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Averaging sums of powers of integers and Faulhaber polynomials

José Luis Cereceda[a]

[a]Distrito Telefónica, Madrid, Spain
jl.cereceda@movistar.es

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Abstract

As an application of Faulhaber's theorem on sums of powers of integers and the associated Faulhaber polynomials, in this article we provide the solution to the following two questions: (1) when is the average of sums of powers of integers itself a sum of the first n integers raised to a power? and (2), when is the average of sums of powers of integers itself a sum of the first n integers raised to a power, times the sum of the first n squares? In addition to this, we derive a family of recursion formulae for the Bernoulli numbers.

Keywords: sums of powers of integers, Faulhaber polynomials, matrix inversion, Bernoulli numbers

MSC: 11C08, 11B68

1. Introduction

Recently Pfaff [1] investigated the solutions of the equation

$$\frac{\sum_{i=1}^{n} i^a + \sum_{i=1}^{n} i^b}{2} = \left(\sum_{i=1}^{n} i\right)^c, \tag{1.1}$$

for positive integers a, b, and c, and found that the only solution (a, b, c) to (1.1) with $a \neq b$ is $(5, 7, 4)$ (the remaining solutions being the trivial one $(1, 1, 1)$ and the well-known solution $(3, 3, 2)$). Furthermore, Pfaff provided some necessary conditions for

$$\frac{\sum_{i=1}^{n} i^{a_1} + \sum_{i=1}^{n} i^{a_2} + \cdots + \sum_{i=1}^{n} i^{a_{m-1}}}{m-1} = \left(\sum_{i=1}^{n} i\right)^{a_m}, \tag{1.2}$$

to hold. Specifically, by assuming that $a_1 \leq a_2 \leq \cdots \leq a_{m-p-1} < a_{m-p} = \cdots = a_{m-1}$ (with $a_1, a_2, \ldots, a_m$ positive integers), Pfaff showed that any solution to (1.2) must fulfil the condition

$$\frac{p}{m-1} = \frac{a_m}{2^{a_m-1}}. \tag{1.3}$$

There are infinitely many solutions to (1.3). For example, for $a_m = 8$, the set of solutions to (1.3) is given by $(a_m, p, m-1) = (8, p, 16p)$, with $p \geq 1$. However, as Pfaff himself pointed out [1], it is not known if any given solution to (1.3) also yields a solution to (1.2), so that solving this problem for $m > 3$ will require some other approach. In this article we show that, for any given value of $a_m \geq 1$, there is indeed a unique solution to equation (1.2), on the understanding that the fraction $\frac{p}{m-1}$ is given in its lowest terms. Interestingly, this is done by exploiting the properties of the coefficients of the so-called Faulhaber polynomials [2, 3, 4, 5, 6]. Although there exist more direct ways to arrive at the solution of equation (1.2) (for example, by means of the binomial theorem or by mathematical induction), our pedagogical approach here will serve to introduce the (relatively lesser known) topic of the Faulhaber polynomials to a broad audience.

In addition to the equation (1.2) considered by Pfaff, we also give the solution to the closely related equation

$$\frac{\sum_{i=1}^{n} i^{a_1} + \sum_{i=1}^{n} i^{a_2} + \cdots + \sum_{i=1}^{n} i^{a_{m-1}}}{m-1} = \left(\sum_{i=1}^{n} i^2\right)\left(\sum_{i=1}^{n} i\right)^{a_m}, \tag{1.4}$$

where now $a_m \geq 0$. Obviously, for $a_m = 0$, we have the trivial solution $a_1 = a_2 = \cdots = a_{m-1} = 2$. In general, it turns out that all the powers $a_1, a_2, \ldots, a_{m-1}$ on the left-hand side of (1.4) must be even integers, whereas those appearing in the left-hand side of (1.2) must be odd integers. This is a straightforward consequence of the following theorem.

2. Faulhaber's theorem on sums of powers of integers

Let us denote by S_r the sum of the first n positive integers each raised to the integer power $r \geq 0$, $S_r = \sum_{i=1}^{n} i^r$. The key ingredient in our discussion is an old result concerning the S_r's which can be traced back to Johann Faulhaber (1580–1635), an early German algebraist who was a close friend of both Johannes Kepler and René Descartes. Faulhaber discovered that, for even powers $r = 2k$ ($k \geq 1$), S_{2k} can be put in the form

$$S_{2k} = S_2\left[F_0^{(2k)} + F_1^{(2k)} S_1 + F_2^{(2k)} S_1^2 + \cdots + F_{k-1}^{(2k)} S_1^{k-1}\right], \tag{2.1}$$

whereas, for odd powers $r = 2k+1$ ($k \geq 1$), S_{2k+1} can be expressed as

$$S_{2k+1} = S_1^2 \big[F_0^{(2k+1)} + F_1^{(2k+1)} S_1 + F_2^{(2k+1)} S_1^2 + \cdots + F_{k-1}^{(2k+1)} S_1^{k-1} \big], \qquad (2.2)$$

where $\{F_j^{(2k)}\}$ and $\{F_j^{(2k+1)}\}$, $j = 0, 1, \ldots, k-1$, are sets of numerical coefficients. Equations (2.1) and (2.2) can be rewritten in compact form as

