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Sums of powers of Fibonacci and Lucas polynomials in terms of Fibopolynomials

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Abstract

We consider sums of powers of Fibonacci and Lucas polynomials of the form $\sum_{n=0}^{q} F_{tsn}^{k}(x)$ and $\sum_{n=0}^{q} L_{tsn}^{k}(x)$, where s,t,k are given natural numbers, together with the corresponding alternating sums $\sum_{n=0}^{q}(-1)^{n} F_{tsn}^{k}(x)$ and $\sum_{n=0}^{q}(-1)^{n} L_{tsn}^{k}(x)$. We give conditions on s,t,k for express these sums as some proposed linear combinations of the s-Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m=1,2,\ldots,tk$.

Keywords: Sums of powers; Fibonacci and Lucas polynomials; Z Transform.

MSC: 11B39

1. Introduction

We use $\mathbb{N}$ for the natural numbers and $\mathbb{N}'$ for $\mathbb{N}\cup\{0\}$. We follow the standard notation $F_n(x)$ for Fibonacci polynomials and $L_n(x)$ for Lucas polynomials. Binet's formulas

$$F_n(x)=\frac{1}{\sqrt{x^2+4}}\left(\alpha^n(x)-\beta^n(x)\right) \quad \text{and} \quad L_n(x)=\alpha^n(x)+\beta^n(x), \tag{1.1}$$

where

$$\alpha(x)=\frac{1}{2}\left(x+\sqrt{x^2+4}\right) \quad \text{and} \quad \beta(x)=\frac{1}{2}\left(x-\sqrt{x^2+4}\right), \tag{1.2}$$

will be used extensively (without further comments). We will use also the identities

$$\frac{F_{(2p-1)s}(x)}{F_s(x)}=\sum_{k=0}^{p-1}(-1)^{sk}L_{2(p-k-1)s}(x)-(-1)^{s(p-1)}, \tag{1.3}$$

$$\frac{F_{2ps}(x)}{F_s(x)} = \sum_{k=0}^{p-1} (-1)^{sk} L_{(2p-2k-1)s}(x), \tag{1.4}$$

$$F_M(x) F_N(x) - F_{M+K}(x) F_{N-K}(x) = (-1)^{N-K} F_{M+K-N}(x) F_K(x), \tag{1.5}$$

where $p \in \mathbb{N}$ in (1.3) and (1.4), and $M, N, K \in \mathbb{Z}$ in (1.5). (Identity (1.5) is a version of the so-called "index-reduction formula"; see [7] for the case $x = 1$.) Two variants of (1.5) we will use in section 5 are

$$F_M(x) L_N(x) - F_{M+K}(x) L_{N-K}(x) = (-1)^{N-K+1} L_{M+K-N}(x) F_K(x), \tag{1.6}$$

$$\left(x^2+4\right) F_M(x) F_N(x) - L_{M+K}(x) L_{N-K}(x) = (-1)^{N-K+1} L_{M+K-N}(x) L_K(x). \tag{1.7}$$

Given $n \in \mathbb{N}'$ and $k \in \{0, 1, \ldots, n\}$, the *s-Fibopolynomial* $\binom{n}{k}_{F_s(x)}$ is defined by $\binom{n}{0}_{F_s(x)} = \binom{n}{n}_{F_s(x)} = 1$, and

$$\binom{n}{k}_{F_s(x)} = \frac{F_{sn}(x) F_{s(n-1)}(x) \cdots F_{s(n-k+1)}(x)}{F_s(x) F_{2s}(x) \cdots F_{ks}(x)}. \tag{1.8}$$

(These mathematical objects were used before in [19], where we called them "*s*-polyfibonomials". However, we think now that "*s*-Fibopolynomials" is a better name to describe them.)

Plainly we have symmetry for *s*-Fibopolynomials: $\binom{n}{k}_{F_s(x)} = \binom{n}{n-k}_{F_s(x)}$. We can use the identity

$$F_{s(n-k)+1}(x) F_{sk}(x) + F_{sk-1}(x) F_{s(n-k)}(x) = F_{sn}(x),$$

(which comes from (1.5) with $M = sn$, $N = 1$ and $K = -sk+1$), to conclude that

$$\binom{n}{k}_{F_s(x)} = F_{s(n-k)+1}(x) \binom{n-1}{k-1}_{F_s(x)} + F_{sk-1}(x) \binom{n-1}{k}_{F_s(x)}. \tag{1.9}$$

Formula (1.9) and a simple induction argument, show that *s*-Fibopolynomials are indeed polynomials (with $\deg \binom{n}{k}_{F_s(x)} = sk(n-k)$). The case $s = x = 1$ corresponds to Fibonomials $\binom{n}{k}_F$, introduced by V. E. Hoggatt, Jr. [5] in 1967 (see also [23]), and the case $x = 1$ corresponds to *s*-Fibonomials $\binom{n}{k}_{F_s}$, first mentioned also in [5], and studied recently in [18]. We comment in passing that Fibonomials are important mathematical objects involved in many interesting research works during the last few decades (see [4, 8, 9, 11, 24]).

The well-known identity

$$\sum_{n=0}^{q} F_n^2 = F_q F_{q+1} = \binom{q+1}{2}_F, \tag{1.10}$$

was the initial motivation for this work. We will see that (1.10) is just a particular case of the following polynomial identities ((4.20) in section 4)

$$(-1)^{(s+1)q} L_s(x) \sum_{n=0}^{q} (-1)^{(s+1)n} F_{sn}^2(x)$$
$$= (-1)^{sq} F_s(x) \sum_{n=0}^{q} (-1)^{sn} F_{2sn}(x) = F_{s(q+1)}(x) F_{sq}(x).$$

To find closed formulas for sums of powers of Fibonacci and Lucas numbers $\sum_{n=0}^{q} F_n^k$ and $\sum_{n=0}^{q} L_n^k$, and for the corresponding alternating sums of powers $\sum_{n=0}^{q} (-1)^n F_n^k$ and $\sum_{n=0}^{q} (-1)^n L_n^k$, is a challenging problem that has been in the interest of many mathematicians along the years (see [1, 3, 12, 13, 15, 16, 21], to mention some). There are also some works considering variants of these sums and/or generalizations (in some sense) of them (see [2, 10, 14, 22], among many others).

This work presents, on one hand, a generalization of the problem mentioned above, considering Fibonacci and Lucas polynomials (instead of numbers) and involving more parameters in the sums. On the other hand, we are not interested in any closed formulas for these sums, but only in sums that can be written as certain linear combinations of certain s-Fibopolynomials (as in (1.10)). More precisely, in this work we obtain sufficient conditions (on the positive integer parameters t, k, s), for the polynomial sums of powers $\sum_{n=0}^{q} F_{tsn}^k(x)$, $\sum_{n=0}^{q} L_{tsn}^k(x)$, $\sum_{n=0}^{q} (-1)^n F_{tsn}^k(x)$ and $\sum_{n=0}^{q} (-1)^n L_{tsn}^k(x)$, can be expressed as linear combinations of the s-Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m = 1, 2, \ldots, tk$, according to some proposed expressions ((3.3), (3.15), (4.5) and (4.6), respectively). (We conjecture that these sufficient conditions are also necessary, see remark 3.2.)

