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Primary classes of compositions of numbers*

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Dedicated to the memory of P. A. MacMahon on the occasion of the 158th anniversary of his birth

Abstract

The compositions, or ordered partitions, of integers, fall under certain natural classes. In this expository paper we highlight the most important classes by means of bijective proofs. Some of the proofs rely on the properties of zig-zag graphs - the graphical representations of compositions introduced by Percy A. MacMahon in his classic book *Combinatory Analysis*.

Keywords: composition, conjugate, zig-zag graph, line graph, bit-encoding, direct detection.

MSC: 05A19.

1. Introduction

A composition of a positive integer n is a representation of n as a sequence of positive integers which sum to n. The terms are called parts of the composition.

Denote the number of compositions of n by $c(n)$. The formula for $c(n)$ may be obtained from the classical recurrence relation:

$$c(n+1) = 2c(n), \quad c(1) = 1. \tag{1.1}$$

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Indeed a composition of $n+1$ may be obtained from a composition of n either by adding 1 to the first part, or by inserting 1 to the left of the previous first part. The recurrence gives the well-known formula: $c(n) = 2^{n-1}$.

For example, the following are the compositions of $n = 1, 2, 3, 4$:

$$(1)$$
$$(2), (1,1)$$
$$(3), (1,2), (2,1), (1,1,1)$$
$$(4), (1,3), (2,2), (3,1), (1,1,2), (1,2,1), (2,1,1), (1,1,1,1)$$

When the order of parts is fixed we obtain the partitions of n. For example, 4 has just 5 partitions – $(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$.

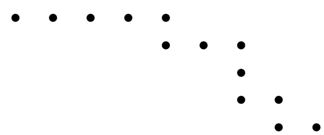
This is an expository paper devoted to a classification of compositions according to certain natural criteria afforded by their rich symmetry. We will mostly employ the extensive beautiful machinery developed by P. A. MacMahon in his classic text [3]. His original analysis of the properties of compositions seems to have received scarce attention in the literature during the last half-century.

Percy Alexander MacMahon was born in Malta on 26 September 1854, the son of brigadier general. He attended a military academy and later became an artillery officer, attaining the rank of Major, all the while doing top-class mathematics research.

According to his posthumous contemporary biographer, Paul Garcia [2],

"*MacMahon did pioneering work in invariant theory, symmetric function theory, and partition theory. He brought all these strands together to bring coherence to the discipline we now call combinatorial analysis. ...*"

MacMahon's study of compositions was influenced by his pioneering work in partitions. For instance, he devised a graphical representation of a composition, called a *zig-zag graph*, which resembles the partition Ferrers graph except that the first dot of each part is aligned with the last part of its predecessor. Thus the zig-zag graph of the composition $(5,3,1,2,2)$ is



(1.2)

The conjugate of a composition is obtained by reading its graph by columns, from left to right: the graph (1.2) gives the conjugate of the composition $(5,3,1,2,2)$ as $(1,1,1,1,2,1,3,2,1)$.

The zigzag graph possesses a rich combinatorial structure providing several equivalent paths to the conjugate composition. The latter are outlined in Section 2.

We will sometimes write $C \models n$ to indicate that C is a composition of n, and the integer n will be referred to as the *weight* of C. A k-composition is a composition

with k parts. The conjugate of C will be denoted by C'.
Now following MacMahon, we define, relative to a composition $C = (c_1, c_2, \dots, c_k)$:

The *inverse* of C is the reversal composition $\overline{C} = (c_k, c_{k-1}, \dots, c_2, c_1)$.

C is called *self-inverse* if $C = \overline{C}$.

C is *inverse-conjugate* if it's inverse coincides with its conjugate: $C' = \overline{C}$.

The zigzag graph of a composition C can be read in four ways to give generally different compositions namely $C, C', \overline{C}, \overline{C}'$. Exceptions occur when C is self-inverse, or when C is inverse-conjugate, in which case only two readings are obtained.

We deliberately refrain from applying generating function techniques in this paper for the simple reason that the apparent efficacy of their use has largely been responsible for obscuring the methods discussed.

