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Algebraic independence results for the infinite products generated by Fibonacci numbers

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Abstract

The aim of this paper is to investigate the algebraic independence between two infinite products generated by the Fibonacci numbers $\{F_n\}_{n\geq 0}$ whose indices run in certain geometric progressions or binary recurrent sequences. As an application, we determine all the integers $m \geq 1$ such that the infinite products

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{F_{2^k}}\right) \quad \text{and} \quad \prod_{k=1}^{\infty}\left(1+\frac{1}{F_{2^k+m}}\right)$$

are algebraically independent over $\mathbb{Q}$.

Keywords: Algebraic independence, Infinite products, Fibonacci numbers, Mahler-type functional equation

MSC: 11J85

1. Introduction and the results

Let $\{R_n\}_{n\geq 0}$ be the binary recurrence defined by

$$R_{n+2} = A_1 R_{n+1} + A_2 R_n, \quad n \geq 0, \tag{1.1}$$

where A_1 and A_2 are nonzero integers and the initial values R_0 and R_1 are integers, not both zero. Suppose that $|A_2| = 1$ and $A_1^2 + 4A_2 > 0$. If $A_1 = A_2 = 1$ and $R_0 = 0, R_1 = 1$, then we have $R_n = F_n$ $(n \geq 0)$, where F_n is the nth Fibonacci number.

Let $d \geq 2$ be a fixed integer. The second author [6] investigated necessary and sufficient conditions for the infinite product generated by the sequence (1.1) to be algebraic. As an application, the transcendence of the infinite product $\prod_{k=1}^{\infty}(1 + \frac{1}{F_{d^k}})$ was deduced. In [3], the algebraic independence over $\mathbb{Q}$ of the sets of infinite products

$$\prod_{\substack{k=1 \\ F_{d^k} \neq -b_i}}^{\infty} \left(1 + \frac{b_i}{F_{d^k}}\right) \quad (i = 1, \ldots, m)$$

was proved for any nonzero distinct integers $b_1, \ldots, b_m$. In particular, the numbers

$$\prod_{k=1}^{\infty}\left(1 + \frac{1}{F_{2^k}}\right) \quad \text{and} \quad \prod_{k=2}^{\infty}\left(1 - \frac{1}{F_{2^k}}\right)$$

are algebraically independent over $\mathbb{Q}$. Recently, the authors [4] proved algebraic independence results for the infinite products generated by two distinct binary recurrences; for example, the two numbers

$$\prod_{k=1}^{\infty}\left(1 + \frac{1}{F_{2^k}}\right) \quad \text{and} \quad \prod_{k=1}^{\infty}\left(1 + \frac{1}{L_{2^k}}\right)$$

are algebraically independent over $\mathbb{Q}$, where the sequence $\{L_n\}_{n\geq 0}$ is the Lucas companion of the Fibonacci sequence defined by

$$L_{n+2} = L_{n+1} + L_n \quad (n \geq 0), \quad L_0 = 2, \quad L_1 = 1.$$

In what follows, let $\{R_n\}_{n\geq 0}$ be the binary recurrence given by (1.1) with $A_1 = A_2 = 1$. Then the sequence $\{R_n\}_{n\geq 0}$ is expressed as

$$R_n = g_1\alpha^n + g_2\beta^n, \quad n \geq 0, \tag{1.2}$$

where $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, and

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -\beta & 1 \\ \alpha & -1 \end{pmatrix} \begin{pmatrix} R_0 \\ R_1 \end{pmatrix}.$$

In this paper, we prove some algebraic independence results for the infinite products generated by Fibonacci numbers and the sequence (1.2). We state our results.

Theorem 1.1. *Let $d \geq 2$ be a fixed integer and $\{R_n\}_{n\geq 0}$ be the sequence defined by* (1.2) *with* $(R_0, R_1) \neq (0, 1)$*. Let*

$$\eta := \prod_{k=1}^{\infty} \left(1 + \frac{1}{F_{d^k}}\right) \quad \text{and} \quad \nu := \prod_{\substack{k=1 \\ R_{d^k} \neq 0, -1}}^{\infty} \left(1 + \frac{1}{R_{d^k}}\right).$$

