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Linear recurrence relations with the coefficients in progression

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Abstract

The aim of this paper is to solve the linear recurrence relation

$$x_{n+1} = a_0 x_n + a_1 x_{n-1} + \cdots + a_{n-1} x_1 + a_n x_0,\ n = 0, 1, 2, \ldots,$$

when its constant coefficients are in arithmetic, respective geometric progression. Rather surprising, when the coefficients are in arithmetic progression, the solution is a sequence of certain generalized Fibonacci numbers, but not of usual Fibonacci numbers, while if they are in geometric progression the solution is again a geometric progression, with different ratio. In both cases the solution will be found by generating function method. Alternatively, in the first case it will be obtained by reduction to a generalized Fibonacci equation and in the second case by mathematical induction. Finally, the case is considered when both the coefficients and solutions form geometric progressions with generalized Fibonacci numbers as terms. The paper has a didactical purpose, being intended to familiarize the students with the usual procedures for solving linear recurrence relations. Another algebraic, differential and integral recurrence relations were considered by the author in the papers cited in the references.

Keywords: linear recurrence relations, arithmetic and geometric progressions, generalized Fibonacci numbers.

MSC: 11C08, 11B39.

1. Introduction

In this paper we apply the usual methods for solving linear recurrence relations with constant coefficients of special form - progressions. The method of characteristic equation, of generating function and of mathematical induction are used. The relationship between the considered relations and the generalized Fibonacci numbers is also specified. The numbers considered in this paper are complex. We remember that one calls *generalized Fibonacci numbers* or *Horadam numbers* (see [5, 6, 7]) of orders α and β, the numbers $x_n,\ n = 0, 1, 2, \ldots$, satisfying the *generalized Fibonacci recurrence relation* $x_{n+1} = \alpha x_n + \beta x_{n-1},\ n = 1, 2, \ldots$, with arbitrary initial data x_0 and x_1. If $\alpha = \beta = 1$, hence when the numbers x_n satisfy the *usual Fibonacci recurrence relation* $x_{n+1} = x_n + x_{n-1}$, these numbers are called *Fibonacci type numbers*. Particularly, when the initial data are $x_0 = 0$ and $x_1 = 1$, the *usual Fibonacci numbers* are obtained. When the coefficients of the linear recurrence relation of order n are in arithmetic progression, then its solutions are generalized Fibonacci numbers of certain orders. When the coefficients are in geometric progression, then the solutions are also in such a progression. In the final Section, this last situation is particularly considered when both the coefficients and solutions are generalized Fibonacci numbers. Aspects of the theory of recurrence relations and Fibonacci numbers can be found in the works listed in References.

2. Linear recurrence relations with coefficients in arithmetic progression

Theorem 2.1. *The numbers x_n are solutions of the linear recurrence relation with the coefficients in arithmetic progression*

$$x_{n+1} = ax_n + (a+r)x_{n-1} + \cdots + (a+(n-1)r)x_1 + (a+nr)x_0,\ n = 0, 1, 2, \ldots, \tag{2.1}$$

with initial data x_0, if and only if they are the generalized Fibonacci numbers given by the Binet type formula

$$x_n = \frac{x_0}{\lambda_1 - \lambda_2}\left[(b - a\lambda_2)\lambda_1^{n-1} - (b - a\lambda_1)\lambda_2^{n-1}\right],\ n = 1, 2, \ldots, \tag{2.2}$$

where

$$b = a^2 + a + r,\ \lambda_{1,2} = \frac{a + 2 \pm \sqrt{a^2 + 4r}}{2}. \tag{2.3}$$

