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Problem proposals

compiled by Clark Kimberling

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Problem 1 (posed by Heiko Harborth).
For $F_{13} = 233$ and $F_{18} = 2584$, this holds:

$$\sigma(F_{13}) + \sigma(F_{18}) = 2(F_{13} + F_{18}).$$

Are there further pairs of Fibonacci numbers equalizing their abundance and deficiency?

Problem 2 (posed by Heiko Harborth).
For 5 and 14, this holds: 5 is 14-perfect and 14 is 5-perfect, where n is h-perfect if

$$\sigma(n) + \sigma(nh) = 2(n + hn).$$

Are there further pairs a, b such that a is b-perfect and b is a-perfect?

Problem 3 (posed by Heiko Harborth).
Find numbers n that are h-perfect for more than one value of h, where n is h-perfect if

$$\sigma(n) + \sigma(nh) = 2(n + hn).$$

Examples: 135 is 7-perfect and 55-perfect, and 5 is h-perfect for $h \in \{14, 806, 1166\}$.

Problem 4 (posed by Clark Kimberling).
Let r_n be the greatest eigenvalue of the n^{th} principal submatrix of the Fibonacci self-fusion matrix, M. Let s_n be the greatest eigenvalue of the n^{th} principal submatrix of the Fibonacci self-fission matrix, $\widetilde{M}$. Prove or disprove:

$$\lim_{n\to\infty} \frac{r_{n+1}}{r_n} = \lim_{n\to\infty} \frac{s_{n+1}}{s_n} = \frac{3+\sqrt{5}}{2}$$

(The matrices M and $\widetilde{M}$ are presented in the *Online Encyclopedia of Integer Sequences* at A202453 and A202503.)

Problem 5 (posed by Bill Webb).
A monic polynomial, all of whose coefficients are negative, will be called a negative polynomial. Characterize polynomials that divide some negative polynomial. (For example, every linear polynomial divides a negative polynomial.)

Problem 6 (posed by Joseph Lahr).
Evaluate these sums:

$$\sum_{n=1}^{k} F_{n^2} \quad \text{and} \quad \sum_{n=1}^{k} L_{n^2}.$$

These sums are comparable to $\sum_{n=1}^{k} e^{n^2}$, which occurs in the Fourier transform of chirp-signals, as typifed by the equation $S_n = A\cos(an^2)$.

Problem 7 (posed by Larry Ericksen).
Let $p(n)$ denote the n^{th} prime, and let n_k denote the k^{th} value of n for which $p(n)+2$ is prime. Find all k such that $k(k+1)$ divides $p(n_k)+1$. Example: $k=8$, $n_8 = 20$, $p(20) = 71$, $p(20)+1 = 8\cdot 9$. In other words, $k(k+1)$ divides the average of the twin primes $p(n_k)$ and $p(n_k)+2$.

Problem 8 (posed by Larry Ericksen).
Let $p(m)$ denote the m^{th} prime. Find all pairs (m,n) such that reversing the digits of m yields n and reversing the digits of $p(m)$ yields $p(n)$. Example: $m = 12$, $n = 21$, $p(m) = 37$, $p(n) = 73$.

Problem 9 (posed by Lawrence Somer).
Let $ax^2+bxy+cy^2$ be a binary quadratic form with a,b,c integers and discriminant $D = b^2-4ac \neq 0$. Suppose that p is a prime such that $p \nmid D$.

(a) Do there exist integers x_0, y_0 such that

$$\left(\frac{ax_0^2+bx_0y_0+cy_0^2}{p}\right) = -1,$$

where $\left(\frac{n}{p}\right)$ denotes the Legendre symbol?

(b) Answer (a) with $a = 1$.

(c) Answer (a) with $a = 1$ and $c = \pm 1$.

(d) Answer (a) with $a = 1$ and p such that $\left(\frac{-D}{p}\right) = 1$.

Problem 10 (posed by Neville Robbins).
A *Wilf partition* of n is a partition such that all distinct parts have distinct multiplicities, as in $6 = 4+1+1$. Let $f(n)$ be the number of Wilf partitions of n, as typified by

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$f(n)$	1	1	2	2	4	5	7	10	13	15	21	28	31

and sequence A098859 in the *Online Encyclopedia of Integer Sequences.*

(a) Prove that $f(n)$ is strictly increasing for $n \geq 3$.

(b) Obtain an explicit formula or recurrence for $f(n)$.