Then the following conditions are equivalent:
(i) *The numbers η and ν are algebraically dependent over $\mathbb{Q}$.*
(ii) *The number ν is algebraic.*
(iii) *$d = 2$ and either the condition $g_1 + g_2 = 1$ or the condition $g_1 = g_2 = -1$ is satisfied.*

Corollary 1.2. *Let $d \geq 2$ and $\{R_n\}_{n\geq 0}$ the sequence defined by* (1.2)*. If $d \geq 3$, then the numbers*

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{F_{d^k}}\right) \quad \text{and} \quad \prod_{\substack{k=1 \\ R_{d^k} \neq 0, -1}}^{\infty} \left(1 + \frac{1}{R_{d^k}}\right)$$

are algebraically independent over $\mathbb{Q}$. The same holds for the case of $d = 2$ and $R_0 \notin \{-2, 0, 1\}$.

Corollary 1.3. *Let $d \geq 2$ be an integer and let $\gamma \neq 1$ be a nonzero rational number. Then the infinite products*

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{F_{d^k}}\right) \quad \text{and} \quad \prod_{\substack{k=1 \\ F_{d^k} \neq -\gamma}}^{\infty} \left(1 + \frac{\gamma}{F_{d^k}}\right)$$

are algebraically independent over $\mathbb{Q}$.

It should be noted that Corollary 1.3 holds even if γ is a nonzero algebraic number (cf. [1]).

Corollary 1.4. *Let $d \geq 2$ and $m \geq 1$ be integers. Then the infinite products*

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{F_{d^k}}\right) \quad \text{and} \quad \prod_{k=1}^{\infty} \left(1 + \frac{1}{F_{d^k+m}}\right) \tag{1.3}$$

are algebraically dependent over $\mathbb{Q}$ if and only if $(d, m) = (2, 1), (2, 2)$. In the two exceptional cases above, we have

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{F_{2^k+1}}\right) = \frac{3(\sqrt{5}-1)}{2}, \quad \prod_{k=1}^{\infty} \left(1 + \frac{1}{F_{2^k+2}}\right) = 6 - 2\sqrt{5}.$$

The proofs of Theorem 1.1 and the corollaries will be given in Section 3.

2. Lemmas

Let $d \geq 2$ be a fixed integer and let $\{R_n\}_{n\geq 0}$ be the sequence defined by (1.2). Define

$$\Phi(x) := \prod_{k=0}^{\infty}\left(1 + \frac{g_1^{-1}x^{d^k}}{1+(-1)^d g_1^{-1}g_2 x^{2d^k}}\right). \tag{2.1}$$

The function $\Phi(x)$ converges in $|x| < 1$ and satisfies the functional equation

$$\Phi(x^d) = c(x)\Phi(x), \tag{2.2}$$

with

$$c(x) = \frac{1+(-1)^d g_1^{-1}g_2x^2}{1+g_1^{-1}x+(-1)^d g_1^{-1}g_2x^2}.$$

To prove Theorem 1.1, we use the following lemma.

Lemma 2.1 (Special case of [6, Theorem 7])**.** *Let $d \geq 2$ be an integer. Let a and b be nonzero algebraic numbers and*

$$G(x) = \prod_{k=0}^{\infty}\left(1 + \frac{ax^{d^k}}{1-bx^{2d^k}}\right), \quad |x| < 1.$$

Then the function $G(x)$ is a rational function with the algebraic coefficients if and only if $d = 2$ and either the condition $a + b = 1$ or the condition $a = b = -1$ is satisfied.

Lemma 2.2. *Let $\Phi(x)$ be the function given in* (2.1)*. Then the following conditions are equivalent*:
(i) *The function $\Phi(x)$ is algebraic over $\mathbb{Q}(\alpha, x)$.*
(ii) *The function $\Phi(x)$ is a rational function with algebraic coefficients.*
(iii) *$d = 2$ and either the condition $g_1 + g_2 = 1$ or the condition $g_1 = g_2 = -1$ is satisfied.*

Proof. First we prove (i)⇒(ii). Suppose that $\Phi(x)$ is algebraic over $\mathbb{Q}(\alpha, x)$. Then, by the functional equation (2.2) and [5, Theorem 1.3] with $C = \overline{\mathbb{Q}}$, we see that $\Phi(x)$ is a rational function over some algebraic number field $\mathbb{L} \supseteq \mathbb{Q}(\alpha)$. The assertions (ii)⇒(iii) and (iii)⇒(i) follow immediately from Lemma 2.1. □

