Annales Mathematicae et Informaticae
42 (2013) pp. 57–64
http://ami.ektf.hu

On pseudoconformal models of fibrations determined by the algebra of antiquaternions and projectivization of them*

Irina Kuzmina[a], Josef Mikeš[b]

[a]Kazan Federal University, Russian Federation
iranina@mail.ru

[b]Palacky University Olomouc, Czech Republic
josef.mikes@upol.cz

Submitted December 4, 2012 — Accepted January 31, 2013

Abstract

In the present article we study the principal bundles determined by the algebra of antiquaternions in the projective model. The projectivizations of the pseudoconformal models of fibrations determined by the subalgebra of complex numbers is considered as example.

Keywords: pseudoconformal models, conformal mapping, pseudo-euclidean space, metric form

MSC: 53B20; 53B30; 53C21

1. Introduction

A. P. Norden developed the theory of normalization which appeared useful in applications to conformal, non-Euclidean and linear geometry [4]. By means of the normalization theory, A. P. Shirokov [8] succeeded to construct conformal models of non-Euclidean spaces. We show here basic steps of this construction.

*Supported by grant P201/11/0356 of The Czech Science Foundation

Let a real non-degenerate hyperquadric Q be given in the projective space $\mathbb{P}^{n+1}$. Let us choose a projective frame $(E_0, \dots, E_{n+1})$ such that E_{n+1} is the pole of the hyperplane $y^{n+1} = 0$, and the straight line $E_n E_{n+1}$ intersects the hyperquadric Q in two real points N and N', and the points $E_0, \dots, E_{n-1}$ belong to the polar of the straight line $E_n E_{n+1}$.

Then the analytic expression of the hyperquadric Q reads

$$\mathbf{y}^2 = a_{pq} y^p y^q + (y^n)^2 - (y^{n+1})^2 = 0, \tag{1.1}$$

where $p, q = 0, \dots, n-1$. The hyperquadric (1.1) intersects the hyperplane $y^{n+1} = 0$ in a hypersphere $\tilde{Q}$

$$a_{pq} y^p y^q + (y^n)^2 = 0,$$

which can be either real or imaginary.

Let us construct the stereographic projection with the pole $N(0 : \dots : 0 : 1 : 1)$ of the hyperplane $\mathbb{P}^n : y^{n+1} = 0$ into the hyperquadric Q. If $U(y^0 : \dots : y^n : 0) \in \mathbb{P}^n$ take the straight line

$$\lambda U + \mu N = (\lambda y^0 : \dots : \lambda y^{n-1} : \lambda y^n + \mu : \mu);$$

coordinates of its intersection point with Q satisfy

$$\lambda^2 a_{pq} y^p y^q + (\lambda y^n + \mu)^2 - \mu^2 = 0, \quad \lambda \neq 0.$$

Setting $k = \frac{\mu}{\lambda}$ we can write the previous equation as

$$a_{pq} y^p y^q + (y^n)^2 + 2k y^n = 0.$$

If $y^n \neq 0$, i. e. the point $U \notin \mathbb{P}^{n-1}$, then

$$k = -\frac{a_{pq} y^p y^q + (y^n)^2}{2y^n}.$$

Hence the intersection point of the straight line UN with the hyperquadric Q is uniquely determined. In the hyperplane $y^{n+1} = 0$, consider the $(n-1)$–plane $\mathbb{P}^{n-1}$: $y^n = 0$ as an ideal hyperplane; we obtain the structure of affine space $\mathbb{A}^n$ on the rest. In $\mathbb{A}^n$, we can introduce Cartesian coordinates $u^i = y^i / y^n$. Moreover, in $\mathbb{A}^n$ there exists the structure of Euclidean space $\mathbb{E}^n$ with the metric form

$$ds_0^2 = \pm a_{pq} du^p du^q. \tag{1.2}$$

The point $U(u^0 : u^1 : \dots : u^{n-1} : 1 : 0)$ is mapped into the point

$$X_1(2u^0 : \dots : 2u^{n-1} : 1 - a_{pq} u^p u^q : -1 - a_{pq} u^p u^q).$$

Let us normalize the hyperquadric (1.1) self-polar, taking the lines of the sheaf of lines with a fixed center $Z = E_{n+1}$ as normals of the first-order, and their polar

$(n-1)$–planes belonging to the hyperplane $y^{n+1} = 0$ as second-order normals. The straight line $E_{n+1}X_1$ intersects the hyperplane $y^{n+1} = 0$ in the point

$$X(2u^0 : \ldots : 2u^{n-1} : 1 - a_{pq}u^p u^q : 0).$$

Note that the polar of the point X related to the hyperquadric (1.1) intersects the hyperplane $y^{n+1} = 0$ exactly in the $(n-1)$–dimensional second-order normal which corresponds to the first-order normal X_1E_{n+1}. Hence in the hyperplane $y^{n+1} = 0$, a point X in general position is in correspondence with an $(n-1)$–plane, and the hyperplane $y^{n+1} = 0$ appears to be a polary normalized projective space $\mathbb{P}^n$ with the same geometry as the quadric itself.