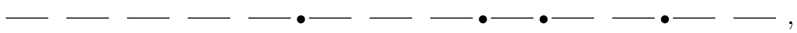
2. The conjugate composition

In this section we outline five different paths to the conjugate composition.

ZG: The Zig-zag Graph, already defined above.

LG: The Line graph (also introduced by MacMahon [3, Sec. IV, Ch. 1, p. 151])

The number n is depicted as a line divided into n equal segments and separated by $n-1$ spaces. A composition $C = (c_1, \dots, c_k)$ then corresponds to a choice of $k-1$ from the $n-1$ spaces, indicated with nodes. The conjugate C' is obtained by placing nodes on the other $n-k$ spaces. Thus the line graph of the composition $(5, 3, 1, 2, 2)$ is



from which we deduce that $C' = (1, 1, 1, 1, 2, 1, 3, 2, 1)$. It follows that C' has $n-k+1$ parts.

SubSum: Subset Partial Sums:

There is a bijection between compositions of n into k parts and $(k-1)$-subsets of $\{1, \dots, n-1\}$ via partial sums (see also [6]) given by

$$C = (c_1, \dots, c_k) \mapsto \{c_1, c_1 + c_2, \dots, c_1 + c_2 + \dots + c_{k-1}\} = L. \qquad (2.1)$$

Hence C' is the composition corresponding to the set $\{1, \dots, n-1\} \setminus L$.

BitS: Encoding by Binary Strings

It is sometimes necessary to express compositions as bit strings. The procedure for such *bit-encoding* consists of converting the set L into a unique bit string $B = (b_1, \dots, b_{n-1}) \in \{0, 1\}^{n-1}$ such that

$$b_i = \begin{cases} 1 & \text{if } i \in L \\ 0 & \text{if } i \notin L. \end{cases}$$

The complementary bit string B', obtained from B by swapping the roles of 1 and 0, is then the bit encoding of C'.

DD: Direct Detection of Conjugates

There is an easily-mastered rule for writing down the conjugate of a composition by inspection. A sequence of x consecutive equal parts $c, \dots, c$ will be abbreviated as c^x. First, the general composition has two forms, subject to inversion:

(1) $C = (1^{a_1}, b_1, 1^{a_2}, b_2, 1^{a_3}, b_3, \dots),\ a_i \geq 0, b_i \geq 2;$

(2) $E = (b_1, 1^{a_1}, b_2, 1^{a_2}, b_3, 1^{a_3}, \dots),\ a_i \geq 0, b_i \geq 2.$

The conjugate, in either case, is given by the rule:

(1c) $C' = (a_1 + 1, 1^{-1+b_1-1}, 1 + a_2 + 1, 1^{-1+b_2-1}, 1 + a_3 + 1, \dots)$
$= (a_1 + 1, 1^{b_1-2}, a_2 + 2, 1^{b_2-2}, a_3 + 2, \dots).$

Similarly,

(2c) $E' = (1^{b_1-1}, a_1 + 2, 1^{b_2-2}, a_2 + 2, \dots).$

For example, $(1, 3, 4, 1^3, 2, 1^2, 6)'$ is given by

$$(1 + 1, 1^{3-2}, 1 + 1, 1^{4-2}, 1 + 1^3 + 1, 1 + 1^2 + 1, 1^{6-1}) = (2, 1, 2, 1^2, 5, 4, 1^5).$$

The various approaches to the conjugate composition obviously have their merits and demerits. The strength of the **DD** method is that it often provides a general form of the conjugate composition explicitly.

3. Special classes of compositions

We will need the following algebraic operations:

If $A = (a_1, \dots, a_i)$ and $B = (b_1, \dots, b_j)$ are compositions, we define the concatenation of the parts of A and B by

$A|B = (a_1, \dots, a_i, b_1, \dots, b_j).$

In particular for a nonnegative integer c, we have $A|(c) = (A, c)$ and $(c)|A = (c, A)$. Define the *join* of A and B as

$A \uplus B = (a_1, \dots, a_{i-1}, a_i + b_1, b_2, \dots, b_j).$

The following rules are easily verified:

1. $\overline{A|B} = \overline{B}|\overline{A}.$
2. $(A|B)' = A' \uplus B'.$

Note that $(A, 0) \uplus B = A \uplus (0, B) = A|B.$

3.1. Equitable decomposition by conjugation

The conjugation operation immediately implies the following identity:

Proposition 3.1. *The number of compositions of n with k parts equals the number of compositions of n with $n - k + 1$ parts.*

The two classes consist of different compositions except when n is odd and $k = (n+1)/2 = n-k+1$. In the latter case the two classes are coincident. Indeed since there are $c(n,k) = \binom{n-1}{k-1}$ compositions of n with k parts, we see at once that $c(n,k) = c(n, n-k+1)$.

