_{i=1}^{n}p_ib_i^q} \tag{2.14}$$
$$\leq\left(\frac{\sum_{i=1}^{n}p_ia_ib_i}{\sum_{i=1}^{n}p_ib_i^q}\right)^p+2\left[\frac{k^p+K^p}{2}-\left(\frac{k+K}{2}\right)^p\right].$$

Taking into the second inequality of (2.14) the power $1/p$ and utilizing the elementary inequality

$$(\alpha+\beta)^{1/p}\leq\alpha^{1/p}+\beta^{1/p}, \alpha,\beta\geq 0 \text{ and } p>1,$$

then we get the following additive reverse of Hölder inequality

$$\left(\sum_{i=1}^{n}p_ia_i^p\right)^{1/p}\left(\sum_{i=1}^{n}p_ib_i^q\right)^{1/q}\le\sum_{i=1}^{n}p_ia_ib_i + 2^{1/p}\left[\frac{k^p+K^p}{2}-\left(\frac{k+K}{2}\right)^p\right]^{1/p}\sum_{i=1}^{n}p_ib_i^q, \tag{2.15}$$

provided

$$0\le k\le\frac{a_i}{b_i^{q-1}}\le K,\ \text{for } i\in\{1,\dots,n\}$$

and $p_i>0$ for $i\in\{1,\dots,n\}$.

3. Power inequalities

We can state the following result for powers:

Theorem 3.1. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. If $p>1$, $0<\alpha<R$ and $0<x\le1$, then*

$$0\le\frac{f(\alpha x^p)}{f(\alpha)}-\left(\frac{f(\alpha x)}{f(\alpha)}\right)^p\le\frac{2^{p-1}-1}{2^{p-1}}\max\left\{1-\frac{f(\alpha x)}{f(\alpha)},\frac{f(\alpha x)}{f(\alpha)}\right\} \le\frac{2^{p-1}-1}{2^{p-1}}. \tag{3.1}$$

Proof. Let $m\ge1$ and $0<\alpha<R$, $0<x\le1$. If we write the inequality (2.7) for

$$w_j=\frac{a_j\alpha^j}{\sum_{k=0}^{m}a_k\alpha^k}\ \text{and}\ z_j:=x^j\in[0,1],\ j\in\{0,\dots,m\},$$

then we get

$$\begin{aligned}0&\le\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^jx^{pj}-\left(\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^jx^j\right)^p \\ &\le\frac{2^{p-1}-1}{2^{p-1}}\max\left\{1-\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^jx^j,\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^jx^j\right\} \\ &\le\frac{2^{p-1}-1}{2^{p-1}}.\end{aligned} \tag{3.2}$$

Since all series whose partial sums involved in the inequality (3.2) are convergent, then by letting $m\to\infty$ in (3.2) we deduce (2.15). □

Corollary 3.2. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $u,v>0$ with $v^p\le u^q<R$, then*

$$0\le \frac{f(v^p)}{f(u^q)}-\left(\frac{f(uv)}{f(u^q)}\right)^p\le \frac{2^{p-1}-1}{2^{p-1}}\max\left\{1-\frac{f(uv)}{f(u^q)},\frac{f(uv)}{f(u^q)}\right\} \tag{3.3}$$

and

$$0\le [f(v^p)]^{1/p}[f(u^q)]^{1/q}-f(uv)\le\left(\frac{2^{p-1}-1}{2^{p-1}}\right)^{1/p}f(u^q). \tag{3.4}$$

Proof. The inequality (3.3) follows by taking into (3.1) $\alpha=u^q$ and $x=\frac{v}{u^{q/p}}$. The details are omitted.

Taking the power $1/p$ and using the inequality $(a+b)^{1/p}\le a^{1/p}+b^{1/p}$, $p\ge 1$ we get from

$$\frac{f(v^p)}{f(u^q)}\le\left(\frac{f(uv)}{f(u^q)}\right)^p+\frac{2^{p-1}-1}{2^{p-1}}$$

the desired inequality (3.4). □

Example 3.3. a) If we write the inequality (3.1) for the function $\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n$, $z\in D(0,1)$, then we have

$$0\le\frac{1-\alpha}{1-\alpha x^p}-\left(\frac{1-\alpha}{1-\alpha x}\right)^p\le\frac{2^{p-1}-1}{2^{p-1}}\max\left\{\frac{\alpha(1-x)}{1-\alpha x},\frac{1-\alpha}{1-\alpha x}\right\} \tag{3.5}$$

for any $\alpha,x\in(0,1)$ and $p\ge 1$.

b) If we write the inequality (3.1)) for the function $\exp z=\sum_{n=0}^{\infty}\frac{1}{n!}z^n$, $z\in\mathbb{C}$, then we have

$$\begin{aligned}0&\le \exp[\alpha(x^p-1)]-\exp[p\alpha(x-1)] \qquad (3.6)\\ &\le\frac{2^{p-1}-1}{2^{p-1}}\max\{1-\exp[\alpha(x-1)],\exp[\alpha(x-1)]\}\end{aligned}$$

for any $\alpha,p>0$ and $x\in(0,1)$.

