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On geodesic mappings of Riemannian spaces with cyclic Ricci tensor

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Abstract

An n-dimensional Riemannian space V^n is called a Riemannian space with cyclic Ricci tensor [2, 3], if the Ricci tensor R_{ij} satisfies the following condition

$$R_{ij,k} + R_{jk,i} + R_{ki,j} = 0,$$

where R_{ij} the Ricci tensor of V^n, and the symbol "," denotes the covariant derivation with respect to Levi-Civita connection of V^n.

In this paper we would like to treat some results in the subject of geodesic mappings of Riemannian space with cyclic Ricci tensor.

Let $V^n = (M^n, g_{ij})$ and $\overline{V}^n = (M^n, \overline{g}_{ij})$ be two Riemannian spaces on the underlying manifold M^n. A mapping $V^n \to \overline{V}^n$ is called geodesic, if it maps an arbitrary geodesic curve of V^n to a geodesic curve of $\overline{V}^n$.[4]

At first we investigate the geodesic mappings of a Riemannian space with cyclic Ricci tensor into another Riemannian space with cyclic Ricci tensor.

Finally we show that, the Riemannian - Einstein space with cyclic Ricci tensor admit only trivial geodesic mapping.

Keywords: Riemannian spaces, geodesic mapping.

MSC: 53B40.

1. Introduction

Let an n-dimensional V^n Riemannian space be given with the fundamental tensor $g_{ij}(x)$. V^n has the Riemannian curvature tensor R^i_{jkl} in the following form:

$$R^h_{ijk}(x) = \partial_j \Gamma^h_{ik}(x) + \Gamma^\alpha_{ik}(x)\Gamma^h_{j\alpha}(x) - \partial_k \Gamma^h_{ij}(x) - \Gamma^\alpha_{ij}(x)\Gamma^h_{k\alpha}(x), \quad (1.1)$$

where $\Gamma^i_{jk}(x)$ are the coefficients of Levi-Civita connection of V^n.

The Ricci curvature tensor we obtain from the Riemannian curvature tensor using of the following transvection: $R^\alpha_{jk\alpha}(x) = R_{jk}(x)$[1].

Definition 1.1. [2, 3] A Riemannian space V^n is called a Riemannian space with cyclic Ricci tensor, if the Ricci tensor of V^n satisfies the following equation:

$$R_{ij,k} + R_{jk,i} + R_{ki,j} = 0, \quad (1.2)$$

where the symbol "," means the covariant derivation with respect to Levi-Civita connection of V^n.

Definition 1.2. [4] Let two Riemannian spaces V^n and $\overline{V}^n$ be given on the underlying manifold M_n . The maps: $\gamma : V^n \to \overline{V}^n$ is called geodesic (projective) mappings, if any geodesic curve of V^n coincides with a geodesic curve of $\overline{V}^n$.

It is wellknown, that the the geodesic curve $x^i(t)$ of V^n is a result of the second order ordinary differential equations in a canonical parameter t:

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{\alpha\beta}(x)\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0. \quad (1.3)$$

We need in the investigations the next:

Theorem 1.3. *[4] The maps: $V^n \to \overline{V}^n$ is geodesic if and only if exits a gradient vector field $\psi_i(x)$, which satisfies the following condition:*

$$\overline{\Gamma}^i_{jk}(x) = \Gamma^i_{jk}(x) + \delta^i_j\psi_k(x) + \delta^i_k\psi_j(x), \quad (1.4)$$

and

Definition 1.4. [1] A Riemannian space V^n is called Einstein space, if exists a $\rho(x)$ scalar function, which satisfies the equation:

$$R_{ij} = \rho(x)g_{ij}(x). \quad (1.5)$$

[1]The Roman and Greek indices run over the range 1, ..., n; the Roman indices are free but the Greek indices denote summation.

2. Geodesic mappings of Riemannian spaces with cyclic Ricci tensors

It is easy to get the next equations [4]:

$$\overline{R}_{ij} = R_{ij} + (n-1)\psi_{ij}, \tag{2.1}$$

where $\psi_{ij} = \psi_{i,j} - \psi_i\psi_j$ and

$$\overline{R}_{ij,k} = \frac{\partial \overline{R}_{ij}}{\partial x^k} - \overline{\Gamma}^{\alpha}_{ik}(x)\overline{R}_{\alpha j} - \overline{\Gamma}^{\alpha}_{jk}(x)\overline{R}_{\alpha i}, \tag{2.2}$$

where $\overline{\Gamma}^{\alpha}_{ik}(x)$ are components of Levi-Civita connection if $\overline{V}^n$.