$$S_{2k} = S_2 F^{(2k)}(S_1), \qquad (2.3)$$

$$S_{2k+1} = S_1^2 F^{(2k+1)}(S_1), \qquad (2.4)$$

where both $F^{(2k)}(S_1)$ and $F^{(2k+1)}(S_1)$ are polynomials in S_1 of degree $k-1$. Following Edwards [2] we refer to them as Faulhaber polynomials and, by extension, we call $F_j^{(2k)}$ and $F_j^{(2k+1)}$ the Faulhaber coefficients. Next we quote the first instances of S_{2k} and S_{2k+1} in Faulhaber form as

$$\begin{aligned}
S_2 &= S_2, \\
S_3 &= S_1^2, \\
S_4 &= S_2 \big[-\tfrac{1}{5} + \tfrac{6}{5} S_1 \big], \\
S_5 &= S_1^2 \big[-\tfrac{1}{3} + \tfrac{4}{3} S_1 \big], \\
S_6 &= S_2 \big[\tfrac{1}{7} - \tfrac{6}{7} S_1 + \tfrac{12}{7} S_1^2 \big], \\
S_7 &= S_1^2 \big[\tfrac{1}{3} - \tfrac{4}{3} S_1 + 2 S_1^2 \big], \\
S_8 &= S_2 \big[-\tfrac{1}{5} + \tfrac{6}{5} S_1 - \tfrac{8}{3} S_1^2 + \tfrac{8}{3} S_1^3 \big], \\
S_9 &= S_1^2 \big[-\tfrac{3}{5} + \tfrac{12}{5} S_1 - 4 S_1^2 + \tfrac{16}{5} S_1^3 \big], \\
S_{10} &= S_2 \big[\tfrac{5}{11} - \tfrac{30}{11} S_1 + \tfrac{68}{11} S_1^2 - \tfrac{80}{11} S_1^3 + \tfrac{48}{11} S_1^4 \big], \\
S_{11} &= S_1^2 \big[\tfrac{5}{3} - \tfrac{20}{3} S_1 + \tfrac{34}{3} S_1^2 - \tfrac{32}{3} S_1^3 + \tfrac{16}{3} S_1^4 \big].
\end{aligned}$$

Note that, from the expressions for S_5 and S_7, we quickly get that $\frac{S_5 + S_7}{2} = S_1^4$.

Let us now write the equations (1.2) and (1.4) using the notation S_r for the sums of powers of integers,

$$\frac{S_{a_1} + S_{a_2} + \cdots + S_{a_{m-1}}}{m-1} = S_1^{a_m}, \qquad (2.5)$$

and

$$\frac{S_{a_1} + S_{a_2} + \cdots + S_{a_{m-1}}}{m-1} = S_2 S_1^{a_m}. \qquad (2.6)$$

Since $S_1 = n(n+1)/2$ and $S_2 = (2n+1)S_1/3$, from (2.1) and (2.2) we retrieve the well-known result that S_r is a polynomial in n of degree $r+1$. From this result, it in turn follows that the maximum index a_{m-1} on the left-hand side of (2.5) is given by $a_{m-1} = 2a_m - 1$, a condition already established in [1]. In fact, in order for equation (2.5) to hold, it is necessary that all the indices $a_1, a_2, \ldots, a_{m-1}$ appearing in the left-hand side of (2.5) be odd integers. To see this, suppose on the contrary that one of the indices is even, say a_j. Then, from (2.3) and (2.4), the left-hand side of (2.5) can be expressed as follows:

$$L(S_1, S_2) = \frac{S_2 F^{(a_j)}(S_1) + S_1^2 P(S_1)}{m-1},$$

where $F^{(a_j)}(S_1)$ and $P(S_1)$ are polynomials in S_1. On the other hand, for nonnegative integers u and v, it is clear that $S_2 S_1^u \neq S_1^v$ irrespective of the values of u and v, as $S_2 S_1^u$ (S_1^v) is a polynomial in n of odd (even) degree. This means that $S_2 F^{(a_j)}(S_1)$ cannot be reduced to a polynomial in S_1 from which we conclude, in particular, that $L(S_1, S_2) \neq S_1^{a_m}$.

Similarly, using (2.3) and (2.4) it can be seen that, in order for equation (2.6) to hold, all the indices $a_1, a_2, \ldots, a_{m-1}$ in the left-hand side of (2.6) have to be even integers, the maximum index a_{m-1} being given by $a_{m-1} = 2a_m + 2$.

3. Faulhaber's coefficients

The sets of coefficients $\{F_j^{(2k)}\}$ and $\{F_j^{(2k+1)}\}$ satisfy several remarkable properties, a number of which will be described below. As it happens with the binomial coefficients and the Pascal triangle, the properties of the Faulhaber coefficients are better appreciated and explored when they are arranged in a triangular array. In Table 1 we have displayed the set $\{F_j^{(2k+1)}\}$ for $k = 1, 2, \ldots, 10$, while the corresponding coefficients $\{F_j^{(2k)}\}$ (also for $k = 1, 2, \ldots, 10$) are given in Table 2. The numeric arrays in Tables 1 and 2 also can be viewed as lower triangular matrices, with the rows being labelled by k and the columns by j. The following list of properties of the Faulhaber coefficients are readily verified for the coefficients shown in Tables 1 and 2. They are, however, completely general.