In Section 2 we recall some facts about Z transform, since some results related to the Z transform of the sequences $\left\{F_{tsn}^k(x)\right\}_{n=0}^{\infty}$ and $\left\{L_{tsn}^k(x)\right\}_{n=0}^{\infty}$ (obtained in a previous work) are the starting point of the results in this work.

The main results are presented in Section 3 and 4. Propositions 3.1 and 3.3 in section 3 contain, respectively, sufficient conditions on the positive integers t, k, s for the sums of powers $\sum_{n=0}^{q} F_{tsn}^k(x)$ and $\sum_{n=0}^{q} L_{tsn}^k(x)$ can be written as linear combinations of the mentioned s-Fibopolynomials, and propositions 4.1 and 4.3 in section 4 contain, respectively, sufficient conditions on the positive integers t, k, s for the alternating sums of powers $\sum_{n=0}^{q} (-1)^n F_{tsn}^k(x)$ and $\sum_{n=0}^{q} (-1)^n L_{tsn}^k(x)$ can be written as linear combinations of those s-Fibopolynomials. Surprisingly, there are some intersections on the conditions on t and k in Proposition 3.1 and 4.1 (and also in Proposition 3.3 and 4.3), allowing us to write results for sums of powers of the form $\sum_{n=0}^{q} (-1)^{sn} F_{tsn}^k(x)$ or $\sum_{n=0}^{q} (-1)^{(s+1)n} F_{tsn}^k(x)$ (and similar sums for Lucas polynomials), that work at the same time for sums $\sum_{n=0}^{q} F_{tsn}^k(x)$ and alternating sums $\sum_{n=0}^{q} (-1)^n F_{tsn}^k(x)$ as well, depending on the parity of s. These results are presented in section 4: Corollary 4.2 (for the Fibonacci case) and 4.4 (for the Lucas case).

Finally, in Section 5 we show some examples of identities obtained as derivatives of some of the results obtained in previous sections.

2. Preliminaries

The Z transform maps complex sequences $\{a_n\}_{n=0}^{\infty}$ into holomorphic functions $A : U \subset \mathbb{C} \to \mathbb{C}$, defined by the Laurent series $A(z) = \sum_{n=0}^{\infty} a_n z^{-n}$ (also written as $\mathcal{Z}(a_n)$, defined outside the closure of the disk of convergence of the Taylor series $\sum_{n=0}^{\infty} a_n z^n$). We also write $a_n = \mathcal{Z}^{-1}(A(z))$ and we say the the sequence $\{a_n\}_{n=0}^{\infty}$ is the inverse Z transform of $A(z)$. Some basic facts we will need are the following:

(a) $\mathcal{Z}$ is linear and injective (same for $\mathcal{Z}^{-1}$).

(b) If $\{a_n\}_{n=0}^{\infty}$ is a sequence with Z transform $A(z)$, then the Z transform of the sequence $\{(-1)^n a_n\}_{n=0}^{\infty}$ is

$$\mathcal{Z}((-1)^n a_n) = A(-z). \tag{2.1}$$

(c) If $\{a_n\}_{n=0}^{\infty}$ is a sequence with Z transform $A(z)$, then the Z transform of the sequence $\{na_n\}_{n=0}^{\infty}$ is

$$\mathcal{Z}(na_n) = -z\frac{d}{dz}A(z). \tag{2.2}$$

Plainly we have (for given $\lambda \in \mathbb{C}$, $\lambda \neq 0$)

$$\mathcal{Z}(\lambda^n) = \frac{z}{z-\lambda}. \tag{2.3}$$

For example, if $t, k \in \mathbb{N}'$ are given, we can write the generic term of the sequence $\left\{F_{tsn}^k(x)\right\}_{n=0}^{\infty}$ as

$$\begin{aligned} F_{tsn}^k(x) &= \left(\frac{1}{\sqrt{x^2+4}}\left(\alpha^{tsn}(x) - \beta^{tsn}(x)\right)\right)^k \qquad (2.4) \\ &= \frac{1}{(x^2+4)^{\frac{k}{2}}}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left(\alpha^{tsl}(x)\,\beta^{ts(k-l)}(x)\right)^n. \end{aligned}$$

The linearity of $\mathcal{Z}$ and (2.3) give us

$$\mathcal{Z}\left(F_{tsn}^k(x)\right) = \frac{1}{(x^2+4)^{\frac{k}{2}}}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\frac{z}{z-\alpha^{tsl}(x)\,\beta^{ts(k-l)}(x)}. \tag{2.5}$$

Similarly, since the generic term of the sequence $\left\{L_{tsn}^k(x)\right\}_{n=0}^{\infty}$ can be expressed as

$$L_{tsn}^k(x) = \left(\alpha^{tsn}(x) + \beta^{tsn}(x)\right)^k = \sum_{l=0}^{k}\binom{k}{l}\left(\alpha^{tsl}(x)\,\beta^{ts(k-l)}(x)\right)^n, \tag{2.6}$$

we have that

$$\mathcal{Z}\left(L_{tsn}^k(x)\right) = \sum_{l=0}^{k}\binom{k}{l}\frac{z}{z-\alpha^{tsl}(x)\,\beta^{ts(k-l)}(x)}. \tag{2.7}$$

Observe that formulas

$$\mathcal{Z}\left(F_n\left(x\right)\right)=\frac{z}{z^2-xz-1}\qquad\text{and}\qquad\mathcal{Z}\left(L_n\left(x\right)\right)=\frac{z\left(2z-x\right)}{z^2-xz-1},\tag{2.8}$$

are the simplest cases ($k=t=s=1$) of (2.5) and (2.7), respectively.

In a recent work [20] (inspired by [6], among others), we proved that expressions (2.5) and (2.7) can be written in a special form. The result is that (2.5) can be written as

$$\begin{aligned}&\mathcal{Z}\left(F_{tsn}^k\left(x\right)\right)\qquad\qquad(2.9)\\&=z\frac{\sum_{i=0}^{tk}\sum_{j=0}^{i}\left(-1\right)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}F_{ts(i-j)}^k\left(x\right)z^{tk-i}}{\sum_{i=0}^{tk+1}\left(-1\right)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{tk+1}{i}_{F_s(x)}z^{tk+1-i}},\end{aligned}$$

and (2.7) can be written as

$$\begin{aligned}&\mathcal{Z}\left(L_{tsn}^k\left(x\right)\right)\qquad\qquad(2.10)\\&=z\frac{\sum_{i=0}^{tk}\sum_{j=0}^{i}\left(-1\right)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}L_{ts(i-j)}^k\left(x\right)z^{tk-i}}{\sum_{i=0}^{tk+1}\left(-1\right)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{tk+1}{i}_{F_s(x)}z^{tk+1-i}}.\end{aligned}$$