Remark 2.3. If the property (iii) in Lemma 2.2 is satisfied, then the corresponding infinite products $\Phi(x)$ are expressed as rational functions explicitly. Indeed, in the case of $d = 2$ and $g_1 + g_2 = 1$, we have

$$\Phi(x) = \prod_{k=0}^{\infty}\left(1 + \frac{(1-b)x^{2^k}}{1-bx^{2^{k+1}}}\right) = \prod_{k=0}^{\infty}\frac{(1+x^{2^k})(1-bx^{2^k})}{1-bx^{2^{k+1}}} = \frac{1-bx}{1-x} \tag{2.3}$$

with $b = -g_1^{-1}g_2$. If $d = 2$ and $g_1 = g_2 = -1$, then

$$\Phi(x) = \prod_{k=0}^{\infty}\left(1 + \frac{-x^{2^k}}{1 + x^{2^{k+1}}}\right) = \prod_{k=0}^{\infty}\frac{(1+\omega^{2^k}x^{2^k})(1+\omega^{-2^k}x^{2^k})}{1+x^{2^{k+1}}}$$
$$= \frac{1-x^2}{(1-\omega x)(1-\omega^{-1}x)} = \frac{1-x^2}{1+x+x^2}, \quad (2.4)$$

where ω is a primitive cubic root of unity.

Let $\mathbb{K}$ be an algebraic number field. For an integer $d \geq 2$, we define the subgroup H_d of the group $\mathbb{K}(x)^{\times}$ of nonzero elements of $\mathbb{K}(x)$ by

$$H_d = \left\{ \frac{g(x^d)}{g(x)} \;\middle|\; g(x) \in \mathbb{K}(x)^{\times} \right\}.$$

Let $\mathbb{K}[[x]]$ be the ring of formal power series with coefficients in $\mathbb{K}$.

Lemma 2.4 (Kubota [2, Corollary 8]). *Let $f_1(x), \dots, f_m(x) \in \mathbb{K}[[x]] \setminus \{0\}$ satisfy the functional equations*

$$f_i(x^d) = c_i(x)f_i(x), \quad c_i(x) \in \mathbb{K}(x)^{\times} \quad (i = 1, \dots, m). \quad (2.5)$$

Then $f_1(x), \dots, f_m(x)$ are algebraically independent over $\mathbb{K}(x)$ if and only if the rational functions $c_1(x), \dots, c_m(x)$ are multiplicatively independent modulo H_d.

Lemma 2.5 (Kubota [2], see also Nishioka [5, Theorem 3.6.4]). *Suppose that the functions $f_1(x), \dots, f_m(x) \in \mathbb{K}[[x]]$ converge in $|x| < 1$ and satisfy the functional equations* (2.5) *with $c_i(x)$ defined and nonzero at $x = 0$. Let γ be an algebraic number with $0 < |\gamma| < 1$ such that $c_i(\gamma^{d^k})$ are defined and nonzero for all $k \geq 0$. If $f_1(x), \dots . f_m(x)$ are algebraically independent over $\mathbb{K}(x)$, then the values $f_1(\gamma), \dots, f_m(\gamma)$ are algebraically independent over $\mathbb{Q}$.*

3. Proofs of Theorem 1.1 and the corollaries

Putting $g_1 = -g_2 = 1/\sqrt{5}$ in (2.1), we have

$$\Psi(x) := \prod_{k=0}^{\infty}\left(1 + \frac{\sqrt{5}x^{d^k}}{1-(-1)^d x^{2d^k}}\right).$$

By Lemma 2.2, the function $\Psi(x)$ is transcendental over $\mathbb{K}(x)$. Let η and ν be as in Theorem 1.1. Take an integer N such that $|R_{d^k}| > 1$ for all $k \geq N > 1$. Then, using (2.2), we get

$$\eta = p_N\Psi(\alpha^{-d^N}) = p_N\Psi(\alpha^{-1})\prod_{i=0}^{N-1} b(\alpha^{-d^i}), \quad (3.1)$$

$$\nu \quad = q_N \Phi(\alpha^{-d^N}) = q_N \Phi(\alpha^{-1}) \prod_{i=0}^{N-1} c(\alpha^{-d^i}), \tag{3.2}$$

where

$$b(x) = \frac{1-(-1)^d x^2}{1+\sqrt{5}x-(-1)^d x^2}, \quad c(x) = \frac{1+(-1)^d g_1^{-1} g_2 x^2}{1+g_1^{-1}x+(-1)^d g_1^{-1} g_2 x^2}$$

and p_N and q_N are nonzero rational numbers given by

$$p_N = \prod_{k=1}^{N-1} \left(1+\frac{1}{F_{d^k}}\right), \quad q_N = \prod_{\substack{k=1 \\ R_{d^k} \neq 0,-1}}^{N-1} \left(1+\frac{1}{R_{d^k}}\right).$$

