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Some relationships between poly-Cauchy numbers and poly-Bernoulli numbers*

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Abstract

In this paper, we show some relationships between poly-Cauchy numbers introduced by T. Komatsu and poly-Bernoulli numbers introduced by M. Kaneko.

Keywords: Bernoulli numbers; Cauchy numbers; poly-Bernoulli numbers; poly-Cauchy numbers

MSC: 05A15, 11B75

1. Introduction

Let n and k be positive integers. Poly-Cauchy numbers of the first kind $c_n^{(k)}$ are defined by

$$c_n^{(k)} = \underbrace{\int_0^1 \dots \int_0^1}_{k} (x_1 x_2 \dots x_k)(x_1 x_2 \dots x_k - 1) \dots (x_1 x_2 \dots x_k - n + 1) dx_1 dx_2 \dots dx_k$$

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(see in [7]). The concept of poly-Cauchy numbers is a generalization of that of the classical Cauchy numbers $c_n = c_n^{(1)}$ defined by

$$c_n = \int_0^1 x(x-1)\dots(x-n+1)dx$$

(see e.g. [2, 8]). The generating function of poly-Cauchy numbers ([7, Theorem 2]) is given by

$$\mathrm{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}\,,$$

where

$$\mathrm{Lif}_k(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+1)^k}$$

is the k-th polylogarithm factorial function. An explicit formula for $c_n^{(k)}$ ([7, Theorem 1]) is given by

$$c_n^{(k)} = (-1)^n \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-1)^m}{(m+1)^k} \quad (n \ge 0, k \ge 1)\,, \tag{1.1}$$

where $\begin{bmatrix} n \\ m \end{bmatrix}$ are the (unsigned) Stirling numbers of the first kind, arising as coefficients of the rising factorial

$$x(x+1)\dots(x+n-1) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} x^m$$

(see e.g. [4]).

On the other hand, M. Kaneko ([6]) introduced the poly-Bernoulli numbers $B_n^{(k)}$ by

$$\frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}\,,$$

where

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}$$

is the k-th polylogarithm function. When $k = 1$, $B_n = B_n^{(1)}$ is the classical Bernoulli number with $B_1^{(1)} = 1/2$, defined by the generating function

$$\frac{xe^x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}\,.$$

An explicit formula for $B_n^{(k)}$ ([6, Theorem 1]) is given by

$$B_n^{(k)} = (-1)^n \sum_{m=0}^{n} \left\{ n \atop m \right\} \frac{(-1)^m m!}{(m+1)^k} \quad (n \ge 0, k \ge 1)\,, \tag{1.2}$$

where $\left\{ n \atop m \right\}$ are the Stirling numbers of the second kind, determined by

$$\left\{ n \atop m \right\} = \frac{1}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (m-j)^n$$

(see e.g. [4]).

In this paper, we show some relationships between poly-Cauchy numbers and poly-Bernoulli numbers.

2. Main result

Poly-Bernoulli numbers can be expressed by poly-Cauchy numbers ([7, Theorem 8]).

Theorem 2.1. *For $n \ge 1$ we have*

$$B_n^{(k)} = \sum_{l=1}^{n} \sum_{m=1}^{n} m! \left\{ n \atop m \right\} \left\{ m-1 \atop l-1 \right\} c_l^{(k)}\,.$$

On the other hand,

$$\begin{aligned}
c_2^{(k)} &= \frac{1}{2!} B_2^{(k)} + \frac{3}{2} B_1^{(k)} \\
&= \frac{1}{2!} (B_2^{(k)} + 3B_1^{(k)})\,, \\
c_3^{(k)} &= \frac{1}{3!} B_3^{(k)} + 2B_2^{(k)} + \frac{23}{6} B_1^{(k)} \\
&= \frac{1}{3!} (B_3^{(k)} + 12B_2^{(k)} + 23B_1^{(k)})\,, \\
c_4^{(k)} &= \frac{1}{4!} B_4^{(k)} + \frac{5}{4} B_3^{(k)} + \frac{215}{24} B_2^{(k)} + \frac{55}{4} B_1^{(k)} \\
&= \frac{1}{4!} (B_4^{(k)} + 30B_3^{(k)} + 215B_2^{(k)} + 330B_1^{(k)})\,, \\
c_5^{(k)} &= \frac{1}{5!} B_5^{(k)} + \frac{1}{2} B_4^{(k)} + \frac{207}{24} B_3^{(k)} + \frac{95}{2} B_2^{(k)} + \frac{1901}{30} B_1^{(k)} \\
&= \frac{1}{5!} (B_5^{(k)} + 60B_4^{(k)} + 1035B_3^{(k)} + 5700B_2^{(k)} + 7604B_1^{(k)})\,, \\
c_6^{(k)} &= \frac{1}{6!} B_6^{(k)} + \frac{7}{48} B_5^{(k)} + \frac{707}{144} B_4^{(k)} + \frac{1015}{16} B_3^{(k)} + \frac{13279}{45} B_2^{(k)} + \frac{4277}{12} B_1^{(k)}
\end{aligned}$$

$$= \frac{1}{6!}\left(B_6^{(k)} + 105B_5^{(k)} + 3535B_4^{(k)} + 45675B_3^{(k)} + 212464B_2^{(k)} + 256620B_1^{(k)}\right).$$

In general, we have the following identity, expressing poly-Cauchy numbers $c_n^{(k)}$ by using poly-Bernoulli numbers $B_n^{(k)}$.