Proof. (*By reduction to a generalized Fibonacci recurrence relation*) We suppose that the numbers x_n satisfy the recurrence relation (2.1). Then we have $x_1 = ax_0$, $x_2 = bx_0$ and

$$x_{n+1} - x_n = ax_n + rx_{n-1} + rx_{n-2} + \cdots + rx_1 + rx_0,$$
$$x_n - x_{n-1} = ax_{n-1} + rx_{n-2} + \cdots + rx_1 + rx_0,$$

$$(x_{n+1} - x_n) - (x_n - x_{n-1}) = ax_n + (r - a)x_{n-1}.$$

Denoting $c = a - r$, one obtains the generalized Fibonacci recurrence relation

$$x_{n+1} = (a + 2)x_n - (c + 1)x_{n-1},\ n = 2, 3, \ldots. \tag{2.4}$$

This equation has the solution $x_n = C_1\lambda_1^n + C_1\lambda_2^n$, where $\lambda_{1,2}$ are the roots, given by (2.3), of the characteristic equation $\lambda^2 - (a + 2)\lambda + c + 1 = 0$. The initial conditions $x_1 = C_1\lambda_1 + C_2\lambda_2 = ax_0$ and $x_2 = C_1\lambda_1^2 + C_2\lambda_2^2 = bx_0$ give $C_{1,2} = \pm\dfrac{x_0(b - a\lambda_{2,1})}{\lambda_{1,2}(\lambda_1 - \lambda_2)}$, hence the solutions of the recurrence relation (2.1) are given by the formula (2.2). □

Remark. The linear recurrence relation (2.4) fails for $n = 1$ and therefore its initial conditions are x_1 and x_2 instead of x_0 and x_1.

Proof. (*By generating function method*) Working with formal series, we denote by $X(t) = \sum_{n=0}^{\infty} x_n t^n$, the generating function of the sequence x_n. Then the recurrence relation (2.1) takes the form $\sum_{n=0}^{\infty} x_{n+1}t^{n+1} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}(a+kr)x_{n-k}t^{n+1}$. Using the formula for the product of two power series, one obtains

$$X(t) - x_0 = t\sum_{n=0}^{\infty}(a + nr)t^n \sum_{n=0}^{\infty} x_n t^n = t\sum_{n=0}^{\infty}(a + nr)t^n X(t).$$

Because

$$\sum_{n=0}^{\infty}(a + nr)t^n = a\sum_{n=0}^{\infty} t^n + rt\sum_{n=0}^{\infty}\frac{d}{dt}(t^n) = a\frac{1}{1-t} + rt\frac{d}{dt}\left(\frac{1}{1-t}\right) = \frac{a - ct}{(1-t)^2},$$

we have $X(t) - x_0 = \dfrac{t(a - ct)}{(1-t)^2}X(t)$. One obtains

$$X(t) = \frac{x_0(t-1)^2}{(c+1)t^2 - (a+2)t + 1} = \frac{x_0}{c+1} + \frac{x_0((r-c)t + c)}{\sqrt{c+1}(t - t_1)(t - t_2)},$$

where $t_1 = \dfrac{1}{\lambda_1}$ and $t_2 = \dfrac{1}{\lambda_2}$ are the roots of the equation $(c+1)t^2 - (a+2)t + 1 = 0$, the numbers $\lambda_{1,2}$ been given by the relation (2.3). We have

$$\begin{aligned}
X(t) &= \frac{x_0}{c+1} + \frac{x_0}{(c+1)^2(t_1 - t_2)}\left[\frac{(r-c)t_1 + c}{t - t_1} - \frac{(r-c)t_2 + c}{t - t_2}\right] \\
&= \frac{x_0}{\lambda_1\lambda_2} + \frac{x_0}{\lambda_1\lambda_2(\lambda_1 - \lambda_2)}\left[\frac{c\lambda_1 + r - c}{1 - \lambda_1 t} - \frac{c\lambda_2 + r - c}{1 - \lambda_2 t}\right] \\
&= \frac{x_0}{\lambda_1\lambda_2} + \frac{x_0}{\lambda_1\lambda_2(\lambda_1 - \lambda_2)}\left[(c\lambda_1 + r - c)\sum_{n=0}^{\infty}\lambda_1^n t^n - (c\lambda_2 + r - c)\sum_{n=0}^{\infty}\lambda_2^n t^n\right]
\end{aligned}$$

$$= x_0 + \frac{x_0}{\lambda_1\lambda_2(\lambda_1-\lambda_2)}\left[(c\lambda_1+r-c)\sum_{n=1}^{\infty}\lambda_1^n t^n - (c\lambda_2+r-c)\sum_{n=1}^{\infty}\lambda_2^n t^n\right].$$