Thus the set $W(n)$ of compositions of n may be economically stored by keeping only the sets $W(n,k)$ of k-compositions, $k = 1, \ldots, \lfloor \frac{n+1}{2} \rfloor$, whereby the remaining compositions are accessible via conjugation.

Looking closely at this idea, assume that the elements of each set $W(n,k)$ are arranged in lexicographic order, and list the sets in increasing order of lengths of members as follows:

$$\underbrace{W(n,1), W(n,2), \ldots, W(n, \lfloor \tfrac{n+1}{2} \rfloor)}_{\text{generates } W(n) \text{ via conjugation}}, W(n, \lfloor \tfrac{n+1}{2} \rfloor + 1), \ldots, W(n, n-1), W(n,n). \quad (3.1)$$

This arrangement implies one of the beautiful symmetries exhibited by many sets of compositions:

If the set divisions are removed to reveal a single list of all compositions of n, then the j-th composition from the left and the j-th composition from the right are mutual conjugates. In other words, the j-th composition is the conjugate of the $(n-j+1)$-th composition, from either end.

This arrangement is illustrated for compositions of $n = 1, 2, 3, 4$ (see Section 1).

3.2. Equitable four-way decomposition

Define a $1c2$-*composition* as a composition with the first part equal to 1 and last part > 1. The following are analogously defined: $2c1$-*composition,* $1c1$-*composition*, and $2c2$-*composition.*

Then observe that the $2c1$-compositions are inverses of $1c2$-compositions, and that the set of $2c2$-compositions form the set of conjugates of the $1c1$-compositions. It turns out that the set of compositions of n splits naturally into four subsets of equal cardinality corresponding to the four types of compositions.

Theorem 3.2. *Let n be a natural number > 1. Then the following classes of compositions are equinumerous:*

(i) $1c1$-compositions of n.

(ii) $1c2$-compositions of n.

(iii) $2c1$-compositions of n.

(iv) $2c2$-compositions of n.

Each class is enumerated by $c(n-2)$.

Proof. By the remark immediately preceding the theorem, it suffices to establish a bijection: (i) $\iff$ (ii). An object in (ii) has the form $C = (1, c_2, \ldots, c_k), c_k > 1$. Deleting the initial 1 and subtracting 1 from c_k gives $(c_2, \ldots, c_k - 1) = T$, a composition of $n-2$. Now pre-pend and append 1 to obtain $(1, c_2, \ldots, c_k - 1, 1)$, which is a unique composition in (i). Lastly, also note that the passage from C to

T is a bijection from (i) to the class of compositions of $n-2$. In other words the common number of compositions in each of the classes is $c(n-2)$. □

Example. When $n = 5$, the four classes are given by:
(i) $(1,3,1),(1,2,1,1),(1,1,2,1),(1,1,1,1,1)$;
(ii) $(1,4),(1,2,2),(1,1,3),(1,1,1,2)$;
(iii) $(4,1),(2,2,1),(3,1,1),(2,1,1,1)$;
(iv) $(2,1,2),(2,3),(3,2),(5)$.

Remark 3.3. An Application: Since Theorem 3.2 implies $c(n) = 4c(n-2)$, it can be applied to the generation of compositions of n from those of $n-2$ in an obvious way. Such algorithm is clearly more efficient than the classical recursive procedure via the compositions of $n-1$ (see (1.1)). Thus to compute the compositions of 5, for example, it suffices to use the set $W(3) = \{(3),(2,1),(1,2),(1,1,1)\}$, together with the quick generation procedures corresponding to the bijections in the proof of Theorem 3.2.