Proof of Theorem 1.1. The assertion (ii)⇒(i) is trivial. If the condition (iii) holds, then, by Remark 2.3, the function $\Phi(x)$ is a rational function as in (2.3) or (2.4). Hence, by (3.2), we see that the number ν is algebraic and so the property (ii) is satisfied. Thus, we have only to prove (i)⇒(iii).

Suppose that η and ν are algebraically dependent over $\mathbb{Q}$. Then so are the values $\Phi(\alpha^{-1})$ and $\Psi(\alpha^{-1})$ by (3.1) and (3.2). Since $\Psi(x)$ and $\Phi(x)$ satisfy the functional equation (2.2), they are algebraically dependent over $\mathbb{K}(x)$ by Lemma 2.5. Thus, we see by Lemma 2.4, that the rational functions $b(x)$ and $c(x)$ are multiplicatively dependent modulo H_d, namely, there exist integers e_1, e_2, not both zero, and $g(x) \in \mathbb{K}(x)^\times$ such that

$$b(x)^{e_1} c(x)^{e_2} = g(x^d)/g(x), \tag{3.3}$$

where 0 is neither a pole nor a root of $g(x)$ because $b(0)c(0) = 1$. To simplify notations, we rewrite the equation (3.3), as

$$F(x) := \left(\frac{1-(-1)^d x^2}{1+\sqrt{5}x-(-1)^d x^2}\right)^{e_1} \left(\frac{1+g_1^{-1} g_2 (-1)^d x^2}{1+g_1^{-1}x+g_1^{-1} g_2 (-1)^d x^2}\right)^{e_2}, \tag{3.4}$$

where e_1 and e_2 are nonzero integers and

$$F(x) = \frac{A(x^d)B(x)}{A(x)B(x^d)} \tag{3.5}$$

with $A(x)$ and $B(x)$ being the polynomials without common roots with algebraic coefficients such that $g(x) = A(x)/B(x)$. We also assume that $e_1 > 0$, otherwise we replace the pair of exponents (e_1, e_2) by the pair $(-e_1, -e_2)$ and interchange $A(x)$ and $B(x)$. We distinguish four cases.

Case I). $e_1 e_2 > 0$. By (3.4) and (3.5), we have

$$\begin{aligned} A(x)B(x^d)(1-(-1)^d x^2)^{e_1}(1+g_1^{-1} g_2 (-1)^d x^2)^{e_2} \\ = A(x^d)B(x)P(x)^{e_1} Q(x)^{e_2}, \end{aligned} \tag{3.6}$$

where $e_1, e_2 \geq 1$ and

$$P(x) = 1 + \sqrt{5}x - (-1)^d x^2, \quad Q(x) = 1 + g_1^{-1}x + g_1 g_2^{-1}(-1)^d x^2.$$

Let γ_1 and γ_2 be the real roots of $P(x)$. Noting that

$$\gamma_1, \gamma_2 = \frac{(-1)^d\sqrt{5} \pm \sqrt{5 + 4(-1)^d}}{2} = \begin{cases} (\pm 3 + \sqrt{5})/2, & d : \text{even}, \\ (\pm 1 - \sqrt{5})/2, & d : \text{odd}, \end{cases} \tag{3.7}$$

we may put $|\gamma_1| > 1 > |\gamma_2|$.