Therefore the coefficients of the generating function $X(t)$ are given by the relation

$$x_n = \frac{x_0(c\lambda_1+r-c)}{\lambda_2(\lambda_1-\lambda_2)}\lambda_1^{n-1} - \frac{x_0(c\lambda_2+r-c)}{\lambda_1(\lambda_1-\lambda_2)}\lambda_2^{n-1},\ n=1,2,\dots.$$

Taking into account the identities $\dfrac{c\lambda_1+r-c}{\lambda_2} = b-a\lambda_2$ and $\dfrac{c\lambda_2+r-c}{\lambda_1} = b-a\lambda_1$, from the above expression of x_n one obtains formula (2.2). □

Proof. (*Reciprocal*) If the sequence x_n is given by formula (2.2), it satisfies the recurrence relation (2.4). Indeed, using (2.2) and the relation $\lambda_j^2 = (a+2)\lambda_j - (c+1)$, $j=1,2$, which results by the definition of the numbers $\lambda_{1,2}$, one obtain

$$\begin{aligned}
(a+2)x_n - (c+1)x_{n-1} &= (a+2)\frac{x_0}{\lambda_1-\lambda_2}\left[(b-a\lambda_2)\lambda_1^{n-1} - (b-a\lambda_1)\lambda_2^{n-1}\right] - \\
&- (c+1)\frac{x_0}{\lambda_1-\lambda_2}\left[(b-a\lambda_2)\lambda_1^{n-2} - (b-a\lambda_1)\lambda_2^{n-2}\right] \\
&= \frac{x_0}{\lambda_1-\lambda_2}(b-a\lambda_2)\left[(a+2)\lambda_1-(c+1)\right]\lambda_1^{n-2} - \\
&- \frac{x_0}{\lambda_1-\lambda_2}(b-a\lambda_1)\left[(a+2)\lambda_2-(c+1)\right]\lambda_2^{n-2} \\
&= \frac{x_0}{\lambda_1-\lambda_2}\left[(b-a\lambda_2)\lambda_1^{n} - (b-a\lambda_1)\lambda_2^{n}\right] = x_{n+1},\ n=1,2,\dots.
\end{aligned}$$

Now we prove by induction that the sequence x_n given by (2.2) satisfies the recurrence relation (2.1). We first show that (2.1) is satisfied for $n=0,1,2$. Indeed, from (2.2) it follows

$$\begin{aligned}
x_1 &= \frac{x_0}{\lambda_1-\lambda_2}(b-a\lambda_2-b+a\lambda_1) = ax_0, \\
x_2 &= \frac{x_0}{\lambda_1-\lambda_2}\left[(b-a\lambda_2)\lambda_1 - (b-a\lambda_1)\lambda_2\right] = bx_0 \\
&= (a^2+a+r)x_0 = ax_1 + (a+r)x_0, \\
x_3 &= \frac{x_0}{\lambda_1-\lambda_2}\left[(b-a\lambda_2)\lambda_1^2 - (b-a\lambda_1)\lambda_2^2\right] \\
&= \frac{x_0}{\lambda_1-\lambda_2}\left[b(\lambda_1^2-\lambda_2^2) - a\lambda_1\lambda_2(\lambda_1-\lambda_2)\right] \\
&= x_0\left[b(\lambda_1+\lambda_2) - a\lambda_1\lambda_2\right] = x_0\left[b(a+2) - a(c+1)\right] \\
&= abx_0 + x_0\left[2(a^2+a+r) - a(1+a-r)\right] \\
&= ax_2 + x_0(a^2+ar+a+2r) = ax_2 + (a+r)x_1 + (a+2r)x_0.
\end{aligned}$$