A further saving of storage space can be attained by combining this four-way decomposition with the conjugation operation. Then to store the set $W(n)$ of compositions of n it would suffice to hold only one half of $W(n-2)$, arranged as previously described.

As a mixed refinement of Theorem 3.2 we have the following identity, which is a consequence of conjugation.

Proposition 3.4. *The number of compositions of n with one or two 1's which can appear only as a first and/or last part equals the number of compositions of n into 1's and 2's whose first and/or last part is 2.*

For example, when $n = 5$, the two classes of compositions mentioned in the proposition are:

$$(1,4),(4,1),(1,2,2),(1,3,1),(2,2,1);$$

$$(2,1,1,1),(1,1,1,2),(2,2,1),(2,1,2),(1,2,2).$$

3.3. Self-inverse compositions

Self-inverse compositions constitute the next easily distinguishable class of compositions. Their enumeration is usually straightforward. The number of parts of a composition C will also be referred to as its *length*, denoted by $\ell(C)$.

We remark that MacMahon [3] proved most of the results in this sub-section, in the case of k-compositions, using the **LG** method.

Proposition 3.5.

(*i*) *The number of self-inverse compositions of $2n$ is $c(n+1)$.*

(*ii*) *The number of self-inverse compositions of $2n-1$ is $c(n)$.*

Proof. We prove only part (i) (the proof of part (ii) is similar). Firstly, if C is a self-inverse composition with $\ell(C)$ odd, then C has the form:
$C = (c_1, \ldots, c_{k-1}, c_k, c_{k-1}, \ldots, c_1)$, where c_k is even. Thus
$C = (c_1, \ldots, c_{k-1}, c_k/2) \uplus (c_k/2, c_{k-1}, \ldots, c_1) \equiv A \uplus \overline{A}$,
where $A = (c_1, \ldots, c_{k-1}, c_k/2)$ runs over all compositions of n.

If $\ell(C)$ is even, then C has the form $C = (c_1, \ldots, c_{k-1}, c_k, c_k, c_{k-1} \ldots, c_1) \equiv B|\overline{B}$, where $B = (c_1, \ldots, c_{k-1}, c_k)$ runs over all compositions of n.

It follows that there are as many self-inverse compositions of $2n$ into an odd number of parts as into an even number of parts. Using the above notations, a simple bijection is $C \equiv A \uplus \overline{A} \mapsto A|\overline{A}$, and conversely, $C \equiv B|\overline{B} \mapsto B \uplus \overline{B}$. □

The essential results on self-inverse compositions are summarized below.

Theorem 3.6. *The following sets of compositions have the same number of elements:*
(i) self-inverse compositions of $2n - 1$.
(ii) self-inverse compositions of $2n$ *of odd lengths.*
(iii) self-inverse compositions of $2n$ *of even lengths.*
(iv) self-inverse compositions of $2n - 2$.
(v) compositions of n.

Proof. (i) $\iff$ (ii): if $(c_1, \ldots, c_{k-1}, c_k, c_{k-1}, \ldots, c_1)$ is in (i), then

$$(c_1, \ldots, c_{k-1}, c_k + 1, c_{k-1}, \ldots, c_1)$$

is in (ii), and conversely.
(i) $\iff$ (iv): if $(c_1, \ldots, c_{k-1}, c_k, c_{k-1}, \ldots, c_1)$ and $(c_1, \ldots, c_{k-1}, c_k, c_k, c_{k-1}, \ldots, c_1)$ belong to (iv), then (i) contains $(c_1, \ldots, c_{k-1}, c_k + 1, c_{k-1}, \ldots, c_1)$ and

$$(c_1, \ldots, c_{k-1}, c_k, 1, c_k, c_{k-1}, \ldots, c_1),$$

respectively.
Lastly, since the cases (ii) $\iff$ (iii) $\iff$ (v) have been demonstrated with the proof of Proposition 3.5, the theorem follows. □

4. Inverse-conjugate compositions

Let C be a k-composition. If C is inverse-conjugate, then $k = |C| - k + 1$ or $|C| = 2k - 1$. Thus inverse-conjugate compositions are defined only for odd weights. In fact, every odd integer > 1 has a nontrivial inverse-conjugate composition. For instance, $(1, 2^{k-1})$ and $(1^{k-1}, k)$ are both inverse-conjugate compositions of $2k - 1$.