1. The Faulhaber coefficients are nonzero rational numbers.

2. The entries in a row have alternating signs, the sign of the leading coefficient (which is situated on the main diagonal) being positive.

3. The sum of the entries in a row is equal to unity, $\sum_{j=0}^{k-1} F_j^{(2k+1)} = \sum_{j=0}^{k-1} F_j^{(2k)} = 1$.

4. The entries on the main diagonal are given by $F_{k-1}^{(2k+1)} = \frac{2^k}{k+1}$ and $F_{k-1}^{(2k)} = \frac{3 \cdot 2^{k-1}}{2k+1}$, and the entries in the $j = 0$ column are $F_0^{(2k+1)} = 2(2k+1)B_{2k}$ and $F_0^{(2k)} = 6B_{2k}$, where B_{2k} denotes the $2k$-th Bernoulli number. Furthermore, the entries in the $j = 1$ column are connected to those in the $j = 0$ column by the simple relations $F_1^{(2k+1)} = -4F_0^{(2k+1)} = -8(2k+1)B_{2k}$ and $F_1^{(2k)} = -6F_0^{(2k)} = -36B_{2k}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8	9
1	1									
2	$-\frac{1}{3}$	$\frac{4}{3}$								
3	$\frac{1}{3}$	$-\frac{4}{3}$	2							
4	$-\frac{3}{5}$	$\frac{12}{5}$	-4	$\frac{16}{5}$						
5	$\frac{5}{3}$	$-\frac{20}{3}$	$\frac{34}{3}$	$-\frac{32}{3}$	$\frac{16}{3}$					
6	$-\frac{691}{105}$	$\frac{2764}{105}$	$-\frac{944}{21}$	$\frac{4592}{105}$	$-\frac{80}{3}$	$\frac{64}{7}$				
7	35	-140	$\frac{718}{3}$	$-\frac{704}{3}$	$\frac{448}{3}$	-64	16			
8	$-\frac{3617}{15}$	$\frac{14468}{15}$	$-\frac{4948}{3}$	$\frac{24304}{15}$	$-\frac{9376}{9}$	$\frac{1408}{3}$	$-\frac{448}{3}$	$\frac{256}{9}$		
9	$\frac{43867}{21}$	$-\frac{175468}{21}$	$\frac{1500334}{105}$	$-\frac{210656}{15}$	$\frac{45264}{5}$	$-\frac{144512}{35}$	$\frac{6944}{5}$	$-\frac{1024}{3}$	$\frac{256}{5}$	
10	$-\frac{1222277}{55}$	$\frac{4889108}{55}$	$-\frac{5016584}{33}$	$\frac{24655472}{165}$	$-\frac{3180688}{33}$	44096	-15040	$\frac{11776}{3}$	-768	$\frac{1024}{11}$

Table 1: The set of coefficients $\{F_j^{(2k+1)}\}$ for $1 \leq k \leq 10$.

5. There exists a relation between $F_j^{(2k+1)}$ and $F_j^{(2k)}$, namely,

$$F_j^{(2k+1)} = \frac{2(2k+1)}{3(j+2)} F_j^{(2k)}, \quad j = 0, 1, \ldots, k-1. \tag{3.1}$$

   This formula allows us to obtain the k-th row in Table 1 from the k-th row in Table 2, and vice versa.

6. The entries $F_{k-2}^{(2k+1)}, F_{k-3}^{(2k+1)}, \ldots, F_0^{(2k+1)}$ within the k-th row in Table 1 can be successively obtained by the rule

$$\sum_{j=0}^{q} 2^j \binom{k+1-j}{2q+1-2j} F_{k-j-1}^{(2k+1)} = 0, \quad 1 \leq q \leq k-1, \tag{3.2}$$

   given the initial condition $F_{k-1}^{(2k+1)} = \frac{2^k}{k+1}$. By applying the bijection (3.1) to the coefficients $F_{k-j-1}^{(2k+1)}$, one gets the corresponding rule for the entries in the k-th row in Table 2.

7. For any given $k \geq 3$, and for each $j = 0, 1, \ldots, k-3$, we have

$$\sum_{r=1}^{k} \mathrm{odd}(r) \binom{k}{r} F_j^{(2k-r)} = 0, \tag{3.3}$$

   where $\mathrm{odd}(r)$ restricts the summation to odd values of r, i.e., $\mathrm{odd}(r) = 1\,(0)$ for odd (even) r. Similarly, by applying the bijection (3.1) to the coefficients $F_j^{(2k-r)}$, it can be seen that, for any given $k \geq 1$ and for each $j = 0, 1, \ldots, k-1$,

$$\sum_{r=0}^{k+1} \mathrm{even}(r) \binom{k+2}{r+1} (2k+3-r) F_j^{(2k+2-r)} = 0, \tag{3.4}$$