For a fixed index $n \geq 2$, we suppose that the formula (2.1) is true when $k \leq n$, hence we have

$$x_{k+1} = \sum_{j=0}^{k}(a+(k-j)r)x_j,\ k \leq n. \tag{2.5}$$

Using (2.4) and (2.5), one obtains

$$\begin{aligned}
x_{n+2} &= (a+2)x_{n+1} - (c+1)x_n \\
&= (a+2)\sum_{k=0}^{n}(a+(n-k)r)x_k - (c+1)\sum_{k=1}^{n}(a+(n-k)r)x_{k-1} \\
&= \sum_{k=2}^{n}(a+(n-k)r)\big[(a+2)x_k - (c+1)x_{k-1}\big] + (a+2)(a+(n-1)r)x_1 + \\
&\quad + (a+2)(a+nr)x_0 - (c+1)(a+(n-1)r)x_0 = \sum_{k=2}^{n}(a+(n-k)r)x_{k+1} + \\
&\quad + a(a+2)(a+(n-1)r)x_0 + a(a+nr)x_0 + \\
&\quad + 2(a+nr)x_0 + (r-a-1)(a+(n-1)r)x_0 \\
&= \sum_{k=2}^{n}(a+(n-k)r)x_{k+1} + (a+(n-1)r)x_2 + (a+nr)x_1 + (a+(n+1)r)x_0 \\
&= \sum_{k=0}^{n+1}(a+(n+1-k)r)x_k,
\end{aligned}$$

hence formula (2.1) is true for the index $n+1$. According to the induction axiom, (2.1) is true for any natural number n. □

Remarks. 1) The sequence of usual Fibonacci numbers can not be solution of the equation (2.1). Indeed, for this would be that $a+2 = -c-1 = r-a-1 = 1$, for the equation (2.4) to reduce to well-known Fibonacci recurrence relation $x_{n+1} = x_n + x_{n-1}$ and to have the initial conditions $x_1 = ax_0 = 1$ and $x_2 = bx_0 = (a^2+a+r)x_0 = 1$. But these conditions are contradictory, leading to the false equality $x_0 = 1 = -1$.
2) An arithmetic progression x_n cannot be solution of the equation (2.1). Indeed, this requires that $x_{n+1} = 2x_n - x_{n-1}$, therefore $a+2 = 2$ and $-c-1 = r-a-1 = -1$, which leads to the trivial case $a = r = x_n = 0$, for $n = 1, 2, \dots$.

Corollary 2.2. *The linear recurrence relation*

$$x_{n+1} = x_n + 2x_{n-1} + \cdots + nx_1 + (n+1)x_0,\ n = 0, 1, 2, \dots, \tag{2.6}$$

with the initial data $x_0 = 1$, has the solution

$$x_n = \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right],\ n = 1, 2, \dots. \tag{2.7}$$

Proof. For $a = r = x_0 = 1$, from Theorem 2.1 and its proof it results that the recurrence relation (2.6) reduces to the generalized Fibonacci relation

$$x_{n+1} = 3x_n - x_{n-1},\ n = 2, 3, \dots, \tag{2.8}$$

with the initial data $x_1 = 1$ and $x_2 = 3$, hence it has the solution (2.7). Particularly, both (2.6) and (2.7) give $x_3 = 8$, $x_4 = 21$ and so on. □