4. Logarithmic inequalities

If we write the inequality (2.1) for the convex function $f\colon [m,M]\subset(0,\infty)\to\mathbb{R}$, $f(t)=t\ln t$, then we have

$$\begin{aligned}0&\le\sum_{i=1}^{n}w_ix_i\ln x_i-\left(\sum_{i=1}^{n}w_ix_i\right)\ln\left(\sum_{i=1}^{n}w_ix_i\right) \qquad (4.1)\\ &\le 2\max\left\{\frac{M-\sum_{i=1}^{n}w_ix_i}{M-m},\frac{\sum_{i=1}^{n}w_ix_i-m}{M-m}\right\}\end{aligned}$$

$$\times\left[\frac{m\ln m+M\ln M}{2}-\left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right]$$

for any $x_i\in[m,M]$ and $w_i\geq 0$ $(i=1,\dots,n)$ with $W_n:=\sum_{i=1}^n w_i=1$.

This is equivalent to

$$1\leq\frac{\prod_{i=1}^n x_i^{w_ix_i}}{\left(\sum_{i=1}^n w_ix_i\right)^{\left(\sum_{i=1}^n w_ix_i\right)}}\leq\left[\frac{m^mM^M}{\left(\frac{m+M}{2}\right)^{m+M}}\right]^{\max\left\{\frac{M-\sum_{i=1}^n w_ix_i}{M-m},\frac{\sum_{i=1}^n w_ix_i-m}{M-m}\right\}}\tag{4.2}$$

for any $x_i\in[m,M]$ and $w_i\geq 0$ $(i=1,\dots,n)$ with $W_n:=\sum_{i=1}^n w_i=1$.

If we take $M=1$ and let $m\to 0+$ in the inequality (4.1), we have

$$\begin{aligned}0&\leq\sum_{i=1}^n w_ix_i\ln x_i-\left(\sum_{i=1}^n w_ix_i\right)\ln\left(\sum_{i=1}^n w_ix_i\right)\\&\leq\max\left\{1-\sum_{i=1}^n w_ix_i,\sum_{i=1}^n w_ix_i\right\}\ln 2,\end{aligned}\tag{4.3}$$

for any $x_i\in(0,1]$ and $w_i\geq 0$ $(i=1,\dots,n)$ with $W_n:=\sum_{i=1}^n w_i=1$.

This is equivalent to

$$1\leq\frac{\prod_{i=1}^n x_i^{w_ix_i}}{\left(\sum_{i=1}^n w_ix_i\right)^{\left(\sum_{i=1}^n w_ix_i\right)}}\leq 2^{\max\left\{1-\sum_{i=1}^n w_ix_i,\sum_{i=1}^n w_ix_i\right\}},\tag{4.4}$$

for any $x_i\in(0,1]$ and $w_i\geq 0$ $(i=1,\dots,n)$ with $W_n:=\sum_{i=1}^n w_i=1$.

Theorem 4.1. *Let $f(z)=\sum_{n=0}^\infty a_nz^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. If $0<\alpha<R$, $p>0$ and $x\in(0,1)$, then*

$$\begin{aligned}0&\leq\frac{p\alpha x^pf'(\alpha x^p)}{f(\alpha)}\ln x-\frac{f(\alpha x^p)}{f(\alpha)}\ln\left(\frac{f(\alpha x^p)}{f(\alpha)}\right)\\&\leq\max\left\{1-\frac{f(\alpha x^p)}{f(\alpha)},\frac{f(\alpha x^p)}{f(\alpha)}\right\}\ln 2.\end{aligned}\tag{4.5}$$