Theorem 2.2. *For $n \ge 1$ we have*

$$c_n^{(k)} = (-1)^n \sum_{l=1}^{n} \sum_{m=1}^{n} \frac{(-1)^m}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ l \end{bmatrix} B_l^{(k)}.$$

Proof. By (1.1) and (1.2), we have

$$\begin{aligned}
\text{RHS} &= (-1)^n \sum_{l=1}^{n} \sum_{m=1}^{n} \frac{(-1)^m}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ l \end{bmatrix} (-1)^l \sum_{i=0}^{l} \begin{Bmatrix} l \\ i \end{Bmatrix} \frac{(-1)^i i!}{(i+1)^k} \\
&= (-1)^n \sum_{m=1}^{n} \frac{(-1)^m}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \sum_{l=0}^{n} \begin{bmatrix} m \\ l \end{bmatrix} (-1)^l \sum_{i=0}^{l} \begin{Bmatrix} l \\ i \end{Bmatrix} \frac{(-1)^i i!}{(i+1)^k} \\
&= (-1)^n \sum_{m=1}^{n} \frac{(-1)^m}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \sum_{i=0}^{n} \frac{(-1)^i i!}{(i+1)^k} \sum_{l=i}^{n} (-1)^l \begin{bmatrix} m \\ l \end{bmatrix} \begin{Bmatrix} l \\ i \end{Bmatrix} \\
&= (-1)^n \sum_{m=0}^{n} \frac{(-1)^m}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-1)^m m!}{(m+1)^k} (-1)^m \\
&= (-1)^n \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-1)^m}{(m+1)^k} = \text{LHS}.
\end{aligned}$$

Note that $\left[{m \atop 0}\right] = 0$ $(m \ge 1)$ and $\left[{m \atop l}\right] = 0$ $(l > m)$, and

$$\sum_{l=i}^{m} (-1)^{m-l} \begin{bmatrix} m \\ l \end{bmatrix} \begin{Bmatrix} l \\ i \end{Bmatrix} = \begin{cases} 1 & (i = m); \\ 0 & (i \ne m). \end{cases}$$

□

3. Poly-Cauchy numbers of the second kind

Poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)}$ are defined by

$$\hat{c}_n^{(k)} = \underbrace{\int_0^1 \dots \int_0^1}_{k} (-x_1 x_2 \dots x_k)(-x_1 x_2 \dots x_k - 1)$$

$$\dots (-x_1 x_2 \dots x_k - n + 1) dx_1 dx_2 \dots dx_k$$

(see in [7]). If $k = 1$, then $\hat{c}_n^{(1)} = \hat{c}_n$ is the classical Cauchy numbers of the second kind defined by

$$\hat{c}_n = \int_0^k (-x)(-x-1)\dots(-x-n+1)dx$$

(see e.g. [2, 8]). The generating function of poly-Cauchy numbers of the second kind ([7, Theorem 5]) is given by

$$\mathrm{Lif}_k(-\ln(1+x)) = \sum_{n=0}^{\infty} \hat{c}_n^{(k)} \frac{x^n}{n!}\,.$$

An explicit formula for $\hat{c}_n^{(k)}$ ([7, Theorem 4]) is given by

$$\hat{c}_n^{(k)} = (-1)^n \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{1}{(m+1)^k} \quad (n \ge 0, k \ge 1)\,. \tag{3.1}$$

In a similar way, we have a relationship, expressing poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)}$ by using poly-Bernoulli numbers $B_n^{(k)}$. The proof is similar and omitted.

Theorem 3.1. *For $n \ge 1$ we have*

$$\hat{c}_n^{(k)} = (-1)^n \sum_{l=1}^{n} \sum_{m=1}^{n} \frac{1}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ l \end{bmatrix} B_l^{(k)}\,.$$

In addition, we also obtain the corresponding relationship to Theorem 2.1.

Theorem 3.2. *For $n \ge 1$ we have*

$$B_n^{(k)} = (-1)^n \sum_{l=1}^{n} \sum_{m=1}^{n} m! \begin{Bmatrix} n \\ m \end{Bmatrix} \begin{Bmatrix} m \\ l \end{Bmatrix} \hat{c}_l^{(k)}\,.$$

