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On the Fibonacci distances of ab, ac and bc*

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Abstract

For a positive real number x let the Fibonacci distance $\|x\|_F$ be the distance from x to the closest Fibonacci number. Here, we show that for integers $a > b > c \geq 1$, we have the inequality

$$\max\{\|ab\|_F, \|ac\|_F, \|bc\|_F\} > \exp(0.034\sqrt{\log a}).$$

Keywords: Fibonacci distance, Fibonacci diophantine triples

MSC: 11D72

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. For a positive real number x we put

$$\|x\|_F = \min\{|x - F_n| : n \geq 0\}.$$

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In [4], it was shown that there are no positive integers $a > b > c$ such that $ab + 1 = F_\ell,\ ac + 1 = F_m$ and $bc + 1 = F_n$ for some positive integers ℓ, m, n. Note that if such a triple would exist, then $\max\{\|ab\|_F, \|ac\|_F,\ \|bc\|_F\} \le 1$. This suggests investigating the more general problem of the triples of positive integers $a > b > c$ in which all three distances $\|ab\|_F,\ \|ac\|_F$ and $\|bc\|_F$ are small. We have the following result.

Theorem 1.1. *If $a > b > c \ge 1$ are integers then*

$$\max\{\|ab\|_F, \|ac\|_F, \|bc\|_F\} > \exp(0.034\sqrt{\log a}).$$

We have the following numerical corollary.

Corollary 1.2. *If $a > b > c \ge 1$ are positive integers such that*

$$\max\{\|ab\|_F, \|ac\|_F, \|bc\|_F\} \le 2,$$

then $a \le \exp(415.62)$. In fact, the solution with maximal a of the above inequality is the following:

$$(a, b, c) = (235, 11, 1).$$

2. The proof of Theorem 1.1

2.1. Preliminary results

We put $(\alpha, \beta) = ((1 + \sqrt{5})/2, (1 - \sqrt{5})/2)$ and recall the Binet formula

$$F_k = \frac{\alpha^k - \beta^k}{\sqrt{5}} \qquad \text{valid for all} \qquad k \ge 0. \tag{2.1}$$

We write $(L_k)_{k\ge0}$ for the Lucas companion of the Fibonacci sequence $(F_k)_{k\ge0}$ given by $L_0 = 2,\ L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \ge 0$. Its Binet formula is $L_k = \alpha^k + \beta^k$ for all $k \ge 0$. Furthermore, the inequalities

$$\alpha^{k-2} \le F_k \le \alpha^{k-1} \qquad \text{and} \qquad \alpha^{k-1} \le L_k \le \alpha^{k+1} \qquad \text{hold for all} \qquad k \ge 1. \tag{2.2}$$

We put

$$M = \max\{\|ab\|_F, \|ac\|_F, \|bc\|_F\}. \tag{2.3}$$

Lemma 2.1. *We have $M \ge 1$.*

Proof. Assume that $M = 0$. Then

$$6 \le ab = F_n, \quad 3 \le ac = F_m, \quad 2 \le bc = F_\ell$$

for some positive integers $n > m > \ell \ge 3$. If $n > 12$, then, by Carmichael's Primitive Divisor Theorem (see [2]), there exists a prime $p \mid F_n$ which does not divide F_k for any $1 \le k < n$. In particular, p cannot divide $F_m F_\ell = F_n c^2$, which is impossible. Thus, $n \le 12$. A case by case analysis shows that there is no solution. □

We put

$$ab + u = F_n, \qquad ac + v = F_m, \qquad bc + w = F_\ell, \tag{2.4}$$

where $|u| = \|ab\|_F$, $|v| = \|ac\|_F$ and $|w| = \|bc\|_F$. In the above, ℓ, m, n are positive integers and since $F_1 = F_2$, we may assume that $\min\{\ell, m, n\} \geq 2$. Furthermore,

$$\max\{|u|, |v|, |w|\} = M.$$

We treat first the case when $a \leq 4M$.

Lemma 2.2. *If $a \leq 4M$, then*

$$\max\{\ell, m, n\} \leq 5\log(3M).$$

Proof. If $a \leq 4M$, then

$$\alpha^{n-2} \leq F_n = ab + u \leq 4M(4M-1) + M < 16M^2,$$

so

$$\begin{aligned} n &\leq 2 + \frac{2\log(4M)}{\log\alpha} < 2 + 2.1\log(4M) \\ &= 2 + 2.1\log(4/3) + 2.1\log(3M) \\ &< 2.7 + 2.1\log(3M) < 5\log(3M). \end{aligned}$$

A similar argument works for ℓ and m. □

From now on, we assume that $a > 4M$.

Lemma 2.3. *Assume that $a > 4M$. Then*

(i) $n > \max\{\ell, m\}$;

(ii) $a > \sqrt{F_n}$;

(iii) $n \geq 3$.

Proof. (i) Note that

$$F_n = ab + u \geq ab - M > ac + M \geq ac + v = F_m,$$

where the middle inequality $ab - M > ac + M$ holds because it is equivalent to $a(b-c) > 2M$, which holds because $a > 4M$ and $b > c$, so $b - c \geq 1$. Hence, $n > m$. In the same way,

$$F_n = ab + u \geq ab - M > bc + M \geq bc + w = F_\ell.$$

The middle inequality is $ab - M > bc + M$, which is equivalent to $b(a-c) > 2M$. If $a - c \geq 2M$, then indeed $b(a-c) > 2M$ because $b > 1$. If $a - c < 2M$, it follows that $b > c > a - 2M > 2M$ (because $a > 4M$), and $a - c > 1$, so again the inequality $b(a - c) > 2M$ holds. This implies (i).