At now we suppose, that $\overline{V}^n$ in a Riemannian space with cyclic Ricci tensor, that is

$$\overline{R}_{ij,k} + \overline{R}_{jk,i} + \overline{R}_{ki,j} = 0. \tag{2.3}$$

Using (2.2) we can rewrite (2.3) in the following form:

$$\begin{aligned} \frac{\partial \overline{R}_{ij}}{\partial x^k} - \overline{\Gamma}^{\alpha}_{ik}(x)\overline{R}_{\alpha j} - \overline{\Gamma}^{\alpha}_{jk}(x)\overline{R}_{\alpha i} + \\ \frac{\partial \overline{R}_{jk}}{\partial x^i} - \overline{\Gamma}^{\alpha}_{ji}(x)\overline{R}_{\alpha k} - \overline{\Gamma}^{\alpha}_{ki}(x)\overline{R}_{\alpha j} + \\ \frac{\partial \overline{R}_{ki}}{\partial x^j} - \overline{\Gamma}^{\alpha}_{kj}(x)\overline{R}_{\alpha i} - \overline{\Gamma}^{\alpha}_{ij}(x)\overline{R}_{\alpha k} = 0. \end{aligned} \tag{2.4}$$

From (1.4) and (2.1) we can compute:

$$\frac{\partial(R_{ij} + (n-1)\psi_{ij})}{\partial x^k} - (\Gamma^{\alpha}_{ik}(x) + \psi_i(x)\delta^{\alpha}_k + \psi_k(x)\delta^{\alpha}_i)(R_{\alpha j} + (n-1)\psi_{\alpha j}) -$$

$$-(\Gamma^{\alpha}_{jk}(x) + \psi_j(x)\delta^{\alpha}_k + \psi_k(x)\delta^{\alpha}_j)(R_{\alpha i} + (n-1)\psi_{\alpha i}) +$$

$$\frac{\partial(R_{jk} + (n-1)\psi_{jk})}{\partial x^i} - (\Gamma^{\alpha}_{ji}(x) + \psi_j(x)\delta^{\alpha}_i + \psi_i(x)\delta^{\alpha}_j)(R_{\alpha k} + (n-1)\psi_{\alpha k}) -$$

$$-(\Gamma^{\alpha}_{ki}(x) + \psi_k(x)\delta^{\alpha}_i + \psi_i(x)\delta^{\alpha}_k)(R_{\alpha j} + (n-1)\psi_{\alpha j}) +$$

$$\frac{\partial(R_{ki} + (n-1)\psi_{ki})}{\partial x^j} - (\Gamma^{\alpha}_{kj}(x) + \psi_k(x)\delta^{\alpha}_j + \psi_j(x)\delta^{\alpha}_k)(R_{\alpha i} + (n-1)\psi_{\alpha i}) -$$

$$-(\Gamma^{\alpha}_{ij}(x) + \psi_i(x)\delta^{\alpha}_j + \psi_j(x)\delta^{\alpha}_i)(R_{\alpha k} + (n-1)\psi_{\alpha k}) = 0.$$

That is

$$\left.\begin{aligned} \tfrac{\partial R_{ij}}{\partial x^k} - \Gamma^{\alpha}_{ik}(x)R_{\alpha j} - \Gamma^{\alpha}_{jk}(x)R_{\alpha i} + \\ + \tfrac{\partial R_{jk}}{\partial x^i} - \Gamma^{\alpha}_{ji}(x)R_{\alpha k} - \Gamma^{\alpha}_{ki}(x)R_{\alpha j} + \\ + \tfrac{\partial R_{ki}}{\partial x^j} - \Gamma^{\alpha}_{kj}(x)R_{\alpha i} - \Gamma^{\alpha}_{ij}(x)R_{\alpha k} + \end{aligned}\right\} R_{ij,k} + R_{jk,i} + R_{ki,j}$$