Proof. By (1.2) and (3.1), we have

$$\begin{aligned} \mathrm{RHS} &= (-1)^n \sum_{l=1}^{n} \sum_{m=1}^{n} m! \begin{Bmatrix} n \\ m \end{Bmatrix} \begin{Bmatrix} m \\ l \end{Bmatrix} (-1)^l \sum_{i=0}^{l} \begin{bmatrix} l \\ i \end{bmatrix} \frac{1}{(i+1)^k} \\ &= (-1)^n \sum_{m=1}^{n} m! \begin{Bmatrix} n \\ m \end{Bmatrix} \sum_{l=0}^{n} \begin{Bmatrix} m \\ l \end{Bmatrix} (-1)^l \sum_{i=0}^{l} \begin{bmatrix} l \\ i \end{bmatrix} \frac{1}{(i+1)^k} \\ &= (-1)^n \sum_{m=1}^{n} m! \begin{Bmatrix} n \\ m \end{Bmatrix} \sum_{i=0}^{n} \frac{1}{(i+1)^k} \sum_{l=i}^{n} (-1)^l \begin{Bmatrix} m \\ l \end{Bmatrix} \begin{bmatrix} l \\ i \end{bmatrix} \end{aligned}$$

$$= (-1)^n \sum_{m=0}^{n} m! \left\{ {n \atop m} \right\} \frac{1}{(m+1)^k} (-1)^m$$

$$= (-1)^n \sum_{m=0}^{n} \left\{ {n \atop m} \right\} \frac{(-1)^m m!}{(m+1)^k} = \text{LHS}\,.$$

Note that

$$\sum_{l=i}^{m} (-1)^{m-l} \left\{ {m \atop l} \right\} \left[{l \atop i} \right] = \begin{cases} 1 & (i = m); \\ 0 & (i \neq m). \end{cases}$$ □

4. Poly-Cauchy polynomials and poly-Bernoulli polynomials

Poly-Cauchy polynomials of the first kind $c_n^{(k)}(z)$ are defined by

$$c_n^{(k)}(z) = n! \underbrace{\int_0^1 \dots \int_0^1}_{k} (x_1 x_2 \dots x_k - z)(x_1 x_2 \dots x_k - 1 - z) \cdots (x_1 x_2 \dots x_k - (n-1) - z) dx_1 dx_2 \dots dx_k,$$

and are expressed explicitly in terms of Stirling numbers of the first kind ([5, Theorem 1])

$$c_n^{(k)}(z) = \sum_{m=0}^{n} \left[{n \atop m} \right] (-1)^{n-m} \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k}\,.$$

Poly-Cauchy polynomials of the second kind $\hat{c}_n^{(k)}(z)$ are defined by

$$\hat{c}_n^{(k)}(z) = n! \underbrace{\int_0^1 \dots \int_0^1}_{k} (-x_1 x_2 \dots x_k + z)(-x_1 x_2 \dots x_k - 1 + z) \cdots (-x_1 x_2 \dots x_k - (n-1) + z) dx_1 dx_2 \dots dx_k,$$

and are expressed explicitly in terms of Stirling numbers of the first kind ([5, Theorem 4].

$$\hat{c}_n^{(k)}(z) = \sum_{m=0}^{n} \left[{n \atop m} \right] (-1)^{n} \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k}\,.$$

In 2010, Coppo and Candelpergher [3], 2011 Bayad and Hamahata [1, (1.5)] introduced the poly-Bernoulli polynomials $B_n^{(k)}(z)$ given by

$$\frac{\text{Li}_k(1-e^{-x})}{1-e^{-x}} e^{-xz} = \sum_{n=0}^{\infty} B_n^{(k)}(z) \frac{x^n}{n!}\,,$$

and

$$\frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}}e^{xz} = \sum_{n=0}^{\infty} B_n^{(k)}(z)\frac{x^n}{n!}\,,$$

respectively, satisfying $B_n^{(k)}(0) = B_n^{(k)}$.

If we define still different poly-Bernoulli polynomials $B_n^{(k)}$ by

$$B_n^{(k)}(z) = (-1)^n \sum_{m=0}^{n} \left\{ {n \atop m} \right\} (-1)^m m! \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k},$$

satisfying $B_n^{(k)}(0) = B_n^{(k)}$ ($n \ge 0$, $k \ge 1$), then we have relationships between the poly-Bernoulli polynomials and poly-Cauchy polynomials similar to those between the poly-Bernoulli numbers and the poly-Cauchy numbers.

Theorem 4.1. *For $n \ge 1$ we have*

$$B_n^{(k)}(z) = \sum_{l=1}^{n}\sum_{m=1}^{n} m! \left\{ {n \atop m} \right\} \left\{ {m-1 \atop l-1} \right\} c_l^{(k)}(z)\,,$$

$$= (-1)^n \sum_{l=1}^{n}\sum_{m=1}^{n} m! \left\{ {n \atop m} \right\} \left\{ {m \atop l} \right\} \hat{c}_l^{(k)}(z)\,,$$

$$c_n^{(k)}(z) = (-1)^n \sum_{l=1}^{n}\sum_{m=1}^{n} \frac{(-1)^m}{m!} \left[{n \atop m} \right] \left[{m \atop l} \right] B_l^{(k)}(z)$$

$$\hat{c}_n^{(k)}(z) = (-1)^n \sum_{l=1}^{n}\sum_{m=1}^{n} \frac{1}{m!} \left[{n \atop m} \right] \left[{m \atop l} \right] B_l^{(k)}(z)\,.$$

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