From (2.9) and (2.10) we obtained that $F_{tsn}^k\left(x\right)$ and $L_{tsn}^k\left(x\right)$ can be expressed as linear combinations of the s-Fibopolynomials $\binom{n+tk-i}{tk}_{F_s(x)}$, $i=0,1,\ldots,tk$, according to

$$\begin{aligned}&F_{tsn}^k\left(x\right)\qquad\qquad(2.11)\\&=\left(-1\right)^{s+1}\sum_{i=0}^{tk}\sum_{j=0}^{i}\left(-1\right)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}F_{ts(i-j)}^k(x)\binom{n+tk-i}{tk}_{F_s(x)},\end{aligned}$$

and

$$\begin{aligned}&L_{tsn}^k\left(x\right)\qquad\qquad(2.12)\\&=\left(-1\right)^{s+1}\sum_{i=0}^{tk}\sum_{j=0}^{i}\left(-1\right)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}L_{ts(i-j)}^k(x)\binom{n+tk-i}{tk}_{F_s(x)}.\end{aligned}$$

The denominator in (2.9) (or (2.10)) is a $(tk+1)$-th degree z-polynomial, which we denote as $D_{s,tk+1}\left(x,z\right)$, that can be factored as

$$\begin{aligned}&\sum_{i=0}^{tk+1}\left(-1\right)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{tk+1}{i}_{F_s(x)}z^{tk+1-i}\qquad(2.13)\\&=\left(-1\right)^{s+1}\prod_{j=0}^{tk}\left(z-\alpha^{sj}\left(x\right)\beta^{s(tk-j)}\left(x\right)\right).\end{aligned}$$

(See proposition 1 in [20].) Moreover, if tk is even, $tk=2p$ say, then (2.13) can be written as

$$D_{s,2p+1}(x;z) = (-1)^{s+1}\left(z-(-1)^{sp}\right)\prod_{j=0}^{p-1}\left(z^2-(-1)^{sj}L_{2s(p-j)}(x)\,z+1\right), \quad (2.14)$$

and if tk is odd, $tk=2p-1$ say, we have

$$D_{s,2p}(x;z) = (-1)^{s+1}\prod_{j=0}^{p-1}\left(z^2-(-1)^{sj}L_{s(2p-1-2j)}(x)\,z+(-1)^{(2p-1)s}\right). \quad (2.15)$$

(See (40) and (41) in [20].)

3. The main results (I)

Let us consider first the Fibonacci case. From (2.11) we can write the sum $\sum_{n=0}^{q}F_{tsn}^{k}(x)$ in terms of a sum of s-Fibopolynomials in a trivial way, namely

$$\begin{aligned}&\sum_{n=0}^{q}F_{tsn}^{k}(x) & (3.1)\\ &=(-1)^{s+1}\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}F_{ts(i-j)}^{k}(x)\sum_{n=0}^{q}\binom{n+tk-i}{tk}_{F_s(x)}.\end{aligned}$$

The point is that we can write (3.1) as

$$\begin{aligned}&\sum_{n=0}^{q}F_{tsn}^{k}(x) & (3.2)\\ &=(-1)^{s+1}\sum_{m=1}^{tk}\sum_{i=0}^{tk-m}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}F_{ts(i-j)}^{k}(x)\binom{q+m}{tk}_{F_s(x)}\\ &\quad+(-1)^{s+1}\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}F_{ts(i-j)}^{k}(x)\sum_{n=0}^{q}\binom{n}{tk}_{F_s(x)}.\end{aligned}$$

Expression (3.2) tells us that the sum $\sum_{n=0}^{q}F_{tsn}^{k}(x)$ can be written as a linear combination of the s-Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m=1,2,\ldots,tk$, according to

$$\begin{aligned}&\sum_{n=0}^{q}F_{tsn}^{k}(x) & (3.3)\\ &=(-1)^{s+1}\sum_{m=1}^{tk}\sum_{i=0}^{tk-m}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}F_{ts(i-j)}^{k}(x)\binom{q+m}{tk}_{F_s(x)},\end{aligned}$$

if and only if

$$\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}F^k_{ts(i-j)}(x)=0. \tag{3.4}$$

Observe that from (2.5) and (2.9) we can write

$$\begin{aligned}&\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}F^k_{ts(i-j)}(x)\,z^{tk-i} \qquad (3.5)\\ &=\frac{1}{(x^2+4)^{\frac{k}{2}}}\left(\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\frac{1}{z-\alpha^{lts}(x)\,\beta^{(k-l)ts}(x)}\right)\\ &\times\left(\sum_{i=0}^{tk+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{tk+1}{i}_{F_s(x)}z^{tk+1-i}\right).\end{aligned}$$

Let us consider the factors in parentheses of the right-hand side of (3.5), namely

$$\Pi_1(x,z)=\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\frac{1}{z-\alpha^{lts}(x)\,\beta^{(k-l)ts}(x)}, \tag{3.6}$$

and

$$\Pi_2(x,z)=\sum_{i=0}^{tk+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{tk+1}{i}_{F_s(x)}z^{tk+1-i}. \tag{3.7}$$

Clearly any of the conditions

$$\Pi_1(x,1)=0, \tag{3.8}$$

or

$$\Pi_1(x,1)<\infty \text{ and } \Pi_2(x,1)=0. \tag{3.9}$$

imply (3.4).

Proposition 3.1. *The sum $\sum_{n=1}^{q}F^k_{tsn}(x)$ can be written as a linear combination of the s-Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m=1,2,\ldots,tk$, according to (3.3), in the following cases*

	t	k	s
(a)	*even*	*odd*	*even*
(b)	*odd*	$\equiv 2 \bmod 4$	*odd*
(c)	$\equiv 0 \bmod 4$	*odd*	*any*

Proof. Observe that in each of the three cases the product tk is even. Then, according to (2.14) we can write

$$\Pi_2(x,z)=(-1)^{s+1}\left(z-(-1)^{\frac{kts}{2}}\right)\prod_{j=0}^{\frac{tk}{2}-1}\left(z^2-(-1)^{sj}L_{2s\left(\frac{tk}{2}-j\right)}(x)\,z+1\right). \tag{3.10}$$

(a) Let us suppose that t is even, k is odd and s is even. In this case the factor $\left(z-(-1)^{\frac{kts}{2}}\right)$ of the right-hand side of (3.10) is $(z-1)$, so we have $\Pi_2(x,1)=0$. It remains to check that $\Pi_1(x,1)$ is finite. In fact, by writing k as $2k-1$, and using that t and s are even, one can check that

$$\Pi_1(x,1)=\sqrt{x^2+4}\sum_{l=0}^{k-1}\binom{2k-1}{l}(-1)^{l+1}\frac{F_{(2k-1-2l)ts}(x)}{2-L_{(2k-1-2l)ts}(x)}, \tag{3.11}$$

so we have that $\Pi_1(x,1)$ is finite, and then the right-hand side of (3.5) is equal to zero when $z=1$, as wanted.