Let us define a second-order normal by basic points $Y_i = \partial_i X - l_i X$. We find the scalar product $(X, X) = (1 + a_{pq}u^p u^q)^2$. The points X and Y_i are polar conjugate, i. e. the scalar product $(X, Y_i) = 0$. From these conditions we calculate the normalizator l_i:

$$l_i = \frac{2a_{is}u^s}{1 + a_{pq}u^p u^q}.$$

The decompositions

$$\partial_j Y_i = l_j Y_i + \Gamma^s_{ij} Y_s + p_{ij} X$$

determine components of the projective-Euclidean connection Γ^k_{ij} and the tensor p_{ij} [4]. Then the differential equations of the normalized space $\mathbb{P}^n : y^{n+1} = 0$ read

$$\partial_i X = Y_i + l_i X, \qquad \nabla_j Y_i = l_j Y_i + p_{ij} X. \tag{1.3}$$

Covariant differentiation of the equation $(X, Y_i) = 0$ gives

$$(\partial_j X, Y_i) + (X, \nabla_j Y_i) = 0.$$

By (1.3) we get

$$\begin{aligned}(X, \nabla_j Y_i) &= -(\partial_j X, Y_i) = -(Y_j, Y_i) - l_j(X, Y_i)\\ &= -(\partial_i X - l_i X, \partial_j X - l_j X) = -(\partial_i X, \partial_j X) - l_i l_j (X, X).\end{aligned}$$

Therefore

$$p_{ij} = \frac{(X, \nabla_j Y_i)}{(X, X)} = -\frac{(\partial_i X, \partial_j X)}{(X, X)} + l_i l_j = -\frac{4a_{ij}}{(1 + a_{pq}u^p u^q)^2}. \tag{1.4}$$

Hence considering in $\mathbb{A}^n$ the structure of the Euclidean space $\mathbb{E}^n$ with the Cartesian coordinates u^i we obtain a conformal model of a polar normalized projective space $\mathbb{P}^n$, i.e. a non-Euclidean space with the metric tensor

$$ds^2 = g_{ij}du^i du^j = \frac{\pm a_{ij}du^i du^j}{(1 + a_{pq}u^p u^q)^2}. \tag{1.5}$$

As we can see from (1.2) and (1.5), the obtained non-Euclidean space is conformally equivalent to the Euclidean space.

Quadrics of a special type in the projective spaces have been also studied in [1, 2].

2. On pseudoconformal models of fibrations determined by the algebra of antiquaternions and projectivization of them

Consider the associative unital 4-dimensional algebra $\mathbb{A}$ of antiquaternions [5, 6] with the basis $1, f, e, i$ and the multiplication table

	1	f	e	i
1	1	f	e	i
f	f	1	i	e
e	e	$-i$	1	$-f$
i	i	$-e$	f	-1

As well known, any antiquaternion can be uniquely expressed as $\mathbf{x} = x^0 + x^1 f + x^2 e + x^3 i$, conjugation is given by $\mathbf{x} \mapsto \bar{\mathbf{x}} = x^0 - x^1 f - x^2 e - x^3 i$, $\overline{\mathbf{xy}} = \bar{\mathbf{y}}\bar{\mathbf{x}}$ holds, the number $\mathbf{x}\bar{\mathbf{x}} = (x^0)^2 - (x^1)^2 - (x^2)^2 + (x^3)^2$ is real, and $\mathbf{x} \mapsto |\mathbf{x}| = \sqrt{\mathbf{x}\bar{\mathbf{x}}}$ defines a norm corresponding to the scalar product $\mathbf{xy} = \frac{1}{2}(\mathbf{x}\bar{\mathbf{y}} + \mathbf{y}\bar{\mathbf{x}})$ that turns $\mathbb{A}$ into the four-dimensional Pseudoeuclidean space $\mathbb{E}^4_2$. $|1| = |i| = 1$, $|e| = |f| = i$. For any $\mathbf{x}$ with $|\mathbf{x}| \neq 0$ there exists the inverse element $\mathbf{x}^{-1} = \frac{\bar{\mathbf{x}}}{|\mathbf{x}|^2}$. The set of all invertible elements from $\mathbb{A}$

$$\tilde{\mathbb{A}} = \{\mathbf{x} \mid |\mathbf{x}|^2 \neq 0\}$$

is a Lie group [7].