$k\backslash j$	0	1	2	3	4	5	6	7	8	9
1	1									
2	$-\frac{1}{5}$	$\frac{6}{5}$								
3	$\frac{1}{7}$	$-\frac{6}{7}$	$\frac{12}{7}$							
4	$-\frac{1}{5}$	$\frac{6}{5}$	$-\frac{8}{3}$	$\frac{8}{3}$						
5	$\frac{5}{11}$	$-\frac{30}{11}$	$\frac{68}{11}$	$-\frac{80}{11}$	$\frac{48}{11}$					
6	$-\frac{691}{455}$	$\frac{4146}{455}$	$-\frac{1888}{91}$	$\frac{328}{13}$	$-\frac{240}{13}$	$\frac{96}{13}$				
7	7	-42	$\frac{1436}{15}$	$-\frac{352}{3}$	$\frac{448}{5}$	$-\frac{224}{5}$	$\frac{64}{5}$			
8	$-\frac{3617}{85}$	$\frac{21702}{85}$	$-\frac{9896}{17}$	$\frac{12152}{17}$	$-\frac{9376}{17}$	$\frac{4928}{17}$	$-\frac{1792}{17}$	$\frac{384}{17}$		
9	$\frac{43867}{133}$	$-\frac{263202}{133}$	$\frac{3000668}{665}$	$-\frac{105328}{19}$	$\frac{407376}{665}$	$-\frac{216768}{95}$	$\frac{83328}{95}$	$-\frac{4608}{19}$	$\frac{768}{19}$	
10	$-\frac{174611}{55}$	$\frac{1047666}{55}$	$-\frac{10033168}{231}$	$\frac{12327736}{231}$	$-\frac{454384}{11}$	22048	$-\frac{60160}{7}$	$\frac{17664}{7}$	$-\frac{3840}{7}$	$\frac{512}{7}$

Table 2: The set of coefficients $\{F_j^{(2k)}\}$ for $1 \leq k \leq 10$.

where $\mathrm{even}(r) = 1\,(0)$ for even (odd) r picks out the even power terms. Informally, we may call the property embodied in equations (3.3) and (3.4) the *sum-to-zero column property*, as the coefficients $F_j^{(2k-r)}$ $[F_j^{(2k+2-r)}]$ entering the summation in (3.3) [(3.4)] pertain to a given column j. This is to be distinguished from the *sum-to-zero row property* in equation (3.2), where the coefficients $F_{k-j-1}^{(2k+1)}$ belong to a given row k.

8. For completeness, next we write down the explicit formula for $F_j^{(2k+1)}$ which was originally obtained in [7, Section 12]. Adapting the notation in [7] to ours, we have that

$$F_j^{(2k+1)} = (-1)^j \frac{2^{j+2}}{j+2} \sum_{r=0}^{\lfloor j/2 \rfloor} \binom{2j+1-2r}{j+1} \binom{2k+1}{2r+1} B_{2k-2r}, \tag{3.5}$$

for $j = 0, 1, \ldots, k-1$, and where $\lfloor j/2 \rfloor$ denotes the floor function of $j/2$, namely the largest integer not greater than $j/2$. The set of coefficients $\{F_j^{(2k)}\}$ can then be found through relation (3.1). We shall use relation (3.5) in Section 6 to derive a family of recursion formulae for the Bernoulli numbers.

4. Averaging sums of powers of integers

Interestingly enough, the sum-to-zero column property in equations (3.3) and (3.4) provides the solution to the problem of averaging sums of powers of integers in equations (2.5) and (2.6). For the sake of brevity, next we focus on the connection between (3.3) and (2.5). An analogous reasoning can be made to establish the link between (3.4) and (2.6). To grasp the meaning of equation (3.3), consider a concrete example where $k = 7$. Then the column index j takes the values $j = 0, 1, 2, 3, 4$, and (3.3) gives rise to the following five equalities:

$$\begin{aligned}
&\tbinom{7}{1}F_0^{(13)} + \tbinom{7}{3}F_0^{(11)} + \tbinom{7}{5}F_0^{(9)} + \tbinom{7}{7}F_0^{(7)} = 0\\
&\tbinom{7}{1}F_1^{(13)} + \tbinom{7}{3}F_1^{(11)} + \tbinom{7}{5}F_1^{(9)} + \tbinom{7}{7}F_1^{(7)} = 0\\
&\tbinom{7}{1}F_2^{(13)} + \tbinom{7}{3}F_2^{(11)} + \tbinom{7}{5}F_2^{(9)} + \tbinom{7}{7}F_2^{(7)} = 0\\
&\tbinom{7}{1}F_3^{(13)} + \tbinom{7}{3}F_3^{(11)} + \tbinom{7}{5}F_3^{(9)} \qquad\quad\;\;\, = 0\\
&\tbinom{7}{1}F_4^{(13)} + \tbinom{7}{3}F_4^{(11)} \qquad\qquad\qquad\quad\;\;\, = 0.
\end{aligned}$$