Remark. The recurrence relation (2.6) from above corollary was considered as problem 9 (ii) in F. Lazebnik, Combinatorics and Graphs Theory, I, (Math 688). Problems and Solutions, 2006, a work appearing on the Internet at the address `www.math.udel.edu/~lazebnik/papers/688hwsols.pdf`. Unfortunately, in the cited work one obtains the wrong solution

$$x_n = \frac{5-\sqrt{5}}{10}\left(\frac{3+\sqrt{5}}{2}\right)^n + \frac{5+\sqrt{5}}{10}\left(\frac{3-\sqrt{5}}{2}\right)^n,\ n = 0,1,2,\ldots,$$

with particular solutions $x_0 = x_1 = 1$, $x_2 = 2$, $x_3 = 5$ and so on, the last two being false. The explanation of this mistake is that the recurrence relation (2.8) was wrongly considered for $n = 1,2,\ldots,$, with the initial data $x_0 = x_1 = 1$, leading to the wrong solution mentioned above. Indeed, for $n = 1$, the obtined recurrence relation $x_2 = 3x_1 - x_0$ is false. This mistake shows the importance of the correct initialization of the recurrence relations.

3. Linear recurrence relations with coefficients in geometric progression

Theorem 3.1. *The numbers x_n are solutions of the linear recurrence relation with constant coefficients in geometric progression*

$$x_{n+1} = ax_n + aqx_{n-1} + \cdots + aq^{n-1}x_1 + aq^n x_0,\ n = 0,1,2,\ldots, \tag{3.1}$$

with initial data x_0, if and only if they form the geometric progression given by the formula

$$x_n = ax_0(a+q)^{n-1},\ n = 1,2,\ldots \tag{3.2}$$

Proof. (*By induction*). From (3.1) we obtain $x_1 = ax_0$ and $x_2 = ax_0(a+q)$. For a fixed natural number n we suppose formula (3.2) true for every $k \le n$. Therefore we have $x_k = ax_0(a+q)^{k-1}$, for $k \le n$. Then, from the recurrence relation (3.1) one obtains

$$\begin{aligned}
x_{n+1} &= a^2x_0(a+q)^{n-1} + a^2x_0q(a+q)^{n-2} + \cdots + \\
&\quad + a^2x_0q^{n-2}(a+q) + a^2x_0q^{n-1} + ax_0q^n \\
&= a^2x_0(a+q)\big[(a+q)^{n-2} + q(a+q)^{n-3} + \cdots + q^{n-3}(a+q) + q^{n-2}\big] + \\
&\quad + ax_0q^{n-1}(a+q) \\
&= a^2x_0(a+q)\frac{(a+q)^{n-1} - q^{n-1}}{a} + ax_0q^{n-1}(a+q) = ax_0(a+q)^n,
\end{aligned}$$

hence the formula (3.2) is true for $n+1$. According to the induction axiom it results that formula (3.2) is true for every natural number n. □

Proof. (*By generating function method*) Denoting $X(t) = \sum_{n=0}^{\infty} x_n t^n$, from the recurrence relation (3.1) one obtains $\sum_{n=0}^{\infty} x_{n+1} t^{n+1} = at \sum_{n=0}^{\infty} \sum_{k=0}^{n} q^k x_{n-k} t^n$. Using the formula for the product of two power series, one obtains

$$X(t) - x_0 = at \sum_{n=0}^{\infty} q^n t^n \sum_{n=0}^{\infty} x_n t^n = \frac{at}{1-qt} X(t).$$

Therefore

$$\begin{aligned} X(t) &= x_0 \frac{qt-1}{(a+q)t-1} = \frac{x_0 q}{a+q} + \frac{a x_0}{(a+q)(1-(a+q)t)} \\ &= \frac{x_0 q}{a+q} + \frac{a x_0}{a+q} \sum_{n=0}^{\infty} (a+q)^n t^n = x_0 + a x_0 \sum_{n=1}^{\infty} (a+q)^{n-1} t^n, \end{aligned}$$