The group of antiquaternions of the unit norm $\mathbf{x}\bar{\mathbf{x}} = 1$ can be interpreted as the unit sphere $S^3_2(1)$

$$(x^0)^2 - (x^1)^2 - (x^2)^2 + (x^3)^2 = 1 \tag{2.1}$$

in the Pseudoeuclidean space $\mathbb{E}^4_2$.

We extend $\mathbb{E}^4_2$ into $\mathbb{P}^4$; taking

$$x^0 = \frac{y^0}{y^4},\ x^1 = \frac{y^1}{y^4},\ x^2 = \frac{y^2}{y^4},\ x^3 = \frac{y^3}{y^4},$$

we introduce homogeneous coordinates $(y^0 : y^1 : y^2 : y^3 : y^4)$. The quadric $S^3_2(1)$ has coordinate expression

$$\mathbf{y}^2 = (y^0)^2 - (y^1)^2 - (y^2)^2 + (y^3)^2 - (y^4)^2 = 0. \tag{2.2}$$

The quadric (2.2) intersects the hyperplane $y^0 = 0$ in the sphere S^2_1

$$(y^1)^2 + (y^2)^2 - (y^3)^2 + (y^4)^2 = 0.$$

The point E_0 of the projective frame $(E_0, \dots, E_4)$ is the pole of the hyperplane $y^0 = 0$, the straight line E_0E_4 intersects the quadric in two real points $N(1 : 0 : 0 :$

$0:1)$ and $N'(-1:0:0:0:1)$, and the points E_1, E_2, E_3 belong to the polar $\mathbb{P}^2$ of the straight line E_0E_4.

The tangent plane at the point N has the equation $y^0 - y^4 = 0$ and intersects the sphere in the real cone $-(y^1)^2 - (y^2)^2 + (y^3)^2 = 0$. Also it intersects $\mathbb{P}^3$ in the 2-plane $\mathbb{P}^2$: $y^0 = 0$, $y^4 = 0$. Hence in the hyperplane $\mathbb{P}^3$ there is a structure of affine space $\mathbb{A}^3$ for which $\mathbb{P}^2$ is the improper plane. Consequently, under the assumption $y^4 \neq 0$ we can introduce Cartesian coordinates

$$u^i = \frac{y^i}{y^4}, \ i = 1, 2, 3.$$

The sphere S_1^2 determines in $\mathbb{A}^3$ the structure of Pseudoeuclidean space $\mathbb{E}_1^3$ with the metric form

$$ds_0^2 = -(du^1)^2 - (du^2)^2 + (du^3)^2. \tag{2.3}$$

Consider the stereographic projection of the hyperplane $y^0 = 0$ from the pole $N(1:0:0:0:1)$ onto the quadric (2.2). The point $U(0:u^1:u^2:u^3:1)$ is mapped into the point

$$X_1(-1 + r^2 : 2u^1 : 2u^2 : 2u^3 : 1 + r^2),$$

where $r^2 = -(u^1)^2 - (u^2)^2 + (u^3)^2$ is the square of distance of the point U from the origin of the Pseudoeuclidean metric of the space $\mathbb{E}_1^3$.

Let us normalize the quadric (2.2) self-polar, taking as the first-order normals straight lines passing through E_0, and as second-order normals their polar two-planes belonging to the hyperplane $y^0 = 0$. The straight line E_0X_1 intersects the hyperplane $y^0 = 0$ in the point

$$X(0 : 2u^1 : 2u^2 : 2u^3 : 1 + r^2).$$

The polar of the point X related to the quadric (2.2) intersects the hyperplane $y^0 = 0$ in the normal of the second order. Hence in the hyperplane $y^0 = 0$, a point X in general position corresponds to a two-plane, and the hyperplane $y^0 = 0$ is the normalized projective space $\mathbb{P}^3$.