For a reason that will become clear in just a moment, we add to this list of equalities a last one to include the value of $\binom{7}{1}F_5^{(13)}$, namely, $\binom{7}{1}F_5^{(13)} = 2^6$. Furthermore, we multiply the first equality by S_1^0, the second equality by S_1^1, the third equality by S_1^2, and so on, that is,

$$\begin{aligned}
&\tbinom{7}{1}F_0^{(13)}S_1^0 + \tbinom{7}{3}F_0^{(11)}S_1^0 + \tbinom{7}{5}F_0^{(9)}S_1^0 + \tbinom{7}{7}F_0^{(7)}S_1^0 = 0\\
&\tbinom{7}{1}F_1^{(13)}S_1^1 + \tbinom{7}{3}F_1^{(11)}S_1^1 + \tbinom{7}{5}F_1^{(9)}S_1^1 + \tbinom{7}{7}F_1^{(7)}S_1^1 = 0\\
&\tbinom{7}{1}F_2^{(13)}S_1^2 + \tbinom{7}{3}F_2^{(11)}S_1^2 + \tbinom{7}{5}F_2^{(9)}S_1^2 + \tbinom{7}{7}F_2^{(7)}S_1^2 = 0\\
&\tbinom{7}{1}F_3^{(13)}S_1^3 + \tbinom{7}{3}F_3^{(11)}S_1^3 + \tbinom{7}{5}F_3^{(9)}S_1^3 \qquad\qquad\;\; = 0\\
&\tbinom{7}{1}F_4^{(13)}S_1^4 + \tbinom{7}{3}F_4^{(11)}S_1^4 \qquad\qquad\qquad\qquad\;\; = 0\\
&\tbinom{7}{1}F_5^{(13)}S_1^5 \qquad\qquad\qquad\qquad\qquad\qquad\quad\;\; = 2^6 S_1^5.
\end{aligned}$$

Now we can see that the sum of the entries in the first column is just $\binom{7}{1}$ times the Faulhaber polynomial $F^{(13)}(S_1)$, the sum of the entries in the second column is $\binom{7}{3}$ times $F^{(11)}(S_1)$, the sum of the third column is $\binom{7}{5}$ times $F^{(9)}(S_1)$, and the sum of the fourth column is $\binom{7}{7}$ times $F^{(7)}(S_1)$. Then we have

$$\tbinom{7}{1}F^{(13)}(S_1) + \tbinom{7}{3}F^{(11)}(S_1) + \tbinom{7}{5}F^{(9)}(S_1) + \tbinom{7}{7}F^{(7)}(S_1) = 2^6 S_1^5.$$

Next we multiply both sides of this equation by S_1^2 and divide them by 2^6. Thus, taking into account (2.4), we finally obtain

$$\frac{\binom{7}{7}S_7 + \binom{7}{5}S_9 + \binom{7}{3}S_{11} + \binom{7}{1}S_{13}}{2^6} = S_1^7. \tag{4.1}$$

Since $\binom{7}{1} + \binom{7}{3} + \binom{7}{5} + \binom{7}{7} = 2^6$, the identity (4.1) constitutes the solution to (2.5) for the particular case $a_m = 7$. In this case we have that $\frac{p}{m-1} = \frac{7}{2^6}$, in accordance with condition (1.3).

In general, for an arbitrary exponent $a_m \geq 1$, the solution to equation (2.5) is given by

$$\frac{\sum_{r=1}^{a_m} \operatorname{odd}(r) \binom{a_m}{r} S_{2a_m - r}}{2^{a_m - 1}} = S_1^{a_m}. \tag{4.2}$$

A few comments are in order concerning the solution in (4.2). In the first place, by the constructive procedure we have used to obtain the solution (4.1) for the case $a_m = 7$, it should be clear that the solution (4.2) is unique for each $a_m \geq 1$, the quotient $\frac{p}{m-1}$ characterizing the solution being determined (when expressed in lowest terms) by the relation $\frac{p}{m-1} = \frac{a_m}{2^{a_m-1}}$. Secondly, for odd (even) a_m, the numerator of (4.2) involves $\frac{a_m+1}{2}$ $(\frac{a_m}{2})$ different sums S_j with j being an odd integer ranging in $a_m \leq j \leq 2a_m - 1$ $(a_m + 1 \leq j \leq 2a_m - 1)$. Furthermore, the binomial coefficients fulfil the identity $\sum_{r=1}^{a_m} \mathrm{odd}(r) \binom{a_m}{r} = 2^{a_m-1}$, thus ensuring that the overall number of terms appearing in the numerator of (4.2) equals 2^{a_m-1}.

