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Proof of the Tojaaldi sequence conjectures

Russell Jay Hendel[a], **Thomas J. Barrale**[b], **Michael Sluys**[c]

[a]Towson University
RHendel@Towson.Edu

[b]The Kenjya Group
Tom.Barrale@Kenjya.Com

[c]The Kenjya Group
Michael.Sluys@Kenjya.Com

Abstract

Heuristically, the base b, size a Tojaaldi sequence of size k, $\mathcal{T}_k^{(a,b)}$, is the sequence of initial digits of the $(k+1)-$digit Generalized Fibonaaci numbers, defined by $F_0^{(a)} = 0, F_1^{(a)} = 1, F_n^{(a)} = aF_{n-1}^{(a)} + F_{n-2}^{(a)}, n \geq 2$. For example, $\mathcal{T}_2^{(1,10)} = \langle 1, 2, 3, 6, 9 \rangle$ corresponding to the initial digits of the three-digit Fibonacci numbers, 144, 233, 377, 610, 987. In [1] we showed that (eventually) there are at most b Tojaaldi sequences and conjectured that there are exactly b Tojaaldi sequences. Based on computer studies we also conjectued that the Tojaaldi sequences are Benford distributed. We prove these two conjectures

Keywords: Tojaaldi, Fibonacci, initial digits, Benford

MSC: 11B37 11B39

1. Introduction and goals

The goal of this paper is to prove the two conjectures presented in [1]. For purposes of completeness we will repeat the necessary definitions, conventions and theorems from [1]. For pedagogic purposes we will also repeat key illustrative examples. However, the reader should consult [1] for details on proofs and the well-definedness of definitions.

An outline of this paper is as follows: In this section we present all necessary definitions and propositions. In the next section we state the main Theorems of [1] as well as the two conjectures. In the final section we prove the conjectures.

Notational Conventions. Throughout this paper if $\{n \in \mathbb{N} : P(n)\}$ is the set of integers with property P then we notationally indicate the sequence of such integers (with the natural order inherited from the integers) by $\langle n \in \mathbb{N} : P(n)\rangle$. Throughout this paper discrete sequences and sets will be notationally indicated with angle brackets and braces respectively.

Definition 1.1. For integers $a \geq 1, n \geq 0$, the *generalized Fibonacci numbers* are defined by

$$F_0^{(a)} = 0, F_1^{(a)} = 1, F_n^{(a)} = aF_{n-1}^{(a)} + F_{n-2}^{(a)}, \qquad n \geq 2.$$

The generalized Fibonacci numbers can equivalently be defined by their Binet form

$$F_n^{(a)} = \frac{\alpha_a^n - \beta_a^n}{D}, D = \alpha_a - \beta_a = \sqrt{a^2+4}, \alpha_a = \frac{a+D}{2}, \beta_a = \frac{a-D}{2}. \tag{1.1}$$

When speaking about the generalized Fibonacci numbers, if we wish to explicitly note the dependence on a, we will use the phrase *the a-Fibonacci numbers*.

The following identity is useful when making estimates.

Lemma 1.2. *For integers $k \geq 1, m \geq 1$,*

$$F_{m+k}^{(a)} = \alpha_a^k F_m^{(a)} + F_k^{(a)} \beta_a^m. \tag{1.2}$$

Definition 1.3. The base b, *a-Tojaaldi sequence* of size k is defined and notationally indicated by

$$\mathcal{T}_k^{(a,b)} = \left\langle \left\lfloor \frac{F_n^{(a)}}{b^k} \right\rfloor : n \geq 1, b^k \leq F_n^{(a)} < b^{k+1} \right\rangle, \qquad k \geq 0. \tag{1.3}$$

The base b, *a-Tojaaldi set* (of the a-Fibonacci numbers) is defined and notationally indicated by

$$\mathcal{T}^{(a,b)} = \{\mathcal{T}_k^{(a,b)} : 0 \leq k < \infty\}.$$

Example 1.4. Heuristically, a Tojaaldi sequence is the sequence of initial digits of all base b size a Fibonacci numbers, with a fixed number of digits. So, for example, $\mathcal{T}_2^{(1,10)} = \langle 1, 2, 3, 6, 9\rangle$, corresponding to the initial digits of the 3-digit Fibonacci numbers: 144,233,377,610,987.

Remark 1.5. The theorems of this paper carry over to the generalized Lucas numbers with extremely minor modifications.

The Tojaaldi sequences were initially studied by Tom Barrale who manually compiled tables of them from 1997-2007. Michael Sluys then contributed computing resources enabling computation of Tojaaldi sequences for the first (approximately) half million Fibonacci numbers. This computer study was replicated by Hendel using alternate algorithms. This computer study contains important information about the distribution of Tojaaldi sequences which is the basis of the conjecture that the Tojaaldi sequences are Benford distributed.

The name Tojaaldi is an acronym formed from the initial two letters of Barrale's family: Thomas, Jared, Allison, and Dianne, his eldest, second eldest son, daughter and wife respectively. (The third letter of "Thomas" was used rather than the second because it is a vowel.)

