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On factors of sums of consecutive Fibonacci and Lucas numbers

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Abstract

The Problem B-1 in the first issue of the Fibonacci Quarterly is the starting point of an extensive exploration of conditions for factorizations of several types of sums involving Fibonacci and Lucas numbers.

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1. Introduction

Recall the Problem B-1 proposed by I. D. Ruggles of San Jose State College on the page 73 in the initial issue of the journal Fibonacci Quarterly in February 1963.

Problem B-1. *Show that the sum of twenty consecutive Fibonacci numbers is divisible by F_{10}.*

In the third issue of this first volume on pages 76 and 77 there is a solution using induction by Marjorie R. Bicknell also of San Jose State College.

With a little help from computers one can easily solve the above problem (using Maple V or Mathematica) and discover many other similar results. It is the purpose of this paper to present some of these discoveries. The proofs of all our claims could be done by induction. We shall leave them as the challenge to the readers.

There are many nice summation formulas for Fibonacci and Lucas numbers in the literature (see, for example, [1], [2], [3], [4] and [5]). We hope that the readers will find the ones that follow also interesting.

2. Sums of $4\,i+4$ consecutive Fibonacci numbers

In the special case (for $i = 4$) the following theorem provides another solution of the Problem B-1. It shows that the sums $\sum_{j=0}^{4i+3} F_{k+j}$ have the Fibonacci number F_{2i+2} as a common factor.

Theorem 2.1. *For integers $i \geq 0$ and $k \geq 0$, the following identities hold:*

$$\sum_{j=0}^{4i+3} F_{k+j} = F_{2i+2}\, L_{k+2i+3} = F_{k+4i+5} - F_{k+1} = F_{2i}\, L_{k+2i+5} + L_{k+3} =$$
$$L_{2i+1}\, F_{k+2i+4} + F_{k+2} = L_{2i}\, F_{k+2i+5} - 3\, F_{k+3} = F_{2i+1}\, L_{k+2i+4} - L_{k+2}.$$

The other identities in Theorem 1 have some importance in computations because they show that in order to get the big sum we need to know initial terms and two terms in the middle. The second representation is not suitable as the number F_{k+4i+5} is rather large.

3. The alternating sums

It is somewhat surprising that the (opposites of the) alternating sums of $4i+4$ consecutive Fibonacci numbers also have $F_{2\,i+2}$ as a common factor. Hence, the alternating sums of twenty consecutive Fibonacci numbers are all divisible by F_{10}.

Theorem 3.1. *For integers $i \geq 0$ and $k \geq 0$, the following identities hold:*

$$-\sum_{j=0}^{4i+3} (-1)^j\, F_{k+j} = F_{2i+2}\, L_{k+2i} = F_{k+4i+2} - F_{k-2} = L_{2i}\, F_{k+2i+2} - 3\, F_k$$
$$= F_{2i+1}\, L_{k+2i+1} - L_{k-1} = F_{2i-1}\, L_{k+2i+3} - 2\, L_{k+1} = L_{2i-1}\, F_{k+2i+3} + 4\, F_{k+1}.$$

4. Sums of $4\,i+2$ consecutive Fibonacci numbers

Similar results hold also for the (alternating) sums of $4\,i+2$ consecutive Fibonacci numbers. The common factor is the Lucas number L_{2i+1}. Hence, all (alternating) sums of twenty-two consecutive Fibonacci numbers are divisible by L_{11}.

Theorem 4.1. *For integers $i \geq 0$ and $k \geq 0$, the following identities hold:*

$$\sum_{j=0}^{4i+1} F_{k+j} = L_{2i+1}\, F_{k+2i+2} = F_{k+4i+3} - F_{k+1} = L_{2i-1}\, F_{k+2i+4} + L_{k+3}$$
$$= L_{2i+2}\, F_{k+2i+1} - L_k = F_{2i+3}\, L_{k+2i} - 3\, F_{k-1} = F_{2i+5}\, L_{k+2i-1} - 7\, F_{k-3}.$$

$$-\sum_{j=0}^{4i+1}(-1)^j\,F_{k+j} = L_{2i+1}\,F_{k+2i-1} = F_{k+4i} - F_{k-2} = F_{2i-1}\,L_{k+2i+1} - 3\,F_k$$
$$= L_{2i}\,F_{k+2i} - L_{k-1} = L_{2i-2}\,F_{k+2i+2} - 2\,L_{k+1} = F_{2i-2}\,L_{k+2i+2} + 4\,F_{k+1}.$$

5. Sums with $4\,i+1$ and $4\,i+3$ terms

One can ask about the formulas for the (alternating) sums of $4\,i+1$ and $4\,i+3$ consecutive Fibonacci numbers. The answer provides the following theorem. These sums do not have common factors. However, they are sums of two familiar type of products (like $F_{2i}\,F_{k+2i+3}$ and $F_{2i+1}\,F_{k+2i}$).