Consider a general composition,

$$C = (1^{a_1}, b_1, 1^{a_2}, b_2, \ldots, 1^{a_r}, b_r),\ a_i \geq 0, b_i \geq 2.$$

Then, using the **DD** conjugation rule in Section 2, we obtain

$$C' = (a_1 + 1, 1^{b_1 - 2}, a_2 + 2, 1^{b_2 - 2}, \ldots, 1^{b_{r-1} - 2}, a_r + 2, 1^{b_r - 1}).$$

Thus the conditions for C to be inverse-conjugate are

$$b_r = a_1 + 1,\ b_{r-1} = a_2 + 2, \ldots, b_1 = a_r + 2.$$

Hence we have proved:

Lemma 4.1. *An inverse-conjugate composition C (or its inverse) has the form:*

$$C = (1^{b_r-1}, b_1, 1^{b_{r-1}-2}, b_2, 1^{b_{r-2}-2}, b_3, \ldots, b_{r-1}, 1^{b_1-2}, b_r),\ b_i \geq 2. \tag{4.1}$$

Note that the sum of the parts is $2(b_1 + \cdots + b_r) - (r-1)(2) - 1 \equiv 1 \pmod 2$, as expected.

Let $(c_1, \ldots, c_k)$ be an inverse-conjugate composition of $n > 1$. For any index $j < k$ with $c_{j+1} \neq 1$, consider the sub-composition $(c_1, \ldots, c_j)$. First, notice the following relation between the two "halves" of (4.1):

$$\overline{(1^{b_r-1}, b_1, \ldots, b_j, 1^{b_{r-j}-2})} = (b_{r-j} - 1, 1^{b_j-2}, b_{r-j+1}, \ldots, 1^{b_1-2}, b_r)'. \tag{4.2}$$

Therefore, if $|C| = 2k-1$, it is possible for the weight of either side of (4.2) to be exactly $k-1$. The latter case implies an instructive dissection of C:

$$\begin{aligned} C &= (1^{b_r-1}, b_1, \ldots, b_j, 1^{b_{r-j}-2})|(1) \uplus (b_{r-j} - 1, 1^{b_j-2}, b_{r-j+1}, \ldots, 1^{b_1-2}, b_r) \\ &= (1^{b_r-1}, b_1, \ldots, b_j, 1^{b_{r-j}-2})|(1) \uplus \overline{(1^{b_r-1}, b_1, \ldots, b_j, 1^{b_{r-j}-2})'}. \end{aligned}$$

where the last equality follows by conjugating both sides of (4.2).

The gist of the foregoing discussion is summarized in the next theorem.

Theorem 4.2. *If $C = (c_1, \ldots, c_k)$ is an inverse-conjugate composition of $n = 2k-1 > 1$, or its inverse, then there is an index j such that $c_1 + \cdots + c_j = k-1$ and $c_{j+1} + \cdots + c_k = k$ with $c_{j+1} > 1$. Moreover,*

$$\overline{(c_1, \ldots, c_j)} = (c_{j+1} - 1, c_{j+2}, \ldots, c_k)' \tag{4.3}$$

Thus C can be written in the form

$$C = A|(1) \uplus B \quad \text{such that} \quad B' = \overline{A}, \tag{4.4}$$

where A and B are generally different compositions of $k-1$.

It follows that an inverse-conjugate composition C of $n > 1$ cannot be self-inverse, even though $\overline{C}$ is also inverse-conjugate (in contrast with the so-called *self-conjugate* partitions of $n > 2$ [1, 4]).

The theorem implies the following result of MacMahon which he demonstrated using the **LG** method.

Theorem 4.3 (MacMahon)**.** *The number of inverse-conjugate compositions of an odd integer $n > 0$ equals the number of compositions of n which are self-inverse.*

Proof. We describe a bijection α between the two classes of compositions by invoking Theorem 4.2. If $C \models 2k-1$ is inverse-conjugate, then C can be written in the form $C = A|(1) \uplus B$ or $C = A \uplus (1)|B$ for certain compositions A, B, of $k-1$ satisfying $B' = \overline{A}$.