(b) Suppose now that t is odd, $k\equiv 2 \bmod 4$ and that s is odd. In this case the factor $\left(z-(-1)^{\frac{kts}{2}}\right)$ of the right-hand side of (3.10) is $(z+1)$, so $\Pi_2(x,1)\neq 0$. However, by writing k as $2(2k-1)$ and using that t and s are odd, we can see that

$$\Pi_1(x,1)=\sum_{l=0}^{2k-2}\binom{2(2k-1)}{l}(-1)^l-\frac{1}{2}\binom{2(2k-1)}{2k-1}=0.$$

Thus, the right-hand side of (3.5) is equal to 0 when $z=1$, as wanted.

(c) Let us suppose that $t\equiv 0 \bmod 4$, k is odd, and s is any positive integer. In this case the factor $\left(z-(-1)^{\frac{kts}{2}}\right)$ of the right-hand side of (3.10) is $(z-1)$, so we have $\Pi_2(x,1)=0$. By writing k as $2k-1$, and using that t is multiple of 4, we can see that formula (3.11) is valid for any $s\in\mathbb{N}$, so we conclude that $\Pi_1(x,1)$ is finite. Thus the right-hand side of (3.5) is 0 when $z=1$, as wanted. $\square$

An example from the case (c) of proposition 3.1 is the following identity (corresponding to $t=4$ and $k=1$), valid for any $s\in\mathbb{N}$

$$\sum_{n=0}^{q}F_{4sn}(x) \tag{3.12}$$
$$=F_{4s}(x)\left(\binom{q+1}{4}_{F_s(x)}+(-1)^{s+1}L_{2s}(x)\binom{q+2}{4}_{F_s(x)}+\binom{q+3}{4}_{F_s(x)}\right).$$

Remark 3.2. A natural question about proposition 3.1 is if the given conditions on t, k and s are also necessary (for expressing the sum $\sum_{n=1}^{q}F_{tsn}^{k}(x)$ as a linear combination of the s-Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m=1,2,\ldots,tk$, according to (3.3)). We believe that the answer is yes, and we think that this conjecture (together with similar conjectures in propositions 3.3, 4.1 and 4.3) can be a good topic for a future work. Nevertheless, we would like to make some comments about this point in the case of proposition 3.1. The cases where we do not have the conditions on t,k,s stated in proposition 3.1 are the following (e = even, o = odd)

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)
t	e	e	o	$\equiv 2 \bmod 4$	o	o	o
k	e	e	e	o	$\equiv 0 \bmod 4$	o	o
s	e	o	e	o	o	e	o

Thus, in order to prove the necessity of the mentioned conditions we need to show that (3.4) *does not hold* in each of these 7 cases. For example, the case $t=1$ and $k=3$ is included in (vi) and (vii). In this case the left-hand side of (3.4) is (for any $s\in\mathbb{N}$)

$$\sum_{i=0}^{3}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{4}{j}_{F_s(x)}F_{s(i-j)}^{3}(x)=-F_s^3(x)\left(2L_s(x)+1+(-1)^s\right).$$

That is, (3.4) does not hold, which means that the sum of cubes $\sum_{n=0}^{q}F_{sn}^{3}(x)$ can not be written as the linear combination of the s-Fibopolynomials $\binom{q+1}{3}_{F_s(x)}$, $\binom{q+2}{3}_{F_s(x)}$ and $\binom{q+3}{3}_{F_s(x)}$ proposed in (3.3). However, it is known that $\sum_{n=0}^{q}F_n^3=\frac{1}{10}\left(F_{3q+2}-6(-1)^qF_{q-1}+5\right)$ (see [1]). In fact, the case of sums of odd powers of Fibonacci and Lucas numbers has been considered for several authors (see [3], [16], [21]). It turns out that some of their nice results belong to some of the cases (i) to (vii) above, so they can not be written as in (3.3).

Let us consider now the case of sums of powers of Lucas polynomials. From (2.12) we see that

$$\begin{aligned}&(-1)^{s+1}\sum_{n=0}^{q}L_{tsn}^{k}(x) \qquad (3.13)\\ &=\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}L_{ts(i-j)}^{k}(x)\sum_{n=0}^{q}\binom{n+tk-i}{tk}_{F_s(x)},\end{aligned}$$

which can be written as

$$\begin{aligned}&\sum_{n=0}^{q}L_{tsn}^{k}(x) \qquad (3.14)\\ &=(-1)^{s+1}\sum_{m=1}^{tk}\sum_{i=0}^{tk-m}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}L_{ts(i-j)}^{k}(x)\binom{q+m}{tk}_{F_s(x)}\\ &\quad+(-1)^{s+1}\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}L_{ts(i-j)}^{k}(x)\sum_{n=0}^{q}\binom{n}{tk}_{F_s(x)}.\end{aligned}$$

Expression (3.14) tells us that the sum $\sum_{n=0}^{q}L_{tsn}^{k}(x)$ can be written as a linear combination of the s-Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m=1,2,\ldots,tk$, according to

$$\begin{aligned}&\sum_{n=0}^{q}L_{tsn}^{k}(x) \qquad (3.15)\\ &=(-1)^{s+1}\sum_{m=1}^{tk}\sum_{i=0}^{tk-m}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}L_{ts(i-j)}^{k}(x)\binom{q+m}{tk}_{F_s(x)},\end{aligned}$$

if and only if

$$\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}L^{k}_{ts(i-j)}(x)=0. \tag{3.16}$$

From (2.7) and (2.10) we can write

$$\begin{aligned}&\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}L^{k}_{ts(i-j)}(x)\,z^{tk-i} \\ &=\left(\sum_{i=0}^{tk+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{tk+1}{i}_{F_s(x)}z^{tk+1-i}\right) \\ &\quad\times\sum_{l=0}^{k}\binom{k}{l}\frac{1}{z-\alpha^{lts}(x)\,\beta^{(k-l)ts}(x)}.\end{aligned} \tag{3.17}$$

We have again the factor $\Pi_2(x,z)$ considered in the Fibonacci case (see (3.7)), and the factor

$$\widetilde{\Pi}_1(x,z)=\sum_{l=0}^{k}\binom{k}{l}\frac{1}{z-\alpha^{lts}(x)\,\beta^{(k-l)ts}(x)}. \tag{3.18}$$

Plainly any of the conditions: (a) $\widetilde{\Pi}_1(x,1)=0$, or, (b) $\widetilde{\Pi}_1(x,1)<\infty$ and $\Pi_2(x,1)=0$, imply (3.16).

Proposition 3.3. *The sum $\sum_{n=0}^{q}L^{k}_{tsn}(x)$ can be written as a linear combination of the s-Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m=1,2,\ldots,tk$, according to (3.15), in the following cases*

	t	k	s
(a)	*even*	*odd*	*even*
(b)	$\equiv 0 \bmod 4$	*odd*	*any*

Proof. In both cases we have tk even, so it is valid the factorization (3.10) for $\Pi_2(x,z)$.