Theorem 5.1. *For integers $i \geq 0$ and $k \geq 0$, the following identities hold:*

$$\sum_{j=0}^{4i} F_{k+j} = F_{2i}\,F_{k+2i+3} + F_{2i+1}\,F_{k+2i} =$$
$$F_{2i}\,L_{k+2i} + L_{2i+1}\,F_{k+2i} = F_{k+4i+2} - F_{k+1} = L_{2i+2}\,F_{k+2i} - 2\,F_k$$
$$= F_{2i-1}\,L_{k+2i+3} - 2\,F_{k+3} = L_{2i+1}\,F_{k+2i+1} - F_{k-1}.$$

$$\sum_{j=0}^{4i}(-1)^j\,F_{k+j} = F_{2i-1}\,F_{k+2i-1} + F_{2i+2}\,F_{k+2i-2} =$$
$$L_{2i+1}\,F_{k+2i} - F_{2i}\,L_{k+2i} = F_{k+4i-1} + F_{k-2} = L_{2i-1}\,F_{k+2i} + 2\,F_k.$$

$$\sum_{j=0}^{4i+2} F_{k+j} = F_{2i+2}\,F_{k+2i+4} - F_{2i+1}\,F_{k+2i+1} = F_{k+4i+4} - F_{k+1} =$$
$$L_{2i+1}\,F_{k+2i+3} + F_k = L_{2i+2}\,F_{k+2i+2} - F_{k+2} = F_{2i+2}\,L_{k+2i+2} - F_{k-1}.$$

$$\sum_{j=0}^{4i+2}(-1)^j\,F_{k+j} = F_{2i+1}\,F_{k+2i+1} + F_{2i+2}\,F_{k+2i-2} =$$
$$L_{2i+3}\,F_{k+2i+1} - 2\,F_{2i+2}\,L_{k+2i} = F_{k+4i+1} + F_{k-2} = F_{2i}\,L_{k+2i+1} + 2\,F_k.$$

6. Sums of consecutive Lucas numbers

The above results suggests to consider many other sums especially when they are products or when they have very simple values.

The first that come to mind are the same sums of consecutive Lucas numbers. A completely analogous study could be done in this case. Here we only give a sample of two such identities.

$$\sum_{j=0}^{4i+3} L_{k+j} = 5\,F_{2i+2}F_{k+2i+3}, \quad \sum_{i=0}^{4i+1} L_{k+j} = L_{2i+1}L_{k+2i+2}.$$

7. Sums of consecutive products

Let us now consider sums of consecutive products of consecutive Fibonacci numbers. For an even number of summands the Fibonacci number F_{2i+2} is a common factor. Let $A = (-1)^k$.

$$\sum_{j=0}^{2i} F_{k+j}\,F_{k+j+1} = \tfrac{L_{2i+1}L_{2k+2i+1}-A}{5},$$

$$\sum_{j=0}^{2i+1} F_{k+j}\,F_{k+j+1} = F_{2i+2}F_{2k+2i+2}.$$

The same for the Lucas numbers gives the following identities:

$$\sum_{j=0}^{2i} L_{k+j}\,L_{k+j+1} = L_{2i+1}L_{2k+2i+1} + A,$$

$$\sum_{j=0}^{2i+1} L_{k+j}\,L_{k+j+1} = 5\,F_{2i+2}F_{2k+2i+2}.$$

We shall get similar identities in the two cases when Fibonacci and Lucas numbers both appear in each summand on the left hand side.

$$\left(\sum_{j=0}^{2i} F_{k+j}\,L_{k+j+1}\right) + A = \left(\sum_{j=0}^{2i} L_{k+j}\,F_{k+j+1}\right) - A = L_{2i+1}F_{2k+2i+1},$$

$$\sum_{j=0}^{2i+1} F_{k+j}\,L_{k+j+1} = \sum_{j=0}^{2i+1} L_{k+j}\,F_{k+j+1} = F_{2i+2}L_{2k+2i+2}.$$

8. Sums of squares of consecutive numbers

Our next step is to consider sums of squares of consecutive Fibonacci and Lucas numbers. Note that once again the summation of even and odd number of terms each lead to a separate formula. In fact, we consider a more general situation when multiples of a fixed number are used as indices of the terms in the sum. Only the parity of this number determines the form of the formula for the sum.

Theorem 8.1. *For all integers i, $k \geq 0$ and $v \geq 1$, we have*

$$\sum_{j=0}^{2i} F_{k+2vj}^2 = \frac{F_{2v(2i+1)}\, L_{2k+4vi}}{5\, F_{2v}} - \frac{2\, A}{5},$$

$$\sum_{j=0}^{2i+1} F_{k+2vj}^2 = \frac{F_{4v(i+1)}\, L_{2k+2v(2i+1)}}{5\, F_{2v}} - \frac{4\, A}{5},$$

$$\sum_{j=0}^{2i} L_{k+2vj}^2 = \frac{F_{2v(2i+1)}\, L_{2k+4vi}}{F_{2v}} + 2\, A,$$

$$\sum_{j=0}^{2i+1} L_{k+2vj}^2 = \frac{F_{4v(i+1)}\, L_{2k+2v(2i+1)}}{F_{2v}} + 4\, A,$$