from which it results the formula (3.2). □

Proof. (*Reciprocal*) If x_n is given by the formula (3.2), then we have

$$\begin{aligned} & a \sum_{k=0}^{n} q^{n-k} x_k = a^2 x_0 \sum_{k=1}^{n} q^{n-k} (a+q)^{k-1} + a x_0 q^n \\ & = a^2 x_0 q^{n-1} \sum_{k=1}^{n} \left(\frac{a+q}{q}\right)^{k-1} + a x_0 q^n \\ & = a^2 x_0 q^{n-1} \frac{\left(\frac{a+q}{q}\right)^n - 1}{\frac{a+q}{q} - 1} + a x_0 q^n = a x_0 (a+q)^n = x_{n+1},\ n = 1, 2, \ldots, \end{aligned}$$

hence the sequence x_n satisfies the recurrence equation (3.1). □

4. Linear recurrence relations having as coefficients generalized Fibonacci numbers in geometric progression

Lemma 4.1. *The terms $a_n = aq^n$, $n = 0, 1, 2, \ldots$, of a geometric progression are generalized Fibonacci numbers of orders α and β if and only if the progression ratio is given by the formula*

$$q = \frac{\alpha \pm \sqrt{\alpha^2 + 4\beta}}{2}. \tag{4.1}$$

Proof. If the terms a_n of the geometric progression are generalized Fibonacci numbers of orders α and β, then $a_{n+1} = \alpha a_n + \beta a_{n-1}$, relation which becomes $aq^{n+1} = \alpha a q^n + \beta a q^{n-1}$. One obtains the quadratic equation $q^2 - \alpha q - \beta = 0$, with the roots given by formula (4.1). Reciprocally, if the number q is given by formula (4.1), it satisfies the above quadratic equation. Multiplying this equation by aq^{n-1}, one obtains the relation $a_{n+1} = \alpha a_n + \beta a_{n-1}$, hence a_n are generalized Fibonacci numbers of orders α and β. □

Example. If $\alpha = 2i$, with $i = \sqrt{-1}$ and $\beta = 1$, then (4.1) gives $q = i$, therefore the terms of the geometric progression $a_n = ai^n$ are generalized Fibonacci numbers of orders $2i$ and 1. Indeed, we have $2ia_n + a_{n-1} = 2ai^{n+1} + ai^{n-1} = ai^{n+1} = a_{n+1}$.

Theorem 4.2. *The coefficients $a_n = aq^n$, $n = 0.1.2.\ldots$, and the solutions x_n, $n = 1, 2, \ldots$ of the linear recurrence relation* (3.1) *are both generalized Fibonacci numbers of orders α and β if and only if*

$$\alpha = a + 2q,\ \beta = -q(a + q). \tag{4.2}$$

Proof. According to Theorem 3.1 and the above Lemma, the coefficients a_n and the solutions x_n of (3.1) are generalized Fibonacci numbers of orders α and β, if and only if

$$q^2 - \alpha q - \beta = 0,\ (a + q)^2 - \alpha(a + q) - \beta = 0, \tag{4.3}$$

hence the formula (4.2) holds. □

Example. If $a = q = i$, then $a_n = i^{n+1}$ and, according to Theorem 3.1, $x_n = \frac{x_0}{2}(2i)^n$. From Theorem 4.3 it results that both a_n and x_n are generalized Fibonacci numbers of orders $\alpha = a + 2q = 3i$ and $\beta = -q(a + q) = 2$. Indeed, we have

$$3ia_n + 2a_{n-1} = 3i^{n+2} + 2i^n = i^{n+2} = a_{n+1}$$

and

$$3ix_n + 2x_{n-1} = \frac{x_0}{2}[3i(2i)^n + 2(2i)^{n-1}] = \frac{x_0}{2}(2i)^{n+1} = x_{n+1}.$$

Corollary 4.3. *The coefficients a_n and the solutions x_n of the linear recurrence relation* (3.1) *are both Fibonacci type numbers if and only if*

$$a = \mp\sqrt{5},\ q = \frac{1 \pm \sqrt{5}}{2}. \tag{4.4}$$

Proof. For $\alpha = \beta = 1$ it follows from Theorem 4.3 that $a+2q = 1$ and $-q(a+q) = 1$, from which we obtain (4.4). □

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