Proof. If $0<\alpha<R$ and $m\geq 1$, then by (4.3) for $x_j=(x^p)^j$, we have

$$0\leq\frac{1}{\sum_{k=0}^m a_k\alpha^k}\sum_{j=0}^m a_j\alpha^j(x^p)^j\ln(x^p)^j\tag{4.6}$$

$$
\begin{aligned}
&-\left(\frac{1}{\sum_{k=0}^{m} a_k\alpha^k}\sum_{j=0}^{m} a_j\alpha^j (x^p)^j\right)\ln\left(\frac{1}{\sum_{k=0}^{m} a_k\alpha^k}\sum_{j=0}^{m} a_j\alpha^j (x^p)^j\right)\\
&\leq \max\left\{1-\frac{1}{\sum_{k=0}^{m} a_k\alpha^k}\sum_{j=0}^{m} a_j\alpha^j (x^p)^j, \frac{1}{\sum_{k=0}^{m} a_k\alpha^k}\sum_{j=0}^{m} a_j\alpha^j (x^p)^j\right\}\ln 2,
\end{aligned}
$$

for $p > 0$ and $x \in (0,1)$. This is equivalent to:

$$
\begin{aligned}
0 &\leq \frac{p\ln(x)}{\sum_{k=0}^{m} a_k\alpha^k}\sum_{j=0}^{m} j a_j\alpha^j (x^p)^j \qquad (4.7)\\
&-\left(\frac{1}{\sum_{k=0}^{m} a_k\alpha^k}\sum_{j=0}^{m} a_j\alpha^j (x^p)^j\right)\ln\left(\frac{1}{\sum_{k=0}^{m} a_k\alpha^k}\sum_{j=0}^{m} a_j\alpha^j (x^p)^j\right)\\
&\leq \max\left\{1-\frac{1}{\sum_{k=0}^{m} a_k\alpha^k}\sum_{j=0}^{m} a_j\alpha^j (x^p)^j, \frac{1}{\sum_{k=0}^{m} a_k\alpha^k}\sum_{j=0}^{m} a_j\alpha^j (x^p)^j\right\}\ln 2,
\end{aligned}
$$

for $p > 0$ and $x \in (0,1)$.

Since $0 < \alpha < R$, $x \in (0,1)$ and $p > 0$ then $0 < \alpha x^p < R$ and the series

$$
\sum_{k=0}^{\infty} a_k\alpha^k, \sum_{j=0}^{\infty} j a_j\alpha^j (x^p)^j \text{ and } \sum_{j=0}^{\infty} a_j\alpha^j (x^p)^j
$$

are convergent. Therefore by letting $m \to \infty$ in (4.7) we deduce (4.5). □

Example 4.2. a) If we write the inequality (4.5) for the function $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $z \in D(0,1)$, then we have for $\alpha, x \in (0,1)$ and $p > 0$ that

$$
\begin{aligned}
0 &\leq \frac{p\alpha x^p (1-\alpha)}{(1-\alpha x^p)^2}\ln x - \frac{1-\alpha}{(1-\alpha x^p)}\ln\left(\frac{1-\alpha}{1-\alpha x^p}\right) \qquad (4.8)\\
&\leq \max\left\{\frac{\alpha(1-x^p)}{1-\alpha x^p}, \frac{1-\alpha}{1-\alpha x^p}\right\}\ln 2
\end{aligned}
$$

b) If we write the inequality (4.5) for the function $\exp z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$, $z \in \mathbb{C}$, then we have

$$
\begin{aligned}
0 &\leq [p\alpha x^p \ln x - \alpha(x^p - 1)]\exp[\alpha(x^p - 1)] \qquad (4.9)\\
&\leq \max\{1-\exp[\alpha(x^p-1)], \exp[\alpha(x^p-1)]\}\ln 2
\end{aligned}
$$

for $x \in (0,1)$ and $\alpha, p > 0$.

5. Exponential inequalities

If we consider the exponential function $f\colon \mathbb{R} \to (0,\infty)$, $f(t) = \exp(t)$, then from (2.1) we have the inequalities

$$\begin{aligned} 0 &\le \sum_{i=1}^{n} w_i \exp(x_i) - \exp\left(\sum_{i=1}^{n} w_i x_i\right) \\ &\le 2\max\left\{\frac{M - \sum_{i=1}^{n} w_i x_i}{M-m}, \frac{\sum_{i=1}^{n} w_i x_i - m}{M-m}\right\} \\ &\times \left[\frac{\exp(m) + \exp(M)}{2} - \exp\left[\left(\frac{m+M}{2}\right)\right]\right] \end{aligned} \tag{5.1}$$

if $x_i \in [m, M]$ and $w_i \ge 0$ $(i = 1, \dots, n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$.

If we take in (5.1) $M = 0$ and let $m \to -\infty$, then we get

$$0 \le \sum_{i=1}^{n} w_i \exp(x_i) - \exp\left(\sum_{i=1}^{n} w_i x_i\right) \le 1 \tag{5.2}$$

for $x_i \le 0$ and $w_i \ge 0$ $(i = 1, \dots, n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$.