For example, for $a_m = 3$, from (4.2) we get the solution $\frac{S_3+3S_5}{4} = S_1^3$, which was also found in [1]. For $a_m = 4$, noting that $\frac{p}{m-1} = \frac{4}{8} = \frac{1}{2}$, we get the solution $\frac{S_5+S_7}{2} = S_1^4$ which, as we saw, corresponds to the one denoted as $(a, b, c) = (5, 7, 4)$ in [1]. More sophisticated examples are, for instance,

$$\frac{S_9 + 36S_{11} + 126S_{13} + 84S_{15} + 9S_{17}}{256} = S_1^9,$$

and

$$\frac{1}{131072}\big(5S_{21} + 285S_{23} + 3876S_{25} + 19380S_{27} + 41990S_{29} + 41990S_{31} + 19380S_{33} + 3876S_{35} + 285S_{37} + 5S_{39}\big) = S_1^{20}.$$

On the other hand, starting with equation (3.4) and making an analysis similar to that leading to equation (4.2), one can deduce the following general solution to equation (2.6), namely,

$$\frac{\frac{1}{a_m+2}\sum_{r=0}^{a_m+1} \mathrm{even}(r)\binom{a_m+2}{r+1}(2a_m + 3 - r)S_{2a_m+2-r}}{3 \cdot 2^{a_m}} = S_2 S_1^{a_m}. \tag{4.3}$$

Now, for each $a_m \geq 0$, the quotient $\frac{p}{m-1}$ characterizing the solution (4.3) turns out to be $\frac{p}{m-1} = \frac{2a_m+3}{3\cdot 2^{a_m}}$. Further, for odd (even) a_m, the numerator of (4.3) involves $\frac{a_m+3}{2}$ $(\frac{a_m}{2} + 1)$ different sums S_j with j being an even integer ranging in $a_m + 1 \leq j \leq 2a_m + 2$ $(a_m + 2 \leq j \leq 2a_m + 2)$. Moreover, the following identity holds

$$\sum_{r=0}^{a_m+1} \mathrm{even}(r)\binom{a_m+2}{r+1}(2a_m + 3 - r) = 3 \cdot 2^{a_m}(a_m + 2),$$

and then the overall number of terms in the numerator of (4.3) is $3 \cdot 2^{a_m}$. For example, for $a_m = 17$, from equation (4.3) we find

$$\frac{1}{393216}\big(S_{18} + 189S_{20} + 4692S_{22} + 35700S_{24} + 107406S_{26} + 140998S_{28} + 82212S_{30} + 20196S_{32} + 1785S_{34} + 37S_{36}\big) = S_2 S_1^{17}.$$

Finally we note that, by combining (4.2) and (4.3), we obtain the double identity (with $a_m \geq 1$):

$$\begin{aligned}(a_m+2)\sum_{r=1}^{a_m}\operatorname{odd}(r)\binom{a_m}{r}S_2S_{2a_m-r}\\ &= \frac{1}{6}\sum_{r=0}^{a_m+1}\operatorname{even}(r)\binom{a_m+2}{r+1}(2a_m+3-r)S_{2a_m+2-r}\\ &= 2^{a_m-1}(a_m+2)S_2S_1^{a_m}.\end{aligned}$$

5. Matrix inversion

It is worth pointing out that, for any given a_m, we can equally obtain $S_1^{a_m}$ $(S_2S_1^{a_m})$ by inverting the corresponding triangular matrix formed by the Faulhaber coefficients in Table 1 (Table 2). This method was originally introduced by Edwards [3] (see also [2]) to obtain the Faulhaber coefficients themselves by inverting a matrix related to Pascal's triangle. As a concrete example illustrating this fact, consider the equation (2.2) written in matrix format up to $k=6$:

$$\begin{pmatrix} S_3\\ S_5\\ S_7\\ S_9\\ S_{11}\\ S_{13}\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ -\frac{1}{3} & \frac{4}{3} & 0 & 0 & 0 & 0\\ \frac{1}{3} & -\frac{4}{3} & 2 & 0 & 0 & 0\\ -\frac{3}{5} & \frac{12}{5} & -4 & \frac{16}{5} & 0 & 0\\ \frac{5}{3} & -\frac{20}{3} & \frac{34}{3} & -\frac{32}{3} & \frac{16}{3} & 0\\ -\frac{691}{105} & \frac{2764}{105} & -\frac{944}{21} & \frac{4592}{105} & -\frac{80}{3} & \frac{64}{7}\end{pmatrix}\begin{pmatrix} S_1^2\\ S_1^3\\ S_1^4\\ S_1^5\\ S_1^6\\ S_1^7\end{pmatrix}. \tag{5.1}$$

Let us call the square matrix of (5.1) $\mathbf{F}$. Clearly, $\mathbf{F}$ is invertible since all the elements in its main diagonal are nonzero. Then, to evaluate the column vector on the right of (5.1), we pre-multiply by the inverse matrix of $\mathbf{F}$ on both sides of (5.1) to get

$$\begin{pmatrix} S_1^2\\ S_1^3\\ S_1^4\\ S_1^5\\ S_1^6\\ S_1^7\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0\\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0\\ 0 & \frac{1}{16} & \frac{5}{8} & \frac{5}{16} & 0 & 0\\ 0 & 0 & \frac{3}{16} & \frac{5}{8} & \frac{3}{16} & 0\\ 0 & 0 & \frac{1}{64} & \frac{21}{64} & \frac{35}{64} & \frac{7}{64}\end{pmatrix}\begin{pmatrix} S_3\\ S_5\\ S_7\\ S_9\\ S_{11}\\ S_{13}\end{pmatrix},$$

from which we obtain the powers $S_1^2, S_1^3, S_1^4, \dots,$ expressed in terms of the odd power sums $S_3, S_5, S_7, \dots$. Of course the resulting formula for S_1^7 agrees with that in equation (4.1). Conversely, by inverting the matrix $\mathbf{F}^{-1}$ we get the corresponding Faulhaber coefficients. Note that the elements of $\mathbf{F}^{-1}$ are nonnegative, and that the sum of the elements in each of the rows is equal to one. In fact, the row elements of $\mathbf{F}^{-1}$ are given by the corresponding coefficients $\binom{a_m}{r}/2^{a_m-1}$ appearing in the left-hand side of (4.2).

