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On (log-) convexity of power mean

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Abstract

The power mean $M_p(a,b)$ of order p of two positive real values a and b is defined by $M_p(a,b) = ((a^p + b^p)/2)^{1/p}$, for $p \neq 0$ and $M_p(a,b) = \sqrt{ab}$, for $p = 0$. In this short note we prove that the power mean $M_p(a,b)$ is convex in p for $p \leq 0$, log-convex for $p \leq 0$ and log-concave for $p \geq 0$.

Keywords: power mean, logarithmic mean

MSC: 26E60, 26D20

1. Introduction

For $p \in \mathbb{R}$, the power mean $M_p(a,b)$ of order p of two positive real numbers, a and b, is defined by

$$M_p(a,b) = \begin{cases} \left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}, & p \neq 0 \\ \sqrt{ab}, & p = 0. \end{cases}$$

Within the past years, the power mean has been the subject of intensive research. Many remarkable inequalities for $M_p(a,b)$ and other types of means can be found in the literature.

It is well known that $M_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$.

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Note that $M_p(a,b) = aM_p(1, \frac{b}{a})$. Mildorf [3] studied the function

$$f(p,a) = M_p(1,a) = \left(\frac{1+a^p}{2}\right)^{\frac{1}{p}}$$

and proved that for any given real number $a > 0$ the following assertions hold:

(A) for $p \geq 1$ the function $f(p,a)$ is concave in p,

(B) for $p \leq -1$ the function $f(p,a)$ is convex in p.

The aim of this note is to study the log-convexity of the power mean $M_p(a,b)$ in variable p. As a consequence we get several known inequalities and their generalization.

2. Main results

Theorem 2.1. *Let $f(p,a) = M_p(1,a)$. We have*

(i) *for $p \leq 0$ the function $f(p,a)$ is log-convex in p,*

(ii) *for $p \geq 0$ the function $f(p,a)$ is log-concave in p,*

(iii) *for $p \leq 0$ the function $f(p,a)$ is convex in p.*

Proof. Observe that for any real number t there holds

$$f(pt,a)^t = f(p,a^t). \tag{2.1}$$

Let

$$g(p,a) = \ln f(p,a).$$

Taking the logarithm in (2.1) we have

$$tg(pt,a) = g(p,a^t).$$

Calculating partial derivatives of both sides of the above equation we get

$$t^2 g_1'(pt,a) = g_1'(p,a^t)$$

and

$$t^3 g_{11}''(pt,a) = g_{11}''(p,a^t). \tag{2.2}$$

Specially, taking $p = 1$ in (2.2), we have

$$t^3 g_{11}''(t,a) = g_{11}''(1,a^t). \tag{2.3}$$

Taking into account that the function $f(p,a)$ is increasing and concave in p for $p \geq 1$ (see (A)), the function $g(p,a)$ is also increasing and concave in p for $p \geq 1$. For this reason

$$g_{11}''(1,a^t) \leq 0$$

for an arbitrary $a > 0$ and real t. Let us consider the left hand side of (2.3). We have

$$t^3 g_{11}''(t,a) \le 0$$

which yields to the facts that the function $g(p,a)$ is concave for $p > 0$, therefore the function $f(p,a)$ is log-concave in this case and the function $g(p,a)$ is convex for $p < 0$. Hence the assertions (i), (ii) follow. Clearly, the assertion (iii) follows immediately from (i). □

The following result is a consequence of the assertion (iii) Theorem 2.1.

Corollary 2.2. *Inequality*

$$\alpha M_p(a,b) + (1-\alpha) M_q(a,b) \ge M_{\alpha p + (1-\alpha) q}(a,b) \tag{2.4}$$

holds for all $a, b > 0$, $\alpha \in [0,1]$ and $p, q \le 0$.

Let us denote by $G(a,b) = \sqrt{ab}$ and $H(a,b) = \frac{2ab}{a+b}$ the arithmetic mean and harmonic mean of a and b, respectively. For $\alpha = \frac{2}{3}$, $p = 0$, $q = 1$ in (2.4) we get the inequality

$$\frac{2}{3} G(a,b) + \frac{1}{3} H(a,b) \ge M_{-\frac{1}{3}}(a,b)$$

which was proved in [6].

The next result is a consequence of (ii) in Theorem 2.1.

Corollary 2.3. *For $\alpha \in [0,1]$, $p, q \ge 0$ the inequality*

$$M_p^{\alpha}(a,b) M_q^{(1-\alpha)}(a,b) \le M_{\alpha p + (1-\alpha) q}(a,b) \tag{2.5}$$

holds for all $a, b > 0$.

Let us denote by $A(a,b) = \frac{a+b}{2}$, $G(a,b) = \sqrt{ab}$,

$$L(a,b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \ne b \\ a, & a = b. \end{cases}$$

the arithmetic mean, geometric mean and logarithmic mean of two positive numbers a and b, respectively. Taking into account the result of Tung-Po Lin [2]

$$L(a,b) \le M_{\frac{1}{3}}(a,b) \tag{2.6}$$

together with (2.5) we have

$$M_p^{\alpha}(a,b) L^{(1-\alpha)}(a,b) \le M_{\alpha p + (1-\alpha)\frac{1}{3}}(a,b). \tag{2.7}$$

Specially, for $p = 1$ and $p = 0$ in (2.7) we get the inequalities

$$A^{\alpha}(a,b) L^{(1-\alpha)}(a,b) \le M_{\frac{1+2\alpha}{3}}(a,b)$$

and

$$G^{\alpha}(a,b)L^{(1-\alpha)}(a,b) \le M_{\frac{1-\alpha}{3}}(a,b)$$

respectively, which results were published in [5].

Denote by

$$I(a,b) = \begin{cases} \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}}, & a \neq b \\ a, & a = b. \end{cases}$$

the identric mean of two positive integers. It was proved by Pittenger [4] that

$$M_{\frac{2}{3}}(a,b) \le I(a,b) \le M_{\ln 2}(a,b). \tag{2.8}$$

Using (2.5) together with (2.6) and (2.8) we immeditely have

$$I^{\alpha}(a,b)L^{(1-\alpha)}(a,b) \le M_{\alpha \ln 2 + (1-\alpha)\frac{1}{3}}.$$

Note, in the case of $\alpha = \frac{1}{2}$ our result does not improve the inequality

$$\sqrt{I(a,b)L(a,b)} \le M_{\frac{1}{2}}(a,b)$$

which is due to Alzer [1], but our result is a more general one.

With the help of using Theorem 2.1 more similar inequalities can be proved.

3. Open problems

Finally, we propose the following open problem on the convexity of power mean. The problem is to prove our conjecture, namely

$$\inf_{a,b>0}\{p : M_p(a,b) \text{ is concave for variable } p\} = \frac{\ln 2}{2},$$
$$\sup_{a,b>0}\{p : M_p(a,b) \text{ is convex for variable } p\} = \frac{1}{2}.$$

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