In the first case we use (4.3) to get $\alpha(C) = A|[(1) \uplus B]'$, which is a self-inverse composition of the type $A|(1)|\overline{A}$.

The second case, $C = A \uplus (1)|B$, implies that there is a part $m > 1$ such that $C = X|(m)|B$, with $X \models M < k-1$. Now split m between the two compositions as follows: $X|(m-1) \uplus (1)|B = (X, m-1) \uplus (1, B)$, which is in the first-case form. Hence $\alpha(C) = (X, m-1) \uplus (1, B)'$, giving a self-inverse composition of the type $Y|(d)|\overline{Y}$, with d an odd integer > 1.

Conversely given a self-inverse composition, $T = (b_1, \ldots, b_r) \equiv B|(d)|\overline{B}$ of $2k-1$, we first write T as the join of two compositions of $k-1$ and k, by splitting the middle part. The middle part, by weight, is b_{j+1} such that $s_j = b_1 + \cdots + b_j \leq k-1$ and $s_j + b_{j+1} \geq k$. Thus

$T \mapsto (b_1, \ldots, b_j)|(k-1-s_j) \uplus (k-t_j)|(b_{j+2}, \ldots, b_r) \equiv X|(k-1-s_j) \uplus (k-t_j)|\overline{X}$, where $t_j = b_{j+2} + \cdots + b_k$.

Hence $\alpha^{-1}(T) = X|(k-1-s_j) \uplus (k-t_j, \overline{X})'$, which is inverse-conjugate. □

Example. Consider the inverse-conjugate composition of 15 given by

$$C = (1, 1, 1, 2, 3, 1, 2, 4).$$

Then since $1+1+1+2 < 7$ and $1+1+1+2+3 > 7$, we have

$$C = (1,1,1,2)|(3)|(1,2,4) \to (1,1,1,2,2) \uplus (1,1,2,4)' = (1,1,1,2,2) \uplus (3,2,1,1,1),$$

which gives $T = (1,1,1,2,5,2,1,1,1)$, a self-inverse composition of 15. Conversely,

$$(1,1,1,2,5,2,1,1,1) \to (1,1,1,2,2) \uplus (3,2,1,1,1)' = (1,1,1,2,2) \uplus (1,1,2,4),$$

which gives back $(1,1,1,2,3,1,2,4)$.

It can also be verified that $C' = (4,2,1,3,2,1,1,1)$ corresponds to the self-inverse composition $(4,2,1,1,1,2,4) = T'$ under the bijection.

Corollary 4.4. *There are as many inverse-conjugate compositions of* $2n-1$ *as there are compositions of* n.

Proof. The proof can be deduced from Theorem 3.6 and Theorem 4.3, but we give a direct proof. If $n = 1$, the composition (1) belongs trivially to the two classes of compositions. So assume $n > 1$.

Let $(c_1, \ldots, c_n)$ be any inverse-conjugate composition of $2n-1$. Then by (4.4) there is an index j such that $c_1 + \cdots + c_j = n-1$ or $c_{k-j+1} + \cdots + c_k = n-1$.

There are $c(n-1)$ inverse-conjugate compositions $(c_, \ldots, c_n)$ in which $c_1 + \cdots + c_j = n-1$, $n > 1$, and there distinct conjugates (i.e., inverses). Since there are no self-inverse inverse-conjugate compositions, the total number of inverse-conjugate compositions of $2n-1$ is $2c(n-1) = c(n)$, as required.

We can also give a bijection. According to Theorem 4.2 every inverse-conjugate composition $(c_1, \dots, c_n)$ satisfies $c_1 + \cdots + c_j = n - 1$ and $c_{j+1} + \cdots + c_n = n$ with $c_{j+1} > 1$, or $c_1 + \cdots + c_j = n$ and $c_{j+1} + \cdots + c_n = n - 1$ with $c_j > 1$, for a certain index j. Now with each inverse-conjugate composition of the first type associate the composition of n given by $(c_1, \dots, c_j, 1)$, and with each of the second type associate, $(c_1, \dots, c_j)$, which is already a composition of n.