Definition 1.6. For integers $b \geq 2, a \geq 1$, $n_0(a,b)$ is the smallest positive integer such that

$$F_n^{(a)} = i \cdot b^j, \text{ is not solvable for integers } 1 \leq i \leq b-1, n \geq n_0(a,b). \tag{1.4}$$

Example 1.7. Clearly, $n_0(1,10) = 1$, $n_0(2,10) = 1$ and $n_0(1,12) = 13$.

Definition 1.8. For integer k, *n(k) = n(k,a,b)* is the unique integer defined by the equation

$$F_{n(k)}^{(a)} < b^k \leq F_{n(k)+1}^{(a)}, \qquad k \geq 1. \tag{1.5}$$

Definition 1.9. For fixed integers $a \geq 1$ and $b \geq 2$, *j(a,b)* is the unique non-negative integer satisfying the inequality,

$$\alpha_a^{j(a,b)} < b < \alpha_a^{j(a,b)+1}. \tag{1.6}$$

Definition 1.10. Let $k_1(a,b)$ be the smallest positive integer such that for all $k \geq k_1(a,b)$, (i) $n(k) \geq n_0(a,b)$, and (ii) $n(k) \geq j(a,b)$. An integer $k \geq k_1(a,b)$ will be called *non-trivial* while other positive integers will be called *trivial*. Similarly, a Tojaaldi sequence $\mathcal{T}_k^{(a,b)}$ will be called *non-trivial* if k is non-trivial. We notationally indicate the set of all non-trivial, base b, a-Tojaaldi sequences, by $\overline{\mathcal{T}}^{(a,b)}$.

Lemma 1.11. *For non-trivial* k,

$$\#\mathcal{T}_k^{(a,b)} \in \{j(a,b), j(a,b)+1\}. \tag{1.7}$$

Proof. [1, Proposition 2.5]. □

Example 1.12. $j(1,10) = 4, n_0(1,10) = 1$, and $n(1,1,10) = 6$. Hence, by (1.7), $\mathcal{T}_0^{(1,10)}$ is the only base 10, 1-Tojaaldi sequence with 6 elements.

Lemma 1.13. *If k is non-trivial then (i)* $F_{n(k)}^{(a)} \leq i \cdot b^k, 1 \leq i \leq b-1 \Rightarrow F_{n(k)}^{(a)} < i \cdot b^k$ *(ii)* $\#\mathcal{T}_k^{(a,b)} \in \{j(a,b), j(a,b)+1\}$, *(iii)* $F_{n(k)+p}^{(a)} > b^k \Leftrightarrow \alpha_a^p F_{n(k)}^{(a)} > b^k, 1 \leq p \leq j(a,b)+1$.

Proof. [1, Proposition 2.8]. □

Remark 1.14. Non-triviality was introduced to avoid only a few aberrent Tojaaldi sequences such as $\mathcal{T}_0^{(1,10)}$. In general, restricting ourselves to non-trivial sequences is not that restrictive. For example, $k_1(1,10) = 1$ and $k_1(1,12) = 3$.

Definition 1.15. For fixed a, b, and $x \in [\alpha_a^{-1}, 1)$, the *base b, real, a-Tojaaldi sequence of x* is defined by

$$T_x^{(a,b)} = \langle \lfloor \alpha_a^k x \rfloor : 1 \leq k \leq m, \text{ with } m \text{ defined by } \alpha_a^m x < b \leq \alpha_a^{m+1} x \rangle.$$

Remark 1.16. $\mathcal{T}_z^{(a)}$ has different definitions depending on whether z is an integer or non-integer. This should cause no confusion in the sequel since the meaning will always be clear from the context.

Definition 1.17. For integer k, $a \geq 1$, and $b \geq 2$,

$$x = x(k) = x(k, a, b) = \frac{F_{n(k)}^{(a)}}{b^k}, \qquad k \geq 1. \tag{1.8}$$

Lemma 1.18. *For integer* k, $a \geq 1$, *and* $b \geq 2$,

$$\mathcal{T}_{x(k)}^{(a,b)} = \mathcal{T}_k^{(a,b)}, \tag{1.9}$$

and

$$x(k) \in (\alpha_a^{-1}, 1). \tag{1.10}$$

Proof. [1, Proposition 2.14] □

Definition 1.19. For each integer, $1 \leq i \leq b$, *e(i) = e(i,a)* is the unique integer satisfying. $\alpha_a^{e(i)-1} \leq i < \alpha_a^{e(i)}$.

Definition 1.20. The *(a, b)-partition* refers to

$$\langle B_i : 1 \leq i \leq b+1 \rangle = \langle 1, \frac{i}{\alpha_a^{e(i)}} : 1 \leq i \leq b \rangle. \tag{1.11}$$

Remark 1.21. By our notational convention on the use of angle brackets, the B_i simply sequentially order the $\{\frac{j}{\alpha_a^{e(j)}}\}_{1 \leq j \leq b}$. Consequently, the $B_i, 1 \leq i \leq b+1$, partition the interval $[\frac{1}{\alpha_a}, 1)$, into b semi-open intervals with $B_1 = \alpha_a^{-1}$ and $B_{b+1} = 1$.