(a) Suppose that t and s are even and that k is odd. In this case the factor $\left(z-(-1)^{\frac{kts}{2}}\right)$ in $\Pi_2(x,z)$ is $(z-1)$, so we have $\Pi_2(x,1)=0$. It remains to check that $\widetilde{\Pi}_1(x,1)$ is finite. In fact, by writing k as $2k-1$ and using that t and s are even, one can see that $\widetilde{\Pi}_1(x,1)=4^{k-1}$.

(b) Suppose now that $t\equiv 0 \bmod 4$, k is odd and s is any positive integer. In this case the factor $\left(z-(-1)^{\frac{kts}{2}}\right)$ in $\Pi_2(x,z)$ is again $(z-1)$, so we have $\Pi_2(x,1)=0$. With a similar calculation to the case (a), we can see that in this case we have also $\widetilde{\Pi}_1(x,1)=4^{k-1}$. □

An example from the case (b) of proposition 3.3 is the following identity (corresponding to $t = 4$ and $k = 1$), valid for any $s \in \mathbb{N}$

$$\sum_{n=0}^{q} L_{4sn}(x) \tag{3.19}$$
$$= 2\binom{q+4}{4}_{F_s(x)} + \left(-L_{4s}(x) + 2(-1)^{s+1} L_{2s}(x)\right)\binom{q+3}{4}_{F_s(x)}$$
$$+ (-1)^s \left(L_{6s}(x) + L_{2s}(x) + 2(-1)^s\right)\binom{q+2}{4}_{F_s(x)} - L_{4s}(x)\binom{q+1}{4}_{F_s(x)}.$$

Examples from the cases (a) and (b) of proposition 3.1, and from the case (a) of proposition 3.3, will be given in section 4, since some variants of them work also as examples of alternating sums of powers of Fibonacci or Lucas polynomials, to be discussed in section 4 (see corollaries 4.2 and 4.4).

4. The main results (II): alternating sums

According to (2.1), (2.5), (2.7), (2.9) and (2.10), the Z transform of the alternating sequences $\left\{(-1)^n F_{tsn}^k(x)\right\}_{n=0}^{\infty}$ and $\left\{(-1)^n L_{tsn}^k(x)\right\}_{n=0}^{\infty}$ are

$$\mathcal{Z}\left((-1)^n F_{tsn}^k(x)\right) = \frac{1}{(x^2+4)^{\frac{k}{2}}} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \frac{z}{z + \alpha^{lts}(x)\beta^{(k-l)ts}(x)} \tag{4.1}$$
$$= -z \frac{\sum_{i=0}^{tk} \sum_{j=0}^{i} (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x)(-z)^{tk-i}}{\sum_{i=0}^{tk+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{tk+1}{i}_{F_s(x)} (-z)^{tk+1-i}},$$

and

$$\mathcal{Z}\left((-1)^n L_{tsn}^k(x)\right) = \sum_{l=0}^{k} \binom{k}{l} \frac{z}{z + \alpha^{lts}(x)\beta^{(k-l)ts}(x)} \tag{4.2}$$
$$= -z \frac{\sum_{i=0}^{tk} \sum_{j=0}^{i} (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^k(x)(-z)^{tk-i}}{\sum_{i=0}^{tk+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{tk+1}{i}_{F_s(x)} (-z)^{tk+1-i}}.$$

By using (2.11) and (2.12) it is possible to establish expressions, for the case of alternating sums, similar to expressions (3.2) and (3.14), namely

$$\sum_{n=0}^{q}(-1)^{n}F_{tsn}^{k}(x)=(-1)^{s+1+tk+q} \tag{4.3}$$
$$\times\sum_{m=1}^{tk}\sum_{i=0}^{tk-m}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i+m}\binom{tk+1}{j}_{F_s(x)}F_{ts(i-j)}^{k}(x)\binom{q+m}{tk}_{F_s(x)}$$
$$+(-1)^{s+1+tk}\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i}\binom{tk+1}{j}_{F_s(x)}F_{ts(i-j)}^{k}(x)\sum_{n=0}^{q}(-1)^{n}\binom{n}{tk}_{F_s(x)},$$

and

$$\sum_{n=0}^{q}(-1)^{n}L_{tsn}^{k}(x)=(-1)^{s+1+tk+q} \tag{4.4}$$
$$\times\sum_{m=1}^{tk}\sum_{i=0}^{tk-m}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i+m}\binom{tk+1}{j}_{F_s(x)}L_{ts(i-j)}^{k}(x)\binom{q+m}{tk}_{F_s(x)}$$
$$+(-1)^{s+1+tk}\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i}\binom{tk+1}{j}_{F_s(x)}L_{ts(i-j)}^{k}(x)\times$$
$$\times\sum_{n=0}^{q}(-1)^{n}\tbinom{n}{tk}_{F_s(x)},$$

respectively. From (4.3) and (4.4) we see that the alternating sums of powers $\sum_{n=0}^{q}(-1)^{n}F_{tsn}^{k}(x)$ and $\sum_{n=0}^{q}(-1)^{n}L_{tsn}^{k}(x)$ can be written as linear combinations of the s-Fibopolynomials $\tbinom{q+m}{tk}_{F_s(x)}$, $m=1,2,\ldots,tk$, according to

$$\sum_{n=0}^{q}(-1)^{n}F_{tsn}^{k}(x)=(-1)^{s+1+tk+q} \tag{4.5}$$
$$\times\sum_{m=1}^{tk}\sum_{i=0}^{tk-m}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i+m}\binom{tk+1}{j}_{F_s(x)}F_{ts(i-j)}^{k}(x)\binom{q+m}{tk}_{F_s(x)},$$

and

$$\sum_{n=0}^{q}(-1)^{n}L_{tsn}^{k}(x)=(-1)^{s+1+tk+q} \tag{4.6}$$
$$\times\sum_{m=1}^{tk}\sum_{i=0}^{tk-m}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i+m}\binom{tk+1}{j}_{F_s(x)}L_{ts(i-j)}^{k}(x)\binom{q+m}{tk}_{F_s(x)},$$

if and only if we have that

$$\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i}\binom{tk+1}{j}_{F_s(x)}F_{ts(i-j)}^{k}(x)=0, \tag{4.7}$$

and

$$\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i}\binom{tk+1}{j}_{F_s(x)}L_{ts(i-j)}^{k}(x)=0, \tag{4.8}$$

respectively. Observe that, according to (4.1) and (4.2), we have

$$\begin{aligned}
&\left(x^2+4\right)^{\frac{k}{2}}\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i}\binom{tk+1}{j}_{F_s(x)}F_{ts(i-j)}^{k}(x)\,z^{tk-i}\\
&=\left(\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\frac{1}{z+\alpha^{lts}(x)\,\beta^{(k-l)ts}(x)}\right)\\
&\quad\times\left(\sum_{i=0}^{tk+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}+i}\binom{tk+1}{i}_{F_s(x)}z^{tk+1-i}\right),
\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}
&\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i}\binom{tk+1}{j}_{F_s(x)}L_{ts(i-j)}^{k}(x)\,z^{tk-i}\\
&=\left(\sum_{l=0}^{k}\binom{k}{l}\frac{1}{z+\alpha^{lts}(x)\,\beta^{(k-l)ts}(x)}\right)\\
&\quad\times\left(\sum_{i=0}^{tk+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}+i}\binom{tk+1}{i}_{F_s(x)}z^{tk+1-i}\right),
\end{aligned} \tag{4.10}$$