Let us define the second-order normal by basic points $Y_i = \partial_i X - l_i X$. Points X and Y_i are polar conjugate, i. e. $(X, Y_i) = 0$. From this condition and since $(X, X) = -(r^2 - 1)^2$ we find coordinates of the normalizer:

$$l_1 = -\frac{2u^1}{r^2 - 1}, \quad l_2 = -\frac{2u^2}{r^2 - 1}, \quad l_3 = \frac{2u^3}{r^2 - 1}.$$

Then by (1.4) we obtain finally

$$p_{11} = p_{22} = -\frac{4}{(r^2 - 1)^2}, \quad p_{33} = \frac{4}{(r^2 - 1)^2}.$$

Now introducing in $\mathbb{A}^3$ the structure of Pseudoeuclidean space $\mathbb{E}_1^3$ with u^i as Cartesian coordinates we find the pseudoconformal model of the sphere $S_2^3(1)$ with the metric form

$$ds^2 = g_{ij} du^i du^j = \frac{-(du^1)^2 - (du^2)^2 + (du^3)^2}{(r^2 - 1)^2}. \tag{2.4}$$

The corresponding Riemannian (Levi-Civita) connection of this Pseudoriemannian metric form appears. Non-vanishing components (Christoffel symbols) of it are

$$\Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^1_{33} = \Gamma^2_{21} = \Gamma^3_{13} = \frac{2u^1}{r^2-1},$$
$$\Gamma^2_{22} = \Gamma^1_{12} = -\Gamma^2_{11} = \Gamma^2_{33} = \Gamma^3_{32} = \frac{2u^2}{r^2-1},$$
$$-\Gamma^3_{33} = -\Gamma^3_{22} = -\Gamma^1_{13} = -\Gamma^3_{11} = -\Gamma^2_{23} = \frac{2u^3}{r^2-1}.$$

The connection is of constant curvature $K = -1$.

As an example, we obtain equations of fibres in model of the fibration defined by the subalgebra of complex numbers.

3. Example

Let us write an antiquaternion in the form

$$\mathbf{x} = x^0 + x^3 i + f(x^1 + x^2 i) = z_1 + f z_2, \quad z_1, z_2 \in \mathbb{R}(i),$$

where $\mathbb{R}(i)$ is a 2–dimensional subalgebra of complex numbers with basis $\{1, i\}$. The set of its invertible elements

$$\tilde{\mathbb{R}}(i) = \{\lambda = a + bi \,|\, \lambda \neq 0\}, \qquad a,\, b \in \mathbb{R}$$

turns out to be a Lie subgroup of the group $\tilde{\mathbb{A}}$, a 2–plane with exception of one point.

The canonical projection $\pi : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}/\tilde{\mathbb{R}}(i)$ takes the form

$$\pi(\mathbf{x}) = (\bar{z}_1 : z_2).$$

The factorspace $\tilde{\mathbb{A}}/\tilde{\mathbb{R}}(i)$ is a subset M of a complex projective line $P(i)$ and

$$M = \{[z_1 : z_2] \in P(i) \,|\, z_1\bar{z}_1 - z_2\bar{z}_2 \neq 0\}.$$

It is covered by two charts

$$U_1 = \{[z_1 : z_2] \,|\, z_2 \neq 0\} \quad \text{with the coordinate} \quad z = \frac{\bar{z}_1}{z_2}\,,$$

where $|z|^2 \neq 1$, since $z_1\bar{z}_1 - z_2\bar{z}_2 \neq 0$;

$$U_2 = \{[z_1 : z_2] \,|\, z_1 \neq 0\} \quad \text{with the coordinate} \quad \tilde{z} = \frac{z_2}{\bar{z}_1}\,,$$

where $|\tilde{z}|^2 \neq 1$ by the same reason.

Let the point $z = u + iv \in M \subset P(i)$ is in U_1. Then the coordinate expression of the projection π in real coordinates is

$$\pi(z_1, z_2) = z = \left(\frac{x^0x^1 - x^2x^3}{(x^1)^2 + (x^2)^2}, \frac{-(x^0x^2 + x^1x^3)}{(x^1)^2 + (x^2)^2} \right). \tag{3.1}$$

Then $z = \frac{\bar{z}_1}{z_2}$, where in homogeneous coordinates

$$z_1 = \frac{y^0 + y^3 i}{y^4}, \quad z_2 = \frac{y^1 + y^2 i}{y^4}.$$

The projection $\pi(\mathbf{y}) = z$ can be written as

$$\pi(\mathbf{y}) = \left(\frac{y^0y^1 - y^2y^3}{(y^1)^2 + (y^2)^2}, \frac{-(y^0y^2 + y^1y^3)}{(y^1)^2 + (y^2)^2} \right),$$

which is equivalent to (3.1), and 2–planes $L_2 : \bar{z}_1 - zz_2 = 0$ are given by a system of equations

$$\begin{cases} y^0 - uy^1 + vy^2 = 0, \\ y^3 + vy^1 + uy^2 = 0. \end{cases} \tag{3.2}$$