Example 1.22. Table 1 presents the (1,10)-parition and other useful information.

Lemma 1.23. *For a fixed* $a \geq 1, b \geq 2, (a, b) -$ *partition*, $\langle B_i : 1 \leq i \leq b+1 \rangle$, *and a real* $y \in [B_m, B_{m+1}), 1 \leq m \leq b$,

$$\mathcal{T}_{B_m}^{(a,b)} = \mathcal{T}_y^{(a,b)}. \tag{1.12}$$

Proof. [1, Proposition 2.15]. □

Example 1.24. $x = x(1, 1, 10) = 0.8$. Inspecting Table 1,

$$x \in [B_6, B_7) = [0.76, 0.81).$$

It is then straightforward to verify, as shown in Table 1, that

$$\mathcal{T}_{0.8}^{(1,10)} = \langle 1, 2, 3, 5, 8 \rangle = \mathcal{T}_1^{(1,10)}. \tag{1.13}$$

$\frac{1}{\alpha}$	$\frac{7}{\alpha^5}$	$\frac{3}{\alpha^3}$	$\frac{8}{\alpha^5}$	$\frac{5}{\alpha^4}$	$\frac{2}{\alpha^2}$	$\frac{9}{\alpha^5}$	$\frac{6}{\alpha^4}$	$\frac{10}{\alpha^5}$	$\frac{4}{\alpha^3}$	1
0.62	0.63	0.71	0.72	0.73	0.76	0.81	0.88	0.90	0.94	1.00
11246	11247	11347	11348	11358	12358	12359	12369	1236	1246	
3,888	21,250*	3,396*	2,068*	8,515	11,158**	13,980*	5,465*	8,515	10,583*	88,818

Table 1: Row 3 of this table contains the ten base 10, 1-Tojaaldi sequences of size at least 1. Row 4 presents the numerical frequencies of Tojaaldi sequences. Row 1 contains the (1,10)-partition of $[\alpha^{-1}, 1)$ by $B_i, 1 \le i \le b$, defined in Definition 1.17. Row 2 contains two digit numerical approximations of the B_i. In row 4, the number of asterisks indicate the difference between (actual) observed and Benford (predicted) frequencies, $88818 \cdot \frac{\log(B_{i+1}) - \log(B_i)}{\log(1) - \log(\alpha^{-1})}$. To illustrate our notation, there are 11158 occurrences of the Tojaaldi sequence $\langle 1, 2, 3, 5, 8 \rangle$ among the Tojaaldi sequences of sizes 1 to 88818. The Benford densities described in Definition 1.28 and Proposition 1.29, predict there should be $88818 \cdot \frac{\left(\log(9) - \log(\alpha^5)\right) - \left(\log(2) - \log(\alpha^2)\right)}{\log(\alpha)} \approx 11156$ occurrences, and hence we have placed two asterisks on the 11158 entry to indicate a difference of two between the observed and predicted frequencies.

In the sequel we will assume integers a, b are fixed. This will allow us to ease notation and drop the functional dependency on a, b. So for example we will speak about k_1 instead of $k_1(a, b)$.

In the sequel we will speak about an integer $K \ge k_1(a, b)$. In several proofs we will speak about the effect of K growing arbitrarily large.

Definition 1.25. The sequence $\{y(k)\}_{k \ge K}$, is recursively defined by

$$y(K) = x(K) = \frac{F^{(a)}_{n(K)}}{b^K},$$
$$y(k) = y(k-1) \begin{cases} \frac{\alpha_a^{j+1}}{b}, & \text{if } y(k-1)\frac{\alpha_a^{j+1}}{b} < 1, \\ \frac{\alpha_a^j}{b}, & \text{if } y(k-1)\frac{\alpha_a^{j+1}}{b} > 1, \end{cases} \qquad \text{for } k > K. \tag{1.14}$$

Definition 1.26. The sequence $\{n_y(k)\}_{k \ge K}$, is defined by $n_y(k) = 0$, for $k < K$, and

$$n_y(k) = n_y = \#\{K \le i \le k : y(i)\frac{\alpha_a^{j+1}}{b} > 1\}, \qquad k \ge K. \tag{1.15}$$

Lemma 1.27.

$$y(k) = \frac{F^{(a)}_{n(K)}}{b^K} \left(\frac{\alpha_a^j}{b}\right)^{k-K} \alpha_a^{n_y(k-1)}, \qquad \textit{for } k \ge K. \tag{1.16}$$

Proof. A straightforward induction. □

Definition 1.28. The sequence $\{n_x(k)\}_{k \geq K}$, is defind by $n_x(k) = 0$, for $k < K$, and

$$n_x(k) = \#\{K \leq i \leq k : x(i)\frac{\alpha_a^{j+1}}{b} > 1\}, \qquad k \geq K. \tag{1.17}$$

Remark 1.29. The definitions and propositions we have just presented are almost identical to those in [1, Section 3]. The sole difference is that [1] restricts these definitions and propositions to the case $K = k_1$ while here, we have allowed $K > k_1$. It is this small subtlety which will allow us to prove that most $x(k)$ are arbitrarily close to $y(k)$ for large enough $k > K$.