respectively. Then we need to consider now the following factors

$$\Omega_1(x,z)=\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\frac{1}{z+\alpha^{lts}(x)\,\beta^{(k-l)ts}(x)}, \tag{4.11}$$

$$\widetilde{\Omega}_1(x,z)=\sum_{l=0}^{k}\binom{k}{l}\frac{1}{z+\alpha^{lts}(x)\,\beta^{(k-l)ts}(x)}, \tag{4.12}$$

and

$$\Omega_2(x,z)=\sum_{i=0}^{tk+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}+i}\binom{tk+1}{i}_{F_s(x)}z^{tk+1-i}. \tag{4.13}$$

Plainly (4.7) is concluded from any of the conditions: (a) $\Omega_1(x,1)=0$, or, (b) $\Omega_1(x,1)<\infty$ and $\Omega_2(x,1)=0$, and (4.8) is concluded from any of the conditions: (a) $\widetilde{\Omega}_1(x,1)=0$, or, (b) $\widetilde{\Omega}_1(x,1)<\infty$ and $\Omega_2(x,1)=0$. In this section we give conditions on the parameters t, k and s, that imply (4.7) (for the Fibonacci case: proposition 4.1), and that imply (4.8) (for the Lucas case: proposition 4.3).

In the Fibonacci case we have the following result.

Proposition 4.1. *The alternating sum* $\sum_{n=0}^{q}(-1)^n F_{tsn}^{k}(x)$ *can be written as a linear combination of the s-Fibopolynomials* $\binom{q+m}{tk}_{F_s(x)}$, $m=1,2,\ldots,tk$, *according to (4.5), in the following cases*

	t	k	s
(a)	*any*	$\equiv 0 \bmod 4$	*any*
(b)	*any*	*even*	*even*
(c)	$\equiv 2 \bmod 4$	*any*	*odd*
(d)	*even*	*even*	*any*

Proof. Observe that in all the four cases we have tk even. Thus, according to (2.14) (with z replaced by $-z$) we can factor $\Omega_2(x,z)$ as

$$\Omega_2(x,z) = (-1)^s \left(z + (-1)^{\frac{tsk}{2}}\right) \prod_{j=0}^{\frac{tk}{2}-1} \left(z^2 + (-1)^{sj} L_{2s\left(\frac{tk}{2}-j\right)}(x)\, z + 1\right). \quad (4.14)$$

(a) Suppose that $k \equiv 0 \bmod 4$ and that t and s are any positive integers. In this case the factor $\left(z + (-1)^{\frac{tsk}{2}}\right)$ of (4.14) is $z+1$, so we have $\Omega_2(x,1) \neq 0$. However, by setting $z=1$ in (4.11), with k replaced by $4k$, we get

$$\Omega_1(x,1) = \sum_{l=0}^{2k-1} \binom{4k}{l} (-1)^l + \frac{1}{2}\binom{4k}{2k} = 0.$$

Thus (4.7) holds, as wanted.

(b) Suppose that k and s are even, and that t is any positive integer. In this case we have $z + (-1)^{\frac{tsk}{2}} = z+1$, and then $\Omega_2(x,1) \neq 0$. By setting $z=1$ in (4.11), with k and s substituted by $2k$ and $2s$, respectively, we get

$$\begin{aligned}\Omega_1(x,1) &= \sum_{l=0}^{2k} \binom{2k}{l} (-1)^l \frac{1}{1+\alpha^{2lts}(x)\,\beta^{(2k-l)2ts}(x)} \\ &= \sum_{l=0}^{k-1} \binom{2k}{l} (-1)^l + \frac{1}{2}\binom{2k}{k}(-1)^k = 0.\end{aligned}$$

Thus (4.7) holds, as wanted.

(c) Suppose that s is odd, $t \equiv 2 \bmod 4$, and k is any positive integer. If in (4.14) we set $z=1$ and replace t by $2(2t-1)$, we obtain that

$$\Omega_2(x,1) = \left(1 + (-1)^k\right) \prod_{j=0}^{(2t-1)k-1} \left((-1)^j L_{2s((2t-1)k-j)}(x) + 2\right). \quad (4.15)$$

We consider two sub-cases:

(c1) Suppose that k is even. In this case we have $\Omega_2(x,1) \neq 0$. But if we set $z = 1$ in (4.11), replace k by $2k$, and use that $t \equiv 2 \bmod 4$ and that s is odd, we obtain that

$$\Omega_1(x,1) = \sum_{l=0}^{k-1} \binom{2k}{l} (-1)^l + \frac{1}{2}\binom{2k}{k} (-1)^k = 0.$$

Thus (4.7) holds when k is even.

(c2) Suppose that k is odd. In this case we have clearly that $\Omega_2(x,1) = 0$. We check that $\Omega_1(x,1)$ is finite. If we set $z = 1$ in (4.11), substitute k by $2k-1$, and use that $t \equiv 2 \bmod 4$ and that s is odd, we obtain that

$$\Omega_1(x,1) = \sqrt{x^2+4}\sum_{l=0}^{k-1} \binom{2k-1}{l} (-1)^{l+1} \frac{F_{(2k-1-2l)ts}(x)}{2 + L_{(2k-1-2l)ts}(x)}.$$

Then we have $\Omega_1(x,1) < \infty$, as wanted. That is, expression (4.7) holds when k is odd.

(d) Suppose that k and t are even and s is any positive integer. In this case the factor $\left(z + (-1)^{\frac{tsk}{2}}\right)$ of (4.14) is $(z+1)$, so we have $\Omega_2(x,1) \neq 0$. Observe that, replacing k and t by $2k$ and $2t$, respectively (and letting s be any natural number) we obtain the same expression for $\Omega_1(x,1)$ of the case (b), namely

$$\Omega_1(x,1) = \sum_{l=0}^{2k} \binom{2k}{l} (-1)^l \frac{1}{1 + \alpha^{2lts}(x)\beta^{(2k-l)2ts}(x)},$$

which is equal to 0. That is, in this case we have also that $\Omega_1(x,1) = 0$, and we conclude that (4.7) holds. □

Corollary 4.2. *(a) If t is odd and $k \equiv 2 \bmod 4$, we have the following identity valid for any $s \in \mathbb{N}$*

$$\sum_{n=0}^{q} (-1)^{(s+1)n} F_{tsn}^k(x) = (-1)^{(s+1)(tk+q)} \tag{4.16}$$
$$\times \sum_{m=1}^{tk}\sum_{i=0}^{tk-m}\sum_{j=0}^{i} (-1)^{\frac{(sj+2(s+1))(j+1)}{2}+(s+1)(i+m)} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) \binom{q+m}{tk}_{F_s(x)}.$$