Theorem 5.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $x \le 0$ with $\exp(x) < R$ and $0 < \alpha < R$, then*

$$0 \le \frac{f(\alpha \exp(x))}{f(\alpha)} - \exp\left[\frac{\alpha x f'(\alpha)}{f(\alpha)}\right] \le 1. \tag{5.3}$$

Proof. If $0 < \alpha < R$ and $m \ge 1$, then by (5.2) for $x_j = jx$, we have

$$0 \le \frac{1}{\sum_{j=0}^{m} a_j \alpha^j} \sum_{j=0}^{m} a_j \alpha^j [\exp(x)]^j - \exp\left(\frac{x}{\sum_{j=0}^{m} a_j \alpha^j} \sum_{j=0}^{m} j a_j \alpha^j\right) \le 1 \tag{5.4}$$

for $x \in (-\infty, 0)$.

Since all series whose partial sums involved in the inequality (5.4) are convergent, then by letting $m \to \infty$ in (5.4) we deduce (5.3). □

Example 5.2. a) If we write the inequality (5.3) for the function $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $z \in D(0,1)$, then we have for $x \le 0$ and $0 < \alpha < 1$, that

$$0 \le \frac{1-\alpha}{1-\alpha\exp(x)} - \exp\left(\frac{\alpha x}{1-\alpha}\right) \le 1. \tag{5.5}$$

b) If we write the inequality (5.3) for the function $\exp z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$, $z \in \mathbb{C}$, then we have

$$0 \le \exp(\alpha[\exp(x) - 1]) - \exp(\alpha x) \le 1 \tag{5.6}$$

for any $\alpha > 0$ and $x \le 0$.

Acknowledgements. The author would like to thank the anonymous referee for valuable suggestions that have been implemented in the final version of the paper.

References

[1] Cerone P., Dragomir S. S., A refinement of the Grüss inequality and applications, *Tamkang J. Math.* **38**(2007), No. 1, 37-49. Preprint *RGMIA Res. Rep. Coll.,* **5**(2) (2002), Art. 14. [Online `http://rgmia.org/papers/v5n2/RGIApp.pdf`].

[2] Cerone P., Dragomir S. S., Some applications of de Bruijn's inequality for power series. *Integral Transform. Spec. Funct.* **18**(**6**) (2007), 387–396.

[3] Dragomir S. S. , *Discrete Inequalities of the Cauchy-Bunyakovsky-Schwarz Type*, Nova Science Publishers Inc., N.Y., 2004.

[4] Dragomir S. S., Some reverses of the Jensen inequality with applications. *Bull. Aust. Math. Soc.* **87** (2013), no. 2, 177–194.

[5] Dragomir S. S., Inequalities for power series with nonnegative coefficients via a reverse of Jensen inequality, Preprint *RGMIA Res. Rep. Coll.,***17** (2014), Art. 47. [Online `http://rgmia.org/papers/v17/v17a47.pdf`].

[6] Dragomir S. S., Further inequalities for power series with nonnegative coefficients via a reverse of Jensen inequality, Preprint *RGMIA Res. Rep. Coll.,***17** (2014)

[7] Dragomir S. S., Ionescu N. M., and Some converse of Jensen's inequality and applications. *Rev. Anal. Numér. Théor. Approx.* **23** (1994), no. 1, 71–78. MR1325895 (96c:26012).

[8] Ibrahim A., Dragomir S. S., Power series inequalities via Buzano's result and applications. *Integral Transform. Spec. Funct.* **22**(**12**) (2011), 867–878.

[9] Ibrahim A., Dragomir S. S., Power series inequalities via a refinement of Schwarz inequality. *Integral Transform. Spec. Funct.* **23**(**10**) (2012), 769–78.

[10] Ibrahim A., Dragomir S. S., A survey on Cauchy–Bunyakovsky–Schwarz inequality for power series, p. 247-p. 295, in G.V. Milovanović and M.Th. Rassias (eds.), *Analytic Number Theory, Approximation Theory, and Special Functions*, Springer, 2013. DOI 10.1007/978-1-4939-0258-310,

[11] Ibrahim A., Dragomir S. S., Darus M., Some inequalities for power series with applications. *Integral Transform. Spec. Funct.* **24**(**5**) (2013), 364–376.

[12] Ibrahim A., Dragomir S. S, Darus M., Power series inequalities related to Young's inequality and applications. *Integral Transforms Spec. Funct.* **24** (2013), no. 9, 700–714.

[13] Ibrahim A., Dragomir S. S., Darus M., Power series inequalities via Young's inequality with applications. *J. Inequal. Appl.* **2013**, 2013:314, 13 pp.

[14] Simić S., On a global upper bound for Jensen's inequality, *J. Math. Anal. Appl.* **343**(2008), 414-419.