Example 1.30. Let $a = 1, b = 10$. Then $k_1(a, b) = 1$. By (1.14) and (1.8),

$$\{y(1), \ldots, y(4)\} = \{\frac{F_6}{10} = 0.8, 0.8872, 0.9839, 0.6744\} \approx$$
$$\{x(1), \ldots, x(4)\} = \{\frac{8}{10}, \frac{89}{100}, \frac{987}{1000}, \frac{6765}{10000}\}.$$

Note that $x(i) - y(i) \approx 0.003$.

Definition 1.31. An integer $k \geq K$ will be called *exceptional* relative to (a, b) if $n_x(k-1) \neq n_y(k-1)$. Otherwise, k will be called *non-exceptional.*

Example 1.32. let $a = 1, b = 10$. Then $j(a, b) = 4$ and $n(1, a, b) = 6$.

By Definition 1.14, $x(44) = \frac{F_{212}}{10^{44}} = 0.9034$, to four decimal places. By Definition 1.21, $y(44) = 0.9006$. But $y(44)\frac{\alpha_a^5}{10} = 0.9988 < 1$, while $x(44)\frac{\alpha_a^5}{10} = 1.0019 > 1$, and consequently $x(44) \neq y(44)$, implying by Definition 1.26 that 45 is exceptional.

Note, that by Definition 1.21, $y(45) = 0.9988$. while by Definition 1.14, $x(45) = 0.6192$.

Hence, for the exceptional value of 45, $x(45)$ and $y(45)$ are not close. In fact, $y(45) - x(45) > 0.37$. The "spikes" in Figures 1 and 2 correspond to the exceptional integers and show that they are rare.

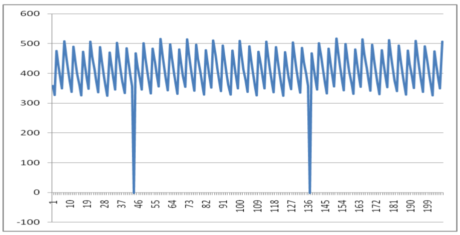

Figure 1: Distribution of $\lfloor \frac{1}{x(n)-y(n)} + 0.5 \rfloor$ for $2 \leq n \leq 200$, for the 1-Fibonacci numbers and base 10. The $x(n)$ and $y(n)$ are defined in Definitions 1.14 and 1.21 respectively.

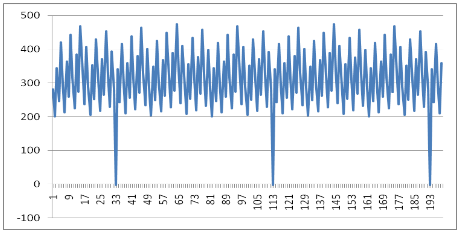

Figure 2: Distribution of $\lfloor \frac{1}{x(n)-y(n)} + 0.5 \rfloor$ for $2 \leq n \leq 200$, for the 2-Fibonacci numbers and base 10. The $x(n)$ and $y(n)$ are defined in Definitions 1.14 and 1.21 respectively.

Definition 1.33. Let $[a, b)$ be an interval on the real line and let $\mathbf{X} \sim Uniform([a, b))$ be a random variable uniformly distributed over this space. If for some constant $c > 1$, the random variable $\mathbf{Y}$ satisfies $\mathbf{Y} = c^{\mathbf{X}}, c > 1$, over the space $[c^a, c^b)$, then we say that $\mathbf{Y}$ is *Benford* distributed over $[c^a, c^b)$, and we notationally indicate this by $\mathbf{Y} \sim Benford([c^a, c^b)$.

Lemma 1.34. *If* $\mathbf{Y} \sim Benford([c^a, c^b)$, *then for* $c^a \leq c_1 \leq c_2 \leq c^b$,

$$Prob(c_1 < \mathbf{Y} < c_2) = \frac{\log_c(\frac{c_2}{c_1})}{b - a}.$$

Remark 1.35. For a proof see [1, Proposition 4.3]. For general references on the Benford distribution see the bibliography in [1]. Notice that the restriction of the spaces and random variables $\mathbf{Y}$ and $\mathbf{X}$ to spaces of countable dense subsets of $[a, b)$ does not change the proposition conclusion.

Example 1.36. Table 1, which presents 88,818 Tojaaldi sequences, allows illustration of the Benford sequence (and Conjecture 2).

Each of these 88,818 Tojaaldi sequences involve 4 or 5 Fibonacci numbers. Thus the 88,818 Tojaaldi sequences involve $3888 \times 5 + 21250 \times 5 + \ldots + 10583 \times 4 = 424992$ Fibonacci numbers. Since the Fibonacci numbers are Benford distributed, we *expect* $\log_{10}(\frac{10}{9}) \times 88818 = 19446.6$ Fibonacci numbers beginning with 9. But $\langle 1, 2, 3, 5, 9 \rangle$ and $\langle 1, 2, 3, 6, 9 \rangle$ are the only Tojaaldi sequences having Fibonacci numbers beginning with 9; so we *observe* $13980 + 5465 = 19445$ Fibonacci numbers beginning with 9.