Theorem 8.2. *For all integers i, $k \geq 0$ and $v \geq 0$, we have*

$$\sum_{j=0}^{2i} F_{k+(2v+1)j}^2 = \frac{L_{(2i+1)(2v+1)}\, L_{2k+2i(2v+1)}}{5\, L_{2v+1}} - \frac{2\, A}{5},$$

$$\sum_{j=0}^{2i+1} F_{k+(2v+1)j}^2 = \frac{F_{2(i+1)(2v+1)}\, F_{2k+(2i+1)(2v+1)}}{L_{2v+1}}.$$

$$\sum_{j=0}^{2i} L_{k+(2v+1)j}^2 = \frac{L_{(2i+1)(2v+1)}\, L_{2k+2i(2v+1)}}{L_{2v+1}} + 2\, A,$$

$$\sum_{j=0}^{2i+1} L_{k+(2v+1)j}^2 = \frac{5\, F_{2(i+1)(2v+1)}\, F_{2k+(2i+1)(2v+1)}}{L_{2v+1}}.$$

In particular, for $v = 0$ and $i = 9$, we conclude that the sums of squares of twenty consecutive Fibonacci numbers are divisible by F_{20} and the same sums of Lucas numbers by $5\, F_{20}$.

9. More sums of products

Here are some additional sums that are products or very close to the products.

$$\sum_{j=1}^{2i} F_j\, F_{k+j} = F_{2i-2}\, F_{k+2i+3} + F_{k+3} = F_{2i}\, F_{k+2i+1},$$

$$\sum_{j=1}^{2i+1} F_j\, F_{k+j} = F_{2i}\, F_{k+2i+3} + F_{k+1} = F_{2i+2}\, F_{k+2i+1}.$$

$$\sum_{j=0}^{2i} L_j\, L_{k+j} = L_{k+4i+1} + L_{k-2} = F_{2i+1}\, L_{k+2i+1} + F_{2i+2}\, L_{k+2i-2},$$

$$\sum_{j=0}^{2i+1} L_j\, L_{k+j} = L_{k+4i+3} - L_{k-1} = 5\, F_{2i+2}\, F_{k+2i+1}.$$

$$\sum_{j=0}^{2i} L_j\, F_{k+j} = F_{2i+2}\, L_{k+2i-1} + F_{k-1} = F_{2i}\, L_{k+2i+1} + 2\, F_k =$$
$$F_{2i+1}\, L_{k+2i-2} + L_{2i+2}\, F_{k+2i-2} = F_{2i}\, L_{k+2i-1} + L_{2i}\, F_{k+2i},$$

$$\sum_{j=0}^{2i+1} L_j\, F_{k+j} = \sum_{j=1}^{2i+1} F_j\, L_{k+j} = F_{2i}\, L_{k+2i+3} + L_{k+1} =$$
$$L_{2i+1}\, F_{k+2i+2} + F_k = F_{2i+2}\, L_{k+2i+1}.$$

$$\sum_{j=1}^{2i} F_j\, L_{k+j} = F_{2i+2}\, L_{k+2i-1} - L_{k-1} = L_{2i+1}\, F_{k+2i} - F_k =$$
$$F_{2i+1}\, L_{k+2i} - L_k = L_{2i}\, F_{k+2i+1} - 2\, F_{k+1} = F_{2i}\, L_{k+2i+1},$$

10. Sums of products of three numbers

In this final section we shall consider two sums of three consecutive Fibonacci and Lucas numbers when once again the common factor appears.

Theorem 10.1. *Let u be either $4i+1$ or $4i+3$. For all integers $i \geq 0$ and $k \geq 0$, we have*

$$\sum_{j=1}^{u} F_{k+j}\, F_{k+2j}\, F_{k+3j} = F_{u+1} \left[\tfrac{P}{4} - \tfrac{Q - A\,S}{10} - \tfrac{A\,R}{6} \right],$$

$$\sum_{j=1}^{u} L_{k+j}\, L_{k+2j}\, L_{k+3j} = 5\, F_{u+1} \left[\tfrac{Q - 2\,P}{4} - \tfrac{A\,R}{2} + \tfrac{A\,S}{6} \right],$$

with

$$P = F_{3k+20i+10} + F_{3k+12i+6} + F_{3k+4i+2}, \qquad R = F_{k+12i+12} + 4\,F_{k+4i+4},$$
$$S = L_{k+12i+12} + 2\,L_{k+4i+4}, \qquad Q = L_{3k+20i+10} + L_{3k+12i+6} + L_{3k+4i+2},$$

if $u = 4i + 1$ *and*

$$P = F_{3k+20i+20} + F_{3k+12i+12} + F_{3k+4i+4}, \qquad R = F_{k+12i+12} + 4\,F_{k+4i+4},$$
$$S = L_{k+12i+12} + 2\,L_{k+4i+4}, \qquad Q = L_{3k+20i+20} + L_{3k+12i+12} + L_{3k+4i+4},$$

if $u = 4i + 3$.

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