Similarly, writting the equation (2.1) in matrix format up to $k = 6$, we have

$$\begin{pmatrix} S_2 \\ S_4 \\ S_6 \\ S_8 \\ S_{10} \\ S_{12} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{5} & \frac{6}{5} & 0 & 0 & 0 & 0 \\ \frac{1}{7} & -\frac{6}{7} & \frac{12}{7} & 0 & 0 & 0 \\ -\frac{1}{5} & \frac{6}{5} & -\frac{8}{3} & \frac{8}{3} & 0 & 0 \\ \frac{5}{11} & -\frac{30}{11} & \frac{68}{11} & -\frac{80}{11} & \frac{48}{11} & 0 \\ -\frac{691}{455} & \frac{4146}{455} & -\frac{1888}{91} & \frac{328}{13} & -\frac{240}{13} & \frac{96}{13} \end{pmatrix} \begin{pmatrix} S_2 \\ S_2 S_1 \\ S_2 S_1^2 \\ S_2 S_1^3 \\ S_2 S_1^4 \\ S_2 S_1^5 \end{pmatrix}. \tag{5.2}$$

Let us call the square matrix of (5.2) $\mathbf{G}$. Then, pre-multiplying both sides of (5.2) by the inverse matrix of $\mathbf{G}$, we obtain

$$\begin{pmatrix} S_2 \\ S_2 S_1 \\ S_2 S_1^2 \\ S_2 S_1^3 \\ S_2 S_1^4 \\ S_2 S_1^5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{5}{6} & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{12} & \frac{7}{12} & 0 & 0 & 0 \\ 0 & \frac{1}{24} & \frac{7}{12} & \frac{3}{8} & 0 & 0 \\ 0 & 0 & \frac{7}{48} & \frac{5}{8} & \frac{11}{48} & 0 \\ 0 & 0 & \frac{1}{96} & \frac{9}{32} & \frac{55}{96} & \frac{13}{96} \end{pmatrix} \begin{pmatrix} S_2 \\ S_4 \\ S_6 \\ S_8 \\ S_{10} \\ S_{12} \end{pmatrix},$$

from which we can determine S_2 times the powers $S_1, S_1^2, S_1^3, \dots$, in terms of the even power sums $S_2, S_4, S_6, \dots$. Likewise, we can see that the elements of $\mathbf{G^{-1}}$ are nonnegative and that the sum of the elements in each row is equal to one.

In view of this example, it is clear that the formulae (4.2) and (4.3) can be regarded as a rule for calculating the inverse of the triangular matrices in Tables 1 and 2, respectively. Moreover, the uniqueness of the solutions in (4.2) and (4.3) follows ultimately from the uniqueness of the inverse of such triangular matrices.

6. A family of recursion formulae for the Bernoulli numbers

As a last important remark we note that the sum-to-zero column property allows us to derive a family of recursive relationships for the Bernoulli numbers. Consider initially the equation (3.3) for the column index $j = 0$. So, recalling that $F_0^{(2k+1)} = 2(2k+1)B_{2k}$, we will have (for odd r) that $F_0^{(2k-r)} = 2(2k-r)B_{2k-r-1}$, and then equation (3.3) becomes (for $j = 0$)

$$\sum_{r=1}^{k} \operatorname{odd}(r) \binom{k}{r} (2k-r) B_{2k-r-1} = 0, \tag{6.1}$$

which holds for any given $k \geq 3$. For example, for $k = 13$, from (6.1) we obtain

$$B_{12} + 90B_{14} + 935B_{16} + 2508B_{18} + 2079B_{20} + 506B_{22} + 25B_{24} = 0,$$

and so, knowing $B_{12}, B_{14}, B_{16}, B_{18}, B_{20}$, and B_{22}, we can get B_{24}. On the other hand, from equation (3.5) we obtain

$$F_2^{(2k+1)} = \frac{4}{3}(2k+1)\big[30B_{2k} + k(2k-1)B_{2k-2}\big],$$

from which we in turn deduce that, for odd r,

$$F_2^{(2k-r)} = 40(2k-r)B_{2k-r-1} + 4\binom{2k-r}{3}B_{2k-r-3}.$$

Therefore, recalling (6.1), from equation (3.3) with $j=2$ we obtain the recurrence relation

$$\sum_{r=1}^{k} \mathrm{odd}(r)\binom{k}{r}\binom{2k-r}{3}B_{2k-r-3} = 0, \tag{6.2}$$

which holds for any given $k \geq 5$. On the other hand, from equation (3.5) we obtain

$$\begin{aligned} F_4^{(2k+1)} = \frac{16}{45}(2k+1)\big[&3780B_{2k} + 210k(2k-1)B_{2k-2} \\ &+ k(k-1)(2k-1)(2k-3)B_{2k-4}\big], \end{aligned}$$

from which we in turn deduce that, for odd r,

$$F_4^{(2k-r)} = 1344(2k-r)B_{2k-r-1} + 224\binom{2k-r}{3}B_{2k-r-3} + \frac{32}{3}\binom{2k-r}{5}B_{2k-r-5}.$$