We can repeat this numerical exercise for each digit (besides 9). We can then compute the $\chi-$square statistic, $\chi^2 = \sum_{i=1}^{9} \frac{(O_i - P_i)^2}{P_i} = 0.0004$ showing a very strong agreement between theory and observed frequency for the Fibonacci-number frequencies.

Similarly, as outlined in the caption to Table 1, we may compute *observed* and *expected* Tojaaldi-sequence frequencies; the associated $\chi-$square statistic is 0.0013, suggesting that the Tojaaldi sequences are Benford distributed. This numerical study motivates Conjecture 2 which will be formally stated in the next section and proven in the final section of this paper.

Definition 1.37. The uniform discrete measure used when making statements about frequency of Tojaaldi sequences on initial segments of integers, is given by the following discrete probablity measure.

$$P_{\mathbf{L}}(\mathcal{T}_{k_0}^{(a,b)}) = \frac{\#\{k : \mathcal{T}_k^{(a,b)} = \mathcal{T}_{k_0}^{(a,b)}, 1 \le k \le L\}}{\#\{\mathcal{T}_k^{(a,b)} : 1 \le k \le L\}}, \qquad L \ge 1, \tag{1.18}$$

with # indicating cardinality and k_0, k, L are integers.

2. Main theorems and conjectures

Conjecture 1. For all $b \ge 2, a \ge 1$, $\#\overline{\mathcal{T}}^{(a,b)} = b$.

Theorem 2.1. *For $b > 1$, and arbitrary $a \ge 1$,*

$$\#\overline{\mathcal{T}}^{(a,b)} \le b.$$

Proof. [1, Theorem 2.9] □

Lemma 2.2. *For given (a, b) let $\langle B_i : 1 \le i \le b+1 \rangle$ be an (a, b)-partition, and let z_0 be an arbitrary point in the real space $[\alpha_a^{-1}, 1)$ with the continuous uniform measure. Then*

$$Prob(\mathcal{T}_z = \mathcal{T}_{z_0}) = \frac{\mu([B_i, B_{i+1}))}{\mu([\alpha_a^{-1}, 1))},$$

where i is picked so that $[B_i, B_{i+1})$ contains z_0.

Proof. [1, Theorem 4.1] □

Theorem 2.3. *For any integer $K \ge k_1$, $\{y(i) : i \ge K\}$ is Benford distributed over the space $[\alpha^{-1}, 1)$.*

Proof. [1, Theorem 4.5] (with K replacing k_1 throughout the proof.) □

Conjecture 2. The $\{x(i)\}_{i \ge k_1}$ are Benford distributed.

3. Proof of the two conjectures

In this section we prove the two main conjectures which we restate as theorems. Prior to doing so we will need some preliminary propositions.

Lemma 3.1. *$\{n_y(k)\}_{k \ge K}$ is non-decreasing and unbounded as k goes to infinity.*

Proof. By Definition 1.22, $n_y(k)$ is non-decreasing. Suppose contrary to the proposition there is a k_0 such that for all $k \geq k_0$, $n_y(k) = n_y(k_0)$. We proceed to derive a contradiction, proving that $n_y(k)$ is unbounded as k goes to infinity.

First, we show, using an inductive argument, that $y(k) \in (\alpha_a^{-1}, 1)$, for $k > K$. The base case, when $k = K$ is established by Definition 1.21 and equation (1.10). The induction step is established by Definitions 1.21 and 1.8.

Returning to the proof of Proposition 3.1, note that according to Definition 1.21, there are two cases to consider, according to whether $y(k_0)\frac{\alpha_a^{j+1}}{b} < 1$, or $y(k_0)\frac{\alpha_a^{j+1}}{b} > 1$. We assume $y(k_0)\frac{\alpha_a^{j+1}}{b} < 1$, the treatment of the other case being almost identical. Then since we assumed $n_y(k) = n_y(k_0), k \geq k_0$, we have $y(k_0)\left(\frac{\alpha_a^{j+1}}{b}\right)^n < 1$, for all integer $n \geq 0$, a contradiction, since by Definition 1.8, $\left(\frac{\alpha_a^{j+1}}{b}\right)^n$ goes to infinity as n gets arbitrary large. This contradiction shows that our original assumption that $y(k)$ is bounded is false. This completes the proof. □

Lemma 3.2. *For non-exceptional $k > K$*

$$|x(k) - y(k)| \in \left(\alpha_a^{-2n(K)-1}, \alpha_a^{-2n(K)}\right). \tag{3.1}$$

Proof. [1, Proposition 3.6] with K replacing k_1 in both the proposition statement and throughout the proof. □

Remark 3.3. As noted in the previous section, because we replaced k_1 by K, the lower bound estimate of the difference in (3.1) is going to 0. Consequently $\{x(i)\}_{i \geq K}$ is asymptotically approaching $\{y(i)\}_{i \geq K}$. Formally, we have the following Corollary.