Problem 11 (posed by Gabriele Gelatti).
Examples gleaned from visual art suggest that if N is a positive integer, then the product

$$F_{n-4}F_{n-3}F_{n-2}F_{n-1}F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}$$

is equal to a polynomial function of $F_n, F_n^2, \dots, F_n^9$. Following the presentation of this problem, Kristóf Huszár sketched a proof that $F_{n-k}F_{n+k} = F_n^2 + (-1)^{n-k+1}F_k^2$, which implies that

$$F_n \prod_{i=1}^{k} F_{n-i}F_{n+i} = F_n \prod_{i=1}^{k} (F_n^2 + (-1)^{n-i+1}F_k^2),$$

a polynomial in F_n of degree $2k+1$. Subsequently, Bill Webb described a general form of identity, as follows. Let $k = 4t+1$, where $t > 0$ (or, one may also start with $k = 4t$ or $k = 4t+2$ or $k = 4t+3$.) For any given $j_1, j_2, \dots, j_k$, the product

$$F_{n+j_1}F_{n+j_2}\cdots F_{n+j_k}$$

can be written in the form

$$\begin{aligned} &a_1F_{kn} + a_2(-1)^nF_{(k-2)n} + a_3F_{(k-3)n} + \cdots + a_{2t+1}F_n \\ &+ b_1F_{k(n+1)} + b_2(-1)^nF_{(k-2)(n+1)} + b_3F_{(k-3)(n+1)} + \cdots + b_{2t}F_{n+1}. \end{aligned} \qquad (1)$$

The values of a_i and b_i are easily calculated as solutions of $k+1$ linear equations. The terms $F_{r(n+1)}$ can be replaced by F_{rn+s_i} or L_{rn+s_i}, and similarly for the terms F_{rn}. It appears likely that the correspondence between $(j_1, j_2, \dots, j_k)$ and the coefficients a_i and b_i includes interesting cases; for example, when is (1) "short"?

Problem 12 (posed by Clark Kimberling, Heiko Harborth, and Peter Moses).
Discuss the triangular arrangements (as indicated by the example below, or of other sorts) of the numbers $1, 2, \dots, n(n+1)/2$ that have interlacing rows; i.e., each term in the first $n-1$ rows is between the two numbers just below it. For $n = 3$:

$$\begin{array}{ccccc} & & 3 & & \\ & 2 & & 5 & \\ 1 & & 4 & & 6 \end{array} \qquad \begin{array}{ccccc} & & 3 & & \\ & 2 & & 5 & \\ 1 & & 4 & & 6 \end{array} \qquad \begin{array}{ccccc} & & 3 & & \\ & 2 & & 5 & \\ 4 & & 1 & & 6 \end{array}$$

$$\begin{array}{ccccc} & & 3 & & \\ & 2 & & 4 & \\ 5 & & 1 & & 6 \end{array} \qquad \begin{array}{ccccc} & & 3 & & \\ & 2 & & 4 & \\ 6 & & 1 & & 5 \end{array}$$

$$\begin{array}{ccccc} & & 4 & & \\ & 2 & & 5 & \\ 1 & & 3 & & 6 \end{array} \qquad \begin{array}{ccccc} & & 4 & & \\ & 2 & & 5 & \\ 1 & & 6 & & 3 \end{array} \qquad \begin{array}{ccccc} & & 4 & & \\ & 2 & & 5 & \\ 3 & & 1 & & 6 \end{array}$$

$$\begin{array}{ccccc} & & 4 & & \\ & 3 & & 5 & \\ 1 & & 6 & & 2 \end{array} \qquad \begin{array}{ccccc} & & 4 & & \\ & 3 & & 5 & \\ 2 & & 6 & & 1 \end{array}$$

Problem 13 (posed by Curtis Cooper)**.**
Find, or prove the nonexistence of, an algebraic identity of the form

$$(r_1x^2 + s_1xy + t_1y^2)^4 + (r_2x^2 + s_2xy + t_2y^2)^4$$
$$= (r_3x^2 + s_3xy + t_3y^2)^4 + (r_4x^2 + s_4xy + t_4y^2)^4 + (r_5x^2 - s_5xy - t_5y^2)^4,$$

where x and y are variables, r_i are positive integers, s_i and t_i are nontrivial integers, $s_5 > 0$, and $t_5 = \pm 1$.

Problem 14 (posed by Augustine Munagi)**.**
Give an explicit bijective proof of the following proposition. The number of compositions of n in which 2 may appear only as a first or last part equals the number of compositions of $n+1$ in which 2 is not a part.

Example: A005251$(n+2)$ is the number of compositions of n having at most two 2s, which may occur only at endpoints; e.g., for $n = 4$, the compositions are (4), $(1,3)$, $(3,1)$, $(1,1,1,1)$, $(2,2)$, $(1,1,2)$, $(2,1,1)$. For the other kind, A005251$(n+1)$ is the number of compositions of n having no 2; e.g., for $n = 5$, the compositions are (5), $(1,4)$, $(4,1)$, $(1,1,3)$, $(1,3,1)$, $(3,1,1)$, $(1,1,1,1)$.