Thus, taking into account (6.1) and (6.2), we see that, for $j=4$, equation (3.3) yields the recurrence relation

$$\sum_{r=1}^{k} \mathrm{odd}(r)\binom{k}{r}\binom{2k-r}{5}B_{2k-r-5} = 0, \tag{6.3}$$

which holds for any given $k \geq 7$. The pattern is now clear. Indeed, by assuming that $F_{2s}^{(2k-r)}$ (for odd r) is of the form

$$F_{2s}^{(2k-r)} = \sum_{q=0}^{s} f_q^{(2k-r)}\binom{2k-r}{2q+1}B_{2k-r-2q-1},$$

with the $f_q^{(2k-r)}$'s being nonzero rational coefficients, from equation (3.3) one readily gets the following general recurrence relation for the Bernoulli numbers:

$$\sum_{r=1}^{k} \mathrm{odd}(r)\binom{k}{r}\binom{2k-r}{2s+1}B_{2k-r-2s-1} = 0, \tag{6.4}$$

which holds for any given $k \geq 2s+3$, with $s = 0, 1, 2, \ldots$. Formulae (6.1), (6.2), and (6.3) are particular cases of the recurrence (6.4) for $s = 0, 1$, and 2, respectively.

Similarly, starting from equation (3.4), it can be shown that

$$\sum_{r=0}^{k+1} \text{even}(r) \binom{k+2}{r+1} \binom{2k+3-r}{2s+1} B_{2k+2-r-2s} = 0, \tag{6.5}$$

which holds for any given $k \geq 2s+1$, with $s = 0, 1, 2, \ldots$. It is easy to see that relations (6.4) and (6.5) are equivalent to each other. Moreover, we note that the recurrence (6.4) is essentially equivalent to the one given in [8, Theorem 1.1].

7. Conclusion

In this article we have tackled the problem of averaging sums of powers of integers as considered by Pfaff [1]. For this purpose, we have expressed the S_r's in the Faulhaber form and then we have used certain properties of the coefficients of the Faulhaber polynomials. Indeed, as we have seen, the sum-to-zero column property in equations (3.3) and (3.4) constitutes the skeleton of the solutions displayed in (4.2) and (4.3). It is to be noted, on the other hand, that the formulae (4.2) and (4.3) can be obtained in a more straightforward way by a proper application of the binomial theorem (for a derivation of the counterpart to the formulae (4.2) and (4.3) using this method, see [5, Subsections 3.2 and 3.3]). Furthermore, a demonstration by mathematical induction of the identities in (4.2) and (4.3) (although expressed in a somewhat different manner) already appeared in [9].

We believe, however, that our approach here is worthwhile since it introduces an important topic concerning the sums of powers of integers that may not be widely known, namely, the Faulhaber theorem and the associated Faulhaber polynomials. We invite the interested reader to prove some of the properties listed above, and to pursue the subject further [3, 4, 5, 6, 10, 11, 12].

References

[1] Pfaff, T. J., Averaging sums of powers of integers, *College Math. J.*, 42 (2011) 402–404.

[2] Edwards, A. W. F., Sums of powers of integers: a little of the history, *Math. Gaz.*, 66 (1982) 22–28.

[3] Edwards, A. W. F., A quick route to sums of powers, *Amer. Math. Monthly*, 93 (1986) 451–455.

[4] Knuth, D. E., Johann Faulhaber and sums of powers, *Math. Comp.*, 61 (1993) 277–294.

[5] Kotiah, T. C. T., Sums of powers of integers—A review, *Int. J. Math. Edu. Sci. Technol.*, 24 (1993) 863–874.

[6] Beardon, A. F., Sums of powers of integers, *Amer. Math. Monthly*, 103 (1996) 201–213.

[7] Gessel, I. M., Viennot, X. G., Determinants, paths, and plane partitions. Preprint, 1989. Available at `http://people.brandeis.edu/~gessel/homepage/papers/pp.pdf`.

[8] Zékiri, A., Bencherif, F., A new recursion relationship for Bernoulli numbers, *Annales Mathematicae et Informaticae*, 38 (2011) 123–126.

[9] Piza, P. A., Powers of sums and sums of powers, *Math. Mag.*, 25 (1952) 137–142.

[10] Krishnapriyan, H. K., Eulerian polynomials and Faulhaber's result on sums of powers of integers, *College Math. J.*, 26 (1995) 118–123.

[11] Shirali, S. A., On sums of powers of integers, *Resonance*, 12(7) (2007) 27–43.

[12] Chen, W. Y. C., Fu, A. M., Zhang, I. F., Faulhaber's theorem on power sums, *Discrete Math.*, 309 (2009) 2974–2981.