Corollary 3.4. *As k varies over non-exceptional k,*

$$\lim_{k \to \infty} |x(k) - y(k)| = 0.$$

Proof. Immediate, by combining Propositions 3.1 and 3.2. □

Lemma 3.5. *Using Definition 1.17, let $\langle B_i : 1 \leq i \leq b+1 \rangle = \langle 1, \frac{i}{\alpha_a^{e(i)}} : 1 \leq i \leq b \rangle$ be an (a, b)-partition. Then the $\#\{T_{B_i}, 1 \leq i \leq b\} = b$, that is, the T_{B_i} are distinct.*

Proof. Following [1, Proposition 2.15], define a $b \times j(a, b) + 1$ matrix, $A(k, l) = B_k \alpha_a^l, 1 \leq k \leq b, 1 \leq l \leq j(a, b) + 1$, so that by Definition 1.13 $T_{B_k} = \langle \lfloor A(k, 1) \rfloor, \ldots, \lfloor A(k, m) \rfloor \rangle$, and by Definitions 1.16, 1.17 and 1.8, m equals $j(a, b)$ or $j(a, b) + 1$. Recall the following facts about the matrix A :
(I) $A(k, e(i(k))) = i(k)$; (II) no other cell entries (besides $(k, e(i(k))))$ can have exact integer values; (III) A is strictly increasing as one goes from top to bottom and left to right, that is, $A(k, l) < A(k', l')$ if either (i)$l < l'$ or (ii) $l = l', k < k'$.

Using these three facts we see that $\lfloor A(k', e(i(k)) \rfloor < A(k, e(i(k))$, for $k' < k$, $1 \leq k \leq b, i(k) \neq b$. Hence, $T_{B_{k'}} \neq T_{B_k}$, for $k' < k$. An almost identical argument applies when $i(k) = b$. Hence the T_{B_i} are distinct as was to be shown. □

Example 3.6. We can illustrate the proof using Table 1. By Table 1, $B_4 = \frac{8}{\alpha^5}$, implying that the 5th member of the sequence T_{B_4} equals 8 and the 5th member of the previous sequences, $T_{B_k}, 1 \leq k < 4$, are strictly less than 8 as confirmed by Table 1.

Note also the special case $B_9 = \frac{10}{\alpha^5}$, implying that the 5th member of the sequence T_{B_9} is empty while the 5th member of the previous sequences, $T_{B_k}, 1 \leq k \leq 8$, are non-empty, as confirmed by Table 1.

The next three propositions show that exceptional k (as defined in Definition 1.26) are rare. First we prove the following proposition, which provides an alternate recursive definition to $x(k)$, defined in Definition 1.14.

Lemma 3.7. *The sequence* $\{x(k)\}_{k \geq K}$, *is recursively defined by*

$$x(K) = \frac{F^{(a)}_{n(K)}}{b^K},$$

$$x(k) = \begin{cases} x(k-1)\frac{\alpha_a^{j+1}}{b} + F^{(a)}_{j+1}\frac{\beta_a^{n(k-1)}}{b^k}, & \text{if } x(k-1)\frac{\alpha_a^{j+1}}{b} < 1, \\ x(k-1)\frac{\alpha_a^{j}}{b} + F^{(a)}_{j}\frac{\beta_a^{n(k-1)}}{b^k}, & \text{if } x(k-1)\frac{\alpha_a^{j+1}}{b} > 1, \qquad \text{for } k > K. \end{cases} \tag{3.2}$$

Proof. If $k = K$ the proposition is true by Definition 1.14. If $k > K$, then by Definitions 1.3, 1.7 and Proposition 1.10

$$n(k) - n(k-1) = \#T^{(a,b)}_{k-1} \in \{j, j+1\}.$$

Consequently, there are two cases to consider. We treat the case $n(k) = n(k-1) + j$, the treatment of the other case, $n(k) = n(k-1) + j + 1$, being similar.

But then, by Proposition 1.12,

$$F^{(a)}_{n(k)} = \alpha_a^j F^{(a)}_{n(k-1)} + F^{(a)}_j \beta_a^{n(k-1)}.$$

Equation (3.2), follows by dividing both sides of this last equation by b^k and applying Defintion 1.14. □

Prior to stating the next two propositions, it may be useful to numerically illustrate the proof method. The following example continues Example 1.27.

Example 3.8. Let $a = 1, b = 10$. Then by Definition 1.8, $j(a,b) = 4$. By Definitions 1.22 and 1.24,

$$n_y(43) = n_x(43),$$

implying by Definition 1.26, that 44 is not exceptional. By Definition 1.21, $y(44) = 0.9006$; by Definition 1.14, $x(44) = 0.9034$. Application of Defintions 1.22 and 1.24 require use of $\frac{\alpha_a^5}{10} = 0.9017$. Observe that

$$y(44) = 0.9006 < 0.9017 < 0.9034 = x(44).$$

Consequently,

$$y(44)\frac{\alpha_a^5}{10} < 1; x(44)\frac{\alpha_a^5}{10} > 1.$$

Therefore, by Definitions 1.22 and 1.24

$$n_y(44) = n_y(43) + 1; n_x(44) = n_x(43).$$

Hence, by Definition 1.26, $k = 45$ is an exceptional value. Notice that $y(44)$ and $x(44)$ are close in value as predicted by Proposition 3.2. The values of $x(45)$ and $y(45)$ may now be computed using Definition 1.22 and Proposition 3.6,

$$y(45) = 0.9988; x(45) = 0.6192.$$

Here, $y(45)$ and $x(45)$ are not close. More precisely, $y(45)$ is close to 1 while $x(45)$ is close to α_a^{-1}.