This gives the required bijection. □

Example. We illustrate the second part of the proof of Corollary 4.4. There are 8 inverse-conjugate compositions of 7:

$$(1,1,1,4), (1,1,2,3), (1,2,2,2), (1,3,1,2),$$
$$(2,1,3,1), (2,2,2,1), (3,2,1,1), (4,1,1,1).$$

The corresponding list of compositions of 4, under the bijection, is:

$$(1,1,1,1), (1,1,2), (1,2,1), (1,3), (2,1,1), (2,2), (3,1), (4).$$

5. Further consequences

The machinery developed here can be used to relate compositions directly with bit strings, that is, finite sequences of 0's and 1's.

Theorem 5.1.
(i) The number of compositions of $n + 1$ without the part m equals the number of n-bit strings that avoid a run of $m - 1$ ones.

(ii) The number of compositions of $n + 1$ in which m may appear only as a first or last part equals the number of n-bit strings that avoid $01^{m-1}0$.

Proof. To prove part (ii) we give a bijection between the two sets, using the **SubSum** and **BitS** conjugation methods. If $C = (m, c_1, c_2, \dots) \models n + 1, c_i \neq m > 1$, then the image of C under the bijection (2.1) is $L = (m, m + c_1, m + c_1 + c_2, \dots)$. Since $c_i \neq m$ for all i, no pair of consecutive terms in L are separated by $m - 1$ elements. So the bit encoding of C avoids $10^{m-1}1$. The same conclusion obviously holds if we start with a composition that does not contain m as a part. Thus the desired bijection is the map that takes a composition C of n with no intermediate m's to the bit encoding of the conjugate C'.

The proof of part (i) is similar. □

It turns out that the two classes of compositions in Theorem 5.1 are equinumerous, for $m = 2$, provided the weights differ by unity.

Theorem 5.2. *The number of compositions of n in which 2 may appear only as a first or last part equals the number of compositions of $n + 1$ without 2's.*

Proof. We provide a recursive proof. Let d_n be the number of compositions of n in which 2 may appear only as a first or last part, and let c_n be the number of compositions of n without 2's.

Then, we first observe that

$$d_n = c_n + 2c_{n-2} + c_{n-4}, \tag{5.1}$$

since d_n enumerates the set consisting of compositions without 2's, those with exactly one 2 at either end, and those with two 2's at both ends.

The enumerator c_n fulfills the following recurrence relations.

$$c_n = 2c_{n-1} - c_{n-2} + c_{n-3}; \tag{5.2}$$

$$c_n = c_{n-1} + c_{n-2} + c_{n-4}; \tag{5.3}$$

with the initial values $c_1 = c_2 = 1$.
For (5.2), we note that a composition counted by c_n can be found in three ways:

(i) by adding 1 to the last part of a composition counted by c_{n-1}, provided we exclude compositions of $n-1$ with last part 1;

(ii) by appending 1 to a composition counted by c_{n-1}; and

(iii) by appending 3 to a composition counted by c_{n-3}, since the previous two types exclude the latter.

The numbers of compositions of n generated are, respectively, $c_{n-1} - c_{n-2}$, c_{n-1} and c_{n-3}. Hence altogether we obtain (5.2).

For (5.3), note that compositions counted by c_n with first part 1 are also counted by c_{n-1}; those with first part > 1, that is, first part ≥ 3, are counted by c_{n-2}, with the exception of those with first part equal to 4. The latter are obtained by appending 4 to compositions of $n-4$ with no 2's. Hence the result.

Now using (5.3) and (5.2), we obtain
$d_n = c_n+2c_{n-2}+c_{n-4} = c_n+2c_{n-2}+c_n-c_{n-1}-c_{n-2} = 2c_n-c_{n-1}+c_{n-2} = c_{n+1}$, as required. □

We are presently unable to give a direct bijection between the two sets of compositions in Theorem 5.2. The theorem can, of course, be formulated in terms of bit strings using the **BitS** conjugation method (cf. Theorem 5.1):

Corollary 5.3. *The number of n-bit strings avoiding* 010 *is equal to the number of* $(n+1)$*-bit strings avoiding isolated* 1*'s.*

However, even in this new form, the difficulty of finding a bijective proof seems to persist. It is possible to give a recursive proof of Corollary 5.3 that is similar to the proof of Theorem 5.2.

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