$$+(n-1)\frac{\partial\psi_{ij}}{\partial x^k}-(n-1)\Gamma^{\alpha}_{ik}(x)\psi_{\alpha j}-\psi_i(x)R_{kj}-(n-1)\psi_i(x)\psi_{kj}-$$
$$-\psi_k(x)R_{ij}-(n-1)\psi_k(x)\psi_{ij}-(n-1)\Gamma^{\alpha}_{jk}(x)\psi_{\alpha i}-\psi_j(x)R_{ki}-$$
$$-(n-1)\psi_j(x)\psi_{ki}-\psi_k(x)R_{ji}-(n-1)\psi_k(x)\psi_{ji}+$$
$$+(n-1)\frac{\partial\psi_{jk}}{\partial x^i}-(n-1)\Gamma^{\alpha}_{ji}(x)\psi_{\alpha k}-\psi_j(x)R_{ik}-(n-1)\psi_j(x)\psi_{ik}-$$
$$-\psi_i(x)R_{jk}-(n-1)\psi_i(x)\psi_{jk}-(n-1)\Gamma^{\alpha}_{ki}(x)\psi_{\alpha j}-\psi_k(x)R_{ij}-$$
$$-(n-1)\psi_k(x)\psi_{ij}-\psi_i(x)R_{kj}-(n-1)\psi_i(x)\psi_{kj}+$$
$$+(n-1)\frac{\partial\psi_{ki}}{\partial x^j}-(n-1)\Gamma^{\alpha}_{kj}(x)\psi_{\alpha i}-\psi_k(x)R_{ji}-(n-1)\psi_k(x)\psi_{ji}-$$
$$-\psi_j(x)R_{ki}-(n-1)\psi_j(x)\psi_{ki}-(n-1)\Gamma^{\alpha}_{ij}(x)\psi_{\alpha k}-\psi_i(x)R_{jk}-$$
$$-(n-1)\psi_i(x)\psi_{jk}-\psi_j(x)R_{ik}-(n-1)\psi_j(x)\psi_{ik}=0.$$

If we suppose, that V^n has cyclic Ricci tensor we have:

$$(n-1)\left(\frac{\partial\psi_{ij}}{\partial x^k}-\Gamma^{\alpha}_{ik}(x)\psi_{\alpha j}-\Gamma^{\alpha}_{jk}(x)\psi_{\alpha i}\right)+$$
$$+(n-1)\left(\frac{\partial\psi_{jk}}{\partial x^i}-\Gamma^{\alpha}_{ji}(x)\psi_{\alpha k}-\Gamma^{\alpha}_{ki}(x)\psi_{\alpha j}\right)+$$
$$+(n-1)\left(\frac{\partial\psi_{ki}}{\partial x^j}-\Gamma^{\alpha}_{kj}(x)\psi_{\alpha i}-\Gamma^{\alpha}_{ij}(x)\psi_{\alpha k}\right)+$$
$$-4\psi_i(x)R_{jk}-4\psi_j(x)R_{ki}-4\psi_k(x)R_{ij}-$$
$$-(n-1)(4\psi_i(x)\psi_{jk}+4\psi_j(x)\psi_{ki}+4\psi_k(x)\psi_{ij})=0.$$

That is

$$\begin{aligned}(n-1)(\psi_{ij,k}+\psi_{jk,i}+\psi_{ki,j})-\\ -4(\psi_i(x)R_{jk}+\psi_j(x)R_{ki}+\psi_k(x)R_{ij})-\\ -4(n-1)(\psi_i(x)\psi_{jk}+\psi_j(x)\psi_{ki}+\psi_k(x)\psi_{ij})=0.\end{aligned}\tag{2.5}$$

Theorem 2.1. *V^n and $\overline{V}^n$ Riemannian spaces with cyclic Ricci tensors have common geodesics, that is V^n and $\overline{V}^n$ have a geodesic mapping if and only if exists a $\psi_i(x)$ gradient vector, which satisfies the condition:*

$$(n-1)(\psi_{ij,k}+\psi_{jk,i}+\psi_{ki,j})-$$
$$-4(\psi_i(x)R_{jk}+\psi_j(x)R_{ki}+\psi_k(x)R_{ij})-$$
$$-4(n-1)(\psi_i(x)\psi_{jk}+\psi_j(x)\psi_{ki}+\psi_k(x)\psi_{ij})=0.$$

3. Consequences

A) If $\psi_{ij} = 0$, then $\overline{R}_{ij} = R_{ij}$, and $\psi_{i,j} = \psi_i \psi_j$, so we obtain:

$$\psi_i(x)R_{jk} + \psi_j(x)R_{ki} + \psi_k(x)R_{ij} = 0. \tag{3.1}$$

B) If the V^n is a Riemannian space with cyclic Ricci tensor and at the same time is a Einstein space, then we get

$$\rho\psi_i(x)g_{jk} + \rho\psi_j(x)g_{ki} + \rho\psi_k(x)g_{ij} = 0$$

that is

$$n\psi_i(x) + 2\psi_i(x) = 0, \tag{3.2}$$

so

$$(n+2)\psi_i(x) = 0. \tag{3.3}$$

It means

Theorem 3.1. *A Riemannian-Einstein space V^n with cyclic Ricci tensor admits into $\overline{V}^n$ with cyclic Ricci tensor only trivial (affin) geodesic mapping.*

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