But by applying Definitions 1.22 and 1.24 we see that

$$n_y(45) = n_y(44); n_x(45) = n_x(44) + 1,$$

implying that

$$n_y(45) = n_x(45),$$

in other words, 46 is not exceptional. We in fact confirm that $y(46)$ and $x(46)$ are indeed close as required.

$$y(46) = 0.6846 < 0.6867 = x(46).$$

We may summarize this numerical example as follows: (I) Most k are non-exceptional. (II) For an exceptional k to occur, one of $x(k-1), y(k-1)$ must be greater than $\frac{\alpha_a^{j+1}}{b}$ while the other is less. (III) This occurs rarely because most k are non-exceptional and hence, by Proposition 3.2, $x(k)$ and $y(k)$ are usually numerically close. (IV) If k is exceptional then $x(k)$ will be close to α_a^{-1} while $y(k)$ will be close to 1. (V) Consequently $k+1$ will not be exceptional and in fact $x(k+1)$ and $y(k+1)$ will again be close to each other.

The next proposition formalizes this example.

Lemma 3.9. *If k is exceptional then $k-1$ and $k+1$ are non-exceptional.*

Proof. Assume that k is exceptional and $k-1$ is not exceptional. This assumption is allowable, since by Definitions 1.21, 1.22 and 1.24, K and $K+1$ are not exceptional and therefore the "first" exceptional k must be preceded by a non-exceptional value. We proceed to show that $k+1$ is not exceptional. Therefore, the "2nd" exceptional k is preceded by a non-exceptional k. Proceeding in this manner we will always be justified if we assume the predecessor of an exceptional k is not exceptional. Consequently, we have left to prove that $k+1$ is not exceptional.

By Definitions 1.26, 1.22 and 1.24, for k to be exceptional we must have one of $x(k-1)\frac{\alpha_a^{j+1}}{b}$ and $y(k-1)\frac{\alpha_a^{j+1}}{b}$ greater than one while the other is less than one. We treat one of these cases, the treatment of the other case being similar.

Accordingly, we assume

$$n_y(k-2) = n_x(k-2) \longrightarrow \ k-1 \text{ is not exceptional}, \tag{3.3}$$

and we further assume

$$y(k-1) < \Big(\frac{\alpha_a^{j+1}}{b}\Big)^{-1} < x(k-1) \longrightarrow y(k-1)\frac{\alpha_a^{j+1}}{b} < 1, x(k-1)\frac{\alpha_a^{j+1}}{b} > 1. \quad (3.4)$$

Combining Proposition 3.2 with (3.4) we obtain

$$y(k-1) > \Big(\frac{\alpha_a^{j+1}}{b}\Big)^{-1} - \frac{1}{\alpha_a^{2n(K)}}, x(k-1) < \Big(\frac{\alpha_a^{j+1}}{b}\Big)^{-1} + \frac{1}{\alpha_a^{2n(K)}}. \quad (3.5)$$

Hence, by Definition 1.21 and Proposition 3.6,

$$y(k) = y(k-1)\frac{\alpha_a^{j+1}}{b}, x(k) = x(k-1)\frac{\alpha_a^{j}}{b} + F_j^{(a)}\frac{\beta_a^{n(k-1)}}{b^k}. \quad (3.6)$$

Using equation (3.4), Definitions 1.26, 1.22 and 1.24, we confirm that

$$n_y(k-1) = n_y(k-2), n_x(k-1) = n_x(k-2)+1 \longrightarrow k \text{ is exceptional.} \quad (3.7)$$

Again, by Definition 1.26, to decide whether $k+1$ is exceptional we need to compute $n_y(k)$ and $n_x(k)$. We first compute $n_y(k)$.

Appplying equations (3.6) and (3.5) to Definition 1.22, we have

$$y(k)\frac{\alpha_a^{j+1}}{b} = y(k-1)\Big(\frac{\alpha_a^{j+1}}{b}\Big)^2 > \frac{\alpha_a^{j+1}}{b} - \frac{\alpha_a^{2j+2}}{b^2\alpha_a^{2n(K)}}. \quad (3.8)$$

j and b are $O(1)$ (relative to the choice of K) while we may chose K arbitrarily large. It follows that as K goes to infinity,

$$y(k)\frac{\alpha_a^{j+1}}{b} > \frac{\alpha_a^{j+1}}{b} - \frac{\alpha_a^{2j+2}}{b^2\alpha^{2n(K)}} \approx \frac{\alpha_a^{j+1}}{b} > 1. \quad (3.9)$$

Consequently by (3.9), Definition 1.22, and (3.7)

$$n_y(k) = n_y(k-1)+1 = n_y(k-2)+1. \quad (3.10)$$

.