These 2–planes are the fibres of this fibration. By intersection with the sphere (2.2), we obtain a 2–parameter family of second order curves

$$\begin{cases} (y^0)^2 - (y^1)^2 - (y^2)^2 + (y^3)^2 - (y^4)^2 = 0, \\ y^0 - uy^1 + vy^2 = 0, \\ y^3 + vy^1 + uy^2 = 0, \end{cases}$$

which define the fibration. Excluding y^0 we find the projection of the family of fibres into the space $\mathbb{E}^3_1$. Passing to the Cartesian coordinates we obtain

$$\begin{cases} -(x^1)^2 - (x^2)^2 + (x^3)^2 + (ux^1 - vx^2)^2 = 1, \\ x^3 + vx^1 + ux^2 = 0. \end{cases} \tag{3.3}$$

There is a correspondence of these equations with the equations (21) ([3, p. 89]). If $\mathbf{y}$ is a point on the quadric distinct from N (i.e. $y^0 - y^4 \neq 0$ holds), the corresponding point ξ in $\mathbb{E}^3$: $y^0 = 0$ is uniquely determined by the homogeneous coordinates $(0 : y^1 : y^2 : y^3 : y^4 - y^0)$, that is

$$\xi(0 : \frac{y^1}{y^4 - y^0} : \frac{y^2}{y^4 - y^0} : \frac{y^3}{y^4 - y^0} : 1),$$

and in the space $\mathbb{A}^3$: $y^4 \neq 0$ the point ξ has the Cartesian coordinates

$$u^1 = \frac{x^1}{1 - x^0}, \quad u^2 = \frac{x^2}{1 - x^0}, \quad u^3 = \frac{x^3}{1 - x^0}.$$

The inverse mapping is characterized by the formulas

$$x^0 = \frac{\xi^2 - 1}{\xi^2 + 1},\ x^1 = \frac{2u^1}{\xi^2 + 1},\ x^2 = \frac{2u^2}{\xi^2 + 1},\ x^3 = \frac{2u^3}{\xi^2 + 1},$$

$$\xi^2 = -(u^1)^2 - (u^2)^2 + (u^3)^2,\ \xi^2 + 1 \neq 0,$$

similar to the formulas (18) (cf. [3, p. 88]). Hence the coordinates of the points $\mathbf{y}$ and ξ are related by the conformal mapping. Substituting these expressions into (3.3) we obtain the equations of the family of fibres in the form

$$\begin{cases} (u^1)^2 + (u^2)^2 - (u^3)^2 + 2(uu^1 - vu^2)^2 + 1 = 0, \\ vu^1 + uu^2 + u^3 = 0. \end{cases} \tag{3.4}$$

These equations coincide with the system (21) (cf. [3, p. 89]). So, we have the following result.

Theorem 3.1. *In the projective model the equations of fibres of the fibration defined by the subalgebra of complex numbers are* (3.4)*.*

References

[1] Jukl, M., On homologies of Klingenberg projective spaces over special commutative local rings, *Publ. Math. Univ. Debrec.*, 55 (1999) 113–121.

[2] Jukl, M., Snášel, V., Projective equivalence of quadrics in Klingenberg projective spaces over a special local ring, *Int. Electr. J. Geom.*, 2 (2009) 34–38.

[3] Kuzmina, I. A., Shapukov, B. N., Conformal and elliptic models of the Hopf fibration, *Tr. Geom. Sem. Kazan. Univ.*, 24 (2003) 81–98.

[4] Norden, A. P., Spaces of Affine Connection, *Nauka, Moscow*, (1976).

[5] Rozenfeld, B. A., Higher-dimensional Spaces, *Nauka, Moscow*, (1966).

[6] Rozenfeld, B. A., Geometry of Lie Groups, *Kluwer, Dordrecht–Boston–London*, (1997).

[7] Shapukov, B. N., Connections on a differential fibred bundle, *Tr. Geom. Sem. Kazan. Univ.*, 12 (1980) 97–109.

[8] Shirokov, A. P., Non-Euclidean Spaces, *Kazan University, Kazan*, (1997).