(b) If $t \equiv 2 \bmod 4$ and k is odd, we have the following identity valid for any $s \in \mathbb{N}$

$$\sum_{n=0}^{q} (-1)^{sn} F_{tsn}^k(x) = (-1)^{s(1+tk+q)+1} \tag{4.17}$$
$$\times \sum_{m=1}^{tk}\sum_{i=0}^{tk-m}\sum_{j=0}^{i} (-1)^{\frac{(sj+2(s+1))(j+1)}{2}+s(i+m)} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) \binom{q+m}{tk}_{F_s(x)}.$$

Proof. (a) When t is odd and $k \equiv 2 \bmod 4$, formula (4.16) with s odd gives the result (3.3) of case (b) of proposition 3.1 (which is valid for t odd, $k \equiv 2 \bmod 4$ and s odd). Similarly, for t odd and $k \equiv 2 \bmod 4$, formula (4.16) with s even, gives the result (4.5) of case (b) of proposition 4.1 (which is valid for k and s even and any t).

(b) When $t \equiv 2 \bmod 4$ and k is odd, formula (4.17) with s even, gives the result (3.3) of case (a) of proposition 3.1 (which is valid for t even, k odd and s even). Similarly, for $t \equiv 2 \bmod 4$ and k odd, formula (4.17) with s odd, gives the result (4.5) of case (c) of proposition 4.1 (which is valid for $t \equiv 2 \bmod 4$, s odd and any k). $\square$

We give examples from the cases considered in corollary 4.2. Beginning with the case (a), by setting $t = 1$ and $k = 2$ in (4.16), we have the following identity, valid for $s \in \mathbb{N}$

$$\sum_{n=0}^{q} (-1)^{(s+1)(n+q)} F_{sn}^2(x) = F_s^2(x) \binom{q+1}{2}_{F_s(x)}. \tag{4.18}$$

The case $t = s = x = 1$ and $k = 6$ of (4.16) is

$$\sum_{n=0}^{q} F_n^6 = \binom{q+1}{6}_F + \binom{q+5}{6}_F - 11\left(\binom{q+2}{6}_F + \binom{q+4}{6}_F\right) - 64\binom{q+3}{6}_F.$$

This identity is mentioned in [17] (p. 259), and previously was obtained in [15] with the much more simple right-hand side $\frac{1}{4}\left(F_q^5 F_{q+3} + F_{2q}\right)$.

An example from the case (b) of corollary 4.2 is the following identity, valid for $s \in \mathbb{N}$ (obtained by setting $t = 2$ and $k = 1$ in (4.17))

$$\sum_{n=0}^{q} (-1)^{s(n+q)} F_{2sn}(x) = F_{2s}(x) \binom{q+1}{2}_{F_s(x)}. \tag{4.19}$$

From (4.18) and (4.19) we see that

$$\begin{aligned} &(-1)^{(s+1)q} L_s(x) \sum_{n=0}^{q} (-1)^{(s+1)n} F_{sn}^2(x) \\ &= (-1)^{sq} F_s(x) \sum_{n=0}^{q} (-1)^{sn} F_{2sn}(x) = F_{s(q+1)}(x) F_{sq}(x). \end{aligned} \tag{4.20}$$

The simplest example from the case (a) of proposition 4.1, corresponding to $k = 4$ and $t = 1$, is the following identity valid for any $s \in \mathbb{N}$

$$\begin{aligned} &\sum_{n=0}^{q} (-1)^{n+q} F_{sn}^4(x) \\ &= F_s^4(x) \left(\binom{q+1}{4}_{F_s(x)} + \binom{q+3}{4}_{F_s(x)} + (3(-1)^s L_{2s}(x) + 4)\binom{q+2}{4}_{F_s(x)}\right). \end{aligned} \tag{4.21}$$

With some patience one can see that the case $x = 1$ of (4.21) is

$$\sum_{n=0}^{q} (-1)^n F_{sn}^4 = \frac{(-1)^q F_{sq} F_{s(q+1)} \left(L_s L_{sq} L_{s(q+1)} - 4L_{2s}\right)}{5 L_s L_{2s}},$$

demonstrated by Melham [13].

An example from the case (d) of proposition 4.1, corresponding to $t = k = 2$, is the following identity valid for any $s \in \mathbb{N}$

$$\begin{aligned} &\sum_{n=0}^{q} (-1)^{n+q} F_{2sn}^2 (x) \qquad (4.22) \\ &= F_{2s}^2 (x) \left(\binom{q+1}{4}_{F_s(x)} + (-1)^{s+1} L_{2s}(x) \binom{q+2}{4}_{F_s(x)} + \binom{q+3}{4}_{F_s(x)} \right). \end{aligned}$$

Now we consider alternating sums of powers of Lucas polynomials.

Proposition 4.3. *The alternating sum* $\sum_{n=0}^{q} (-1)^n L_{tsn}^k (x)$ *can be written as a linear combination of the s-Fibopolynomials* $\binom{q+m}{tk}_{F_s(x)}$, $m = 1, 2, \ldots, tk$, *according to (4.6), if s and k are odd positive integers and* $t \equiv 2 \bmod 4$.

Proof. We will show that in the case stated in the proposition we have $\widetilde{\Omega}_1 (x, 1) < \infty$ and $\Omega_2 (x, 1) = 0$, which implies (4.8). Since s and k are odd, and $t \equiv 2 \bmod 4$, the factor $\left(z + (-1)^{\frac{tsk}{2}}\right)$ of (4.14) is $(z - 1)$, so we have $\Omega_2 (x, 1) = 0$. Let us see that $\widetilde{\Omega}_1 (x, 1) < \infty$. If in (4.12) we set $z = 1$ and replace k by $2k - 1$, we get for $t \equiv 2 \bmod 4$ and s odd that $\widetilde{\Omega}_1 (x, 1) = 4^{k-1}$, as wanted. □

Corollary 4.4. *If $t \equiv 2 \bmod 4$ and k is odd, we have the following identity valid for any* $s \in \mathbb{N}$

$$\begin{aligned} &\sum_{n=0}^{q} (-1)^{sn} L_{tsn}^k (x) = (-1)^{s(1+tk+q)+1} \qquad (4.23) \\ &\times \sum_{m=1}^{tk} \sum_{i=0}^{tk-m} \sum_{j=0}^{i} (-1)^{\frac{(sj+2(s+1))(j+1)}{2} + s(i+m)} \binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^k (x) \binom{q+m}{tk}_{F_s(x)}. \end{aligned}$$

Proof. When $t \equiv 2 \bmod 4$ and k is odd, formula (4.23) with s even, gives the result (3.15) of case (a) of Proposition 3.3 (which is valid for t even, k odd and s even). Similarly, if $t \equiv 2 \bmod 4$ and k is odd, formula (4.23) with s odd, gives the result (4.6) of Proposition 4.3 (which is valid for $t \equiv 2 \bmod 4$, k odd and s odd). □