We now carry out a similar analysis on $x(k)$. By Proposition 3.6 we have

$$x(k)\frac{\alpha_a^{j+1}}{b} = \Big(x(k-1)\frac{\alpha_a^{j}}{b} + F_j^{(a)}\frac{\beta_a^{n(k-1)}}{b^k}\Big)\frac{\alpha_a^{j+1}}{b} \quad (3.11)$$

Applying the upper bound for $x(k-1)$ presented in (3.5) we obtain after some straightforward manipulations

$$x(k)\frac{\alpha_a^{j+1}}{b} < \frac{\alpha_a^{j}}{b} + \alpha_a^{2j+1-2n(K)}\frac{1}{b^2} + F_j^{(a)}\alpha_a^{(j+1)}\frac{\beta_a^{n(k-1)}}{b^{k+1}} \approx \frac{\alpha_a^{j}}{b} < 1. \quad (3.12)$$

Hence, by Definition 1.24 and equation (3.7),

$$n_x(k) = n_x(k-1) = n_x(k-2)+1. \quad (3.13)$$

Equations (3.10) and (3.13) together imply that $n_x(k) = n_y(k)$, and hence, by Definition 1.26, $k+1$ is not exceptinoal as was to be shown.

This completes the proof. □

Lemma 3.10. *$Prob(\{k : k$ is exceptional$\}) = 0$.*

Proof. By Proposition 3.8, exceptional k occur as singletons (that is, two consecutive integers cannot be exceptional). Furthermore, by Proposition 3.2, if k is exceptional $k-1$ is non-exceptional and

$$x(k-1), y(k-1) \in \Big(\big(\frac{\alpha_a^{j+1}}{b}\big)^{-1} - \big(\alpha_a^{2n(K)}\big)^{-1}, \big(\frac{\alpha_a^{j+1}}{b}\big)^{-1} + \big(\alpha_a^{2n(K)}\big)^{-1} \Big).$$

By Theorem 2.3 the $\{y(i)\}_{i \geq K}$ are Benford distributed and hence the probability of $y(k-1)$ being in an open interval whose width is going to 0, may be made as small as we please.

But by Proposition 3.8 every exceptional k is uniquely associated with a non-exceptional k.

This completes the proof. □

We can now prove the two conjectures.

Theorem 3.11. *The $\{x(n)\}_{n \geq 1}$ are Benford distributed.*

Proof. Consider an arbitrary set (of reals), $B \subset (\alpha_a^{-1}, 1)$. To prove the theorem, we must show that $Prob(B \cap \{x(n)\}_{n \geq 1})$ equals the desired Benford-distribution probability.

By Definition 1.28 and Proposition 1.29 we know that $Prob(B \cap \{y(n)\}_{n \geq 1}) = \frac{\log(M_y) - \log(m_y)}{\log(1) - \log(\alpha_a^{-1})}$, with $M_y = \sup(B \cap \{y(n)\}_{n \geq 1})$ and $m_y = \inf(B \cap \{y(n)\}_{n \geq 1})$. Define $M_x = \sup(B \cap \{x(n)\}_{n \geq 1})$ and $m_x = \inf(B \cap \{x(n)\}_{n \geq 1})$. By Corollary 3.3, $|M_y - M_x|$ and $|m_y - m_x|$ can be made arbitrarily small. The result immediately follows. □

Theorem 3.12. *For all $b \geq 2, a \geq 1$, $\#\overline{\mathcal{T}}^{(a,b)} = b$.*

Proof. By Theorem 2.1, $\#\overline{\mathcal{T}}^{(a,b)} \leq b$. It therefore suffices to prove $\#\overline{\mathcal{T}}^{(a,b)} \geq b$. The proof is constructive.

Using Definition 1.17, let $\langle B_i : 1 \leq i \leq b+1 \rangle = \langle 1, \frac{i}{\alpha_a^{e(i)}} : 1 \leq i \leq b \rangle$ be an (a,b)-partition. For $1 \leq i \leq b$, pick a non-exceptional $x(n_i) \in (B_i, B_{i+1})$, for some integer n_i. $x(n_i)$ exists since by Theorem 3.7, $\{x(n)\}_{n \geq 1}\}$ is Benford distributed and hence dense in $(\alpha_a^{-1}, 1)$.

But then by Proposition 1.19, $T_{x(n_i)} = T_{B_i}$; by Proposition 3.4, the T_{B_i} are distinct; and by Proposition 1.15, $T_{x(n_i)} = T_{n_i}$. Hence, we have produced at least b distinct Tojaaldi sequences as was to be shown. □

References

[1] Tom Barrale, Russell Hendel, and Michael Sluys *Sequences of the Initial Digits of Fibonacci Numbers*, Proceedings of the 14th International Conference on Fibonacci Number, (2011), 25-43.