First we suppose $|g_1^{-1}g_2| > 1$. Then the absolute values of the roots of the polynomial

$$(1 - (-1)^d x^2)^{e_1}(1 + g_1^{-1}g_2(-1)^d x^2)^{e_2}$$

appearing in the left hand side in (3.6) are not greater than 1. Let γ ($|\gamma| \geq |\gamma_1| > 1$) be the root of the polynomial appearing in the right hand side in (3.6) with the largest absolute value. Substituting $x = \gamma$ into (3.6), we have $A(\gamma)B(\gamma^d) = 0$, so that $A(\gamma) = 0$ or $B(\gamma^d) = 0$. If $A(\gamma) = 0$, substituting $x = \gamma^{1/d}$ into (3.6) again and noting that $|\gamma^{1/d}| > 1$, we have $A(\gamma^{1/d}) = 0$. Repeating this process, we obtain $A(\gamma^{1/d^k}) = 0$ for all $k \geq 0$, a contradiction. Thus we have $B(\gamma^d) = 0$. Substituting $x = \gamma^d$ into (3.6) and noting that $|\gamma^d| > 1$, we get $B(\gamma^{d^k}) = 0$ for all $k \geq 0$, a contradiction.

A similar contradiction is deduced in the case of $|g_1^{-1}g_2| \leq 1$.

Case II). $e_1 e_2 < 0$. In this case, we have

$$\begin{aligned} &A(x)B(x^d)(1 - (-1)^d x^2)^{h_1} Q(x)^{h_2} \\ &= A(x^d)B(x)(1 + g_1^{-1}g_2(-1)^d x^2)^{h_2} P(x)^{h_1}, \end{aligned} \tag{3.8}$$

where $h_1, h_2 \geq 1$.

First we prove that d is even. Suppose on the contrary that $d \geq 3$ is odd. The assumption $(R_0, R_1) \neq (0, 1)$ in Theorem 1.1 implies that $(g_1, g_2) \neq (1/\sqrt{5}, -1/\sqrt{5})$. Hence, at least one of the roots of $P(x)$ is not a root of $Q(x)$. Let γ ($|\gamma| \neq 1$) be as in (3.7) with $Q(\gamma) \neq 0$. Then, substituting $x = \gamma$ into (3.8), we have $A(\gamma)B(\gamma^d) = 0$, so that $A(\gamma) = 0$ or $B(\gamma^d) = 0$. Assume that $A(\gamma) = 0$. Since $d \geq 3$ and $\deg Q(x) = 2$, there exists a determination of $\gamma^{1/d}$ such that $Q(\gamma^{1/d}) \neq 0$. Hence, substituting $x = \gamma^{1/d}$ into (3.8) again and noting that $|\gamma^{1/d}| \neq 1$, we have $A(\gamma^{1/d}) = 0$. Repeating this process, we find a sequence $\{\gamma^{1/d^k}\}_{k \geq 0}$ of roots of γ such that $A(\gamma^{1/d^k}) = 0$ ($k \geq 0$). This is a contradiction. Thus, we have $B(\gamma^d) = 0$. Let $\zeta_d = e^{2\pi i/d}$ be primitive d-th root of unity. Then the number $\zeta_d\gamma$ is neither real nor purely imaginary because d is odd. Hence, substituting $x = \zeta_d\gamma$ into (3.8), we have $B(\zeta_d\gamma) = 0$, since

$$A(\gamma^d)(1 + g_1^{-1}g_2(-1)^d(\zeta_d\gamma)^2)P(\zeta_d\gamma) \neq 0.$$

Furthermore, noting that $d \geq 3$ and $\deg Q(x) = 2$, we see that there exists a complex nonreal number $\zeta_{d^2}\gamma^{1/d}$ such that

$$A(\zeta_d\gamma)(1 + g_1^{-1}g_2(-1)^d(\zeta_{d^2}\gamma^{1/d})^2)P(\zeta_{d^2}\gamma^{1/d}) \neq 0.$$

Hence, substituting $x = \zeta_{d^2}\gamma^{1/d}$ into (3.8), we get $B(\zeta_{d^2}\gamma^{1/d}) = 0$. Repeating this process, we obtain $B(\zeta_{d^{k+1}}\gamma^{1/d^k}) = 0$ for all $k \geq 0$, a contradiction.