An example of (4.23) is the following identity (corresponding to $t = 2$ and $k = 1$), valid for any $s \in \mathbb{N}$

$$\sum_{n=0}^{q} (-1)^{s(n+q)} L_{2sn} (x) = 2\binom{q+2}{2}_{F_s(x)} - L_{2s} (x) \binom{q+1}{2}_{F_s(x)}, \qquad (4.24)$$

which can be written as

$$\sum_{n=0}^{q}(-1)^{s(n+q)}L_{2sn}(x)=\frac{1}{F_s(x)}L_{sq}(x)F_{s(q+1)}(x). \tag{4.25}$$

5. Further results

To end this work we want to present (in two propositions) some examples of identities obtained as derivatives of some of our previous results. We will use the identities

$$\frac{d}{dx}F_n(x)=\frac{1}{x^2+4}\left(nL_n(x)-xF_n(x)\right). \tag{5.1}$$

$$\frac{d}{dx}L_n(x)=nF_n(x). \tag{5.2}$$

One can see that these formulas are true by checking that both sides of each one have the same Z transform. By using (2.2) and (2.8) we see that the Z transform of both sides of (5.1) is $z^2\left(z^2-xz-1\right)^{-2}$, and that the Z transform of both sides of (5.2) is $z\left(z^2+1\right)\left(z^2-xz-1\right)^{-2}$.

Proposition 5.1. *The following identities hold*

$$(-1)^{(s+1)q}2L_s(x)\sum_{n=0}^{q}(-1)^{(s+1)n}nF_{2sn}(x) \tag{5.3}$$
$$=2qF_{s(2q+1)}(x)+F_{sq}(x)L_{s(q+1)}(x)-\frac{\left(x^2+4\right)F_s(x)}{L_s(x)}F_{s(q+1)}(x)F_{sq}(x).$$

$$(-1)^{sq}2F_s(x)\sum_{n=0}^{q}(-1)^{sn}nL_{2sn}(x) \tag{5.4}$$
$$=2qF_{s(2q+1)}(x)+F_{sq}(x)L_{s(q+1)}(x)-\frac{L_s(x)}{F_s(x)}F_{s(q+1)}(x)F_{sq}(x).$$

Proof. We will use (4.20), which contains two identities, namely

$$(-1)^{(s+1)q}L_s(x)\sum_{n=0}^{q}(-1)^{(s+1)n}F_{sn}^2(x)=F_{s(q+1)}(x)F_{sq}(x), \tag{5.5}$$

and

$$(-1)^{sq}F_s(x)\sum_{n=0}^{q}(-1)^{sn}F_{2sn}(x)=F_{s(q+1)}(x)F_{sq}(x). \tag{5.6}$$

By using that $L_{sq}(x) F_{s(q+1)}(x) + F_{sq}(x) L_{s(q+1)}(x) = 2F_{s(2q+1)}(x)$ (see (1.6)), we can see that

$$\begin{aligned}
&\frac{d}{dx}\left(F_{s(q+1)}(x) F_{sq}(x)\right) \\
&= \frac{1}{x^2+4}\left(2sqF_{s(2q+1)}(x) + sF_{sq}(x) L_{s(q+1)}(x) - 2xF_{s(q+1)}(x) F_{sq}(x)\right).
\end{aligned}$$

The derivative of the left-hand side of (5.5) is

$$\begin{aligned}
&\frac{d}{dx}\left((-1)^{(s+1)q} L_s(x) \sum_{n=0}^{q} (-1)^{(s+1)n} F_{sn}^2(x)\right) \\
&= (-1)^{(s+1)q} L_s(x) \sum_{n=0}^{q} \frac{(-1)^{(s+1)n}}{x^2+4} 2snF_{2sn}(x) \\
&\quad + \left(sF_s(x) - \frac{2xL_s(x)}{x^2+4}\right) \frac{1}{L_s(x)} F_{s(q+1)}(x) F_{sq}(x).
\end{aligned}$$

Then, the derivative of (5.5) is

$$\begin{aligned}
&(-1)^{(s+1)q} L_s(x) \sum_{n=0}^{q} \frac{(-1)^{(s+1)n}}{x^2+4} 2snF_{2sn}(x) \\
&\quad + \left(sF_s(x) - \frac{2xL_s(x)}{x^2+4}\right) \frac{1}{L_s(x)} F_{s(q+1)}(x) F_{sq}(x) \\
&= \frac{1}{x^2+4}\left(2sqF_{s(2q+1)}(x) + sF_{sq}(x) L_{s(q+1)}(x) - 2xF_{s(q+1)}(x) F_{sq}(x)\right)
\end{aligned}$$

from where (5.3) follows.

The derivative of the left-hand side of (5.6) is

$$\begin{aligned}
&\frac{d}{dx}\left((-1)^{sq} F_s(x) \sum_{n=0}^{q} (-1)^{sn} F_{2sn}(x)\right) \\
&= \frac{(-1)^{sq}}{x^2+4} F_s(x) \sum_{n=0}^{q} (-1)^{sn} 2snL_{2sn} + \frac{sL_s(x) - 2xF_s(x)}{F_s(x)(x^2+4)} F_{s(q+1)}(x) F_{sq}(x).
\end{aligned}$$

Thus, the derivative of (5.6) is

$$\begin{aligned}
&\frac{(-1)^{sq}}{x^2+4} F_s(x) \sum_{n=0}^{q} (-1)^{sn} 2snL_{2sn} + \frac{sL_s(x) - 2xF_s(x)}{F_s(x)(x^2+4)} F_{s(q+1)}(x) F_{sq}(x) \\
&= \frac{1}{x^2+4}\left(2sqF_{s(2q+1)}(x) + sF_{sq}(x) L_{s(q+1)}(x) - 2xF_{s(q+1)}(x) F_{sq}(x)\right),
\end{aligned}$$

from where (5.4) follows. $\square$

Proposition 5.2. *The following identity holds*

$$\begin{aligned}(-1)^{sq} F_s(x) \sum_{n=0}^{q} (-1)^{sn} nF_{2sn}(x) \qquad (5.7)\\ = \frac{1}{x^2+4}\left(\frac{(-1)^{s+1}}{F_s(x)} F_{2sq}(x) + qL_{s(2q+1)}(x)\right).\end{aligned}$$

Proof. Identity (5.7) is the derivative of (4.25), together with

$$F_s(x) L_{s(q+1)}(x) - L_s(x) F_{s(q+1)}(x) = 2(-1)^{s+1} F_{sq}(x),$$

and

$$\left(x^2+4\right) F_{sq}(x) F_{s(q+1)}(x) + L_{sq}(x) L_{s(q+1)}(x) = 2L_{s(2q+1)}(x).$$

(See (1.6) and (1.7).) We leave the details of the calculations to the reader. □

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