Thus, we see that d is even and so the equation (3.8) becomes

$$A(x)B(x^d)(1 - x^2)^{h_1}Q(x)^{h_2} = A(x^d)B(x)(1 + g_1^{-1}g_2x^2)^{h_2}P(x)^{h_1}. \qquad (3.9)$$

Comparing the orders at $x = 1$ of both sides of (3.9), we obtain $g_1^{-1}g_2 = -1$ and $h_1 = h_2$. Dividing the both sides of (3.9) by $(1 - x^2)^{h_1}$, we have

$$A(x)B(x^d)(1 + g_1^{-1}x - x^2)^{h_1} = A(x^d)B(x)(1 + \sqrt{5}x - x^2)^{h_1}. \qquad (3.10)$$

Note that the polynomial $Q(x) = 1 + g_1^{-1}x - x^2$ has the real roots ξ_1, ξ_2 with $|\xi_1| > 1 > |\xi_2|$. Let γ_1 and γ_2 be the roots of $1 + \sqrt{5}x - x^2$ given by (3.7). Then $\gamma_i \neq \xi_j$ $(1 \leq i, j \leq 2)$ because $g_1^{-1} \neq \sqrt{5}$. Hence, substituting $x = \gamma_1$ into (3.10), we have $A(\gamma_1)B(\gamma_1^d) = 0$, so that either $A(\gamma_1) = 0$ or $B(\gamma_1^d) = 0$. Assume that $A(\gamma_1) = 0$. Since $|\xi_1| > 1 > |\xi_2|$, we can choose $\gamma_1^{1/d}$ $(|\gamma_1^{1/d}| > 1)$ such that $Q(\gamma_1^{1/d}) \neq 0$. Thus, substituting $x = \gamma_1^{1/d}$ into (3.10), we have $A(\gamma_1^{1/d}) = 0$. Continuing in this way, we create a sequence of complex numbers $\{\gamma^{1/d^k}\}_{k\geq 0}$ which are all roots of $A(x)$, a contradiction. In the case of $B(\gamma_1^d) = 0$, substituting $x = \zeta_d\gamma_1$ $(\neq \gamma_1)$ into (3.10), we get $B(\zeta_d\gamma_1) = 0$. Similarly, we obtain $B(\zeta_{d^{k+1}}\gamma_1^{1/d^k}) = 0$ for all $k \geq 0$, a contradiction.

Case III). $e_1 = 0$. By (2.2) and (3.3)

$$g(x)\Phi(x^{d^k})^{e_2} = \Phi(x)^{e_2}g(x^{d^k}) \qquad (k \geq 0).$$

Taking the limit as $k \to \infty$, we obtain $g(x) = \Phi(x)^{e_2}g(0)$ $(|x| < 1)$, so that $\Phi(x)$ is algebraic over $\mathbb{K}(x)$. Hence, by Lemma 2.2, we see that that $d = 2$ and one of the conditions $g_1 + g_2 = 1$ or $g_1 = g_2 = -1$ is satisfied, which is the property (iii) in Theorem 1.1.

Case IV). $e_2 = 0$. Similarly to the proof in Case III, we see that the function $\Psi(x)$ is algebraic over $\mathbb{K}(x)$. This contradicts Lemma 2.2.

Therefore the proof of Theorem 1.1 is completed. □

Next we prove the corollaries. Corollaries 1.2 and 1.3 follow immediately from Theorem 1.1. We prove Corollary 1.4.

Proof of Corollary 1.4. Let $R_n := F_{n+m}$ $(n \geq 0)$. Then the sequence $\{R_n\}_{n\geq 0}$ is expressed as $R_n = g_1\alpha^n + g_2\beta^n$, where

$$g_1 = \alpha^m/(\alpha - \beta), \quad g_2 = -\beta^m(\alpha - \beta).$$

Note that $g_1, g_2 \neq -1$ for any integer $m \geq 1$. If the infinite products (1.3) are algebraically dependent over $\mathbb{Q}$, then the condition (iii) in Theorem 1.1 is satisfied, namely, $d = 2$ and

$$1 = g_1 + g_2 = \frac{\alpha^m - \beta^m}{\alpha - \beta} = F_m.$$

Thus, we have $m = 1, 2$. Conversely, if $(d, m) = (2, 1)$ or $(2, 2)$, then we have by (2.3) and (3.1)

$$\prod_{k=1}^{\infty}\left(1 + \frac{1}{F_{2^k+1}}\right) = \frac{3(\sqrt{5}-1)}{2}, \quad \prod_{k=1}^{\infty}\left(1 + \frac{1}{F_{2^k+2}}\right) = 6 - 2\sqrt{5}.$$

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