(ii) Here, by the previous argument, we have

$$a^2 > ab + M \geq ab + u = F_n.$$

This implies (ii).

(iii) is a consequence of (i) and of the fact that $\min\{\ell, m\} \geq 2$. □

Lemma 2.4. *When $a > 4M$, it is not possible to have $u = v = 0$.*

Proof. If $u = v = 0$, then, since $n > m$ by (i) of Lemma 2.3, we have

$$a \leq \gcd(ab, ac) = \gcd(F_n, F_m) = F_{\gcd(n,m)} = F_{n/d} \leq \alpha^{n/d-1},$$

where $d > 1$ is some divisor of n and where in the above we used the second inequality in (2.2). Hence, by (ii) of Lemma 2.3 and inequality (2.2), we get

$$\alpha^{n/2-1} \leq \sqrt{F_n} < a \leq \alpha^{n/d-1} \leq \alpha^{n/2-1},$$

a contradiction. □

The following lemma follows immediately by the Pigeon–Hole Principle and is well–known (see Lemma 1 in [3], for example).

Lemma 2.5. *Let $X \geq 3$ be a real number. Let a and b be nonnegative integers with $\max\{a, b\} \leq X$. Then there exist integers λ, ν not both zero with $\max\{|\lambda|, |\nu|\} \leq \sqrt{X}$ such that $|a\lambda + b\nu| \leq 3\sqrt{X}$.*

2.2. Some biquadratic numbers

We write

$$\begin{aligned} F_n - u &= \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right) - u = \frac{1}{\sqrt{5}}\left(\alpha^n - (-\alpha^{-1})^n\right) - u \\ &= \frac{\alpha^{-n}}{\sqrt{5}}\left(\alpha^{2n} - \sqrt{5}u\alpha^n - (-1)^n\right) \\ &= \frac{\alpha^{-n}}{\sqrt{5}}\left(\alpha^n - u_{1,n}\right)\left(\alpha^n - u_{2,n}\right). \end{aligned} \tag{2.5}$$

In the above,

$$u_{i,n} = \frac{\sqrt{5}u + (-1)^i\sqrt{5u^2 + 4(-1)^n}}{2}, \qquad i \in \{1, 2\}. \tag{2.6}$$

In the same way,

$$F_m - v = \frac{\alpha^{-m}}{\sqrt{5}}\left(\alpha^m - v_{1,m}\right)\left(\alpha^m - v_{2,m}\right), \tag{2.7}$$

where

$$v_{j,m} = \frac{\sqrt{5}v + (-1)^j\sqrt{5v^2 + 4(-1)^m}}{2}, \qquad j \in \{1, 2\}. \tag{2.8}$$

Observe that $u_{2,n} = (-1)^{n+1}u_{1,n}^{-1}$ and $v_{2,m} = (-1)^{m+1}v_{1,m}^{-1}$. Furthermore, both $u_{1,n}$, $u_{2,n}$ are roots of the polynomial

$$f_{u,n}(X) = (X^2 - (-1)^n)^2 - 5u^2X^2 = X^4 - (5u^2 + 2(-1)^n)X^2 + 1.$$

Similarly, both $v_{1,m}$ and $v_{2,m}$ are roots of the polynomial

$$f_{v,m}(X) = (X^2 - (-1)^m)^2 - 5v^2X^2 = X^4 - (5v^2 + 2(-1)^m)X^2 + 1.$$

Put $\mathbb{K} = \mathbb{Q}(\sqrt{5}, u_{1,n}, v_{1,m})$. Then the degree $d = [\mathbb{K} : \mathbb{Q}]$ of $\mathbb{K}$ over $\mathbb{Q}$ is a divisor of 32. Further, $\mathbb{K}$ contains α, $u_{1,n}$, $u_{2,n}$, $v_{1,m}$, $v_{2,m}$ and all their conjugates. It follows easily that all conjugates $u_{i,n}^{(s)}$ for $s = 1, \ldots, d$ satisfy

$$u_{i,n}^{(s)} = \frac{1}{2}\left(\pm\sqrt{5}u \pm \sqrt{5u^2 + 4(-1)^n}\right), \qquad i = 1, 2, \qquad s = 1, \ldots, d,$$

therefore the inequality

$$|u_{i,n}^{(s)}| \leq \frac{1}{2}\left(\sqrt{5}|u| + \sqrt{5u^2 + 4}\right) \leq \frac{1}{2}\left(\sqrt{5}M + \sqrt{5M^2 + 4}\right) < 3M \tag{2.9}$$

holds for $i = 1, 2$ and $s = 1, \ldots, d$. Similarly the inequality

$$|v_{j,m}^{(s)}| < 3M \tag{2.10}$$

holds for $j = 1, 2$ and $s = 1, \ldots, d$.

2.3. The first upper bound on n

The key step of the proof is writing

$$a \mid \gcd(ab, ac) = \gcd(F_n - u, F_m - v),$$

and passing in the above relation at the level of principal ideals in $\mathcal{O}_{\mathbb{K}}$. Using relations (2.5) and (2.7), we can write in $\mathcal{O}_{\mathbb{K}}$:

$$\begin{aligned} a\mathcal{O}_{\mathbb{K}} &\mid \gcd\left((\alpha^n - u_{1,n})(\alpha^n - u_{2,n})\mathcal{O}_{\mathbb{K}}, (\alpha^m - v_{1,m})(\alpha^m - v_{2,m})\mathcal{O}_{\mathbb{K}}\right) \\ &\mid \prod_{\substack{1\leq i\leq 2\\ 1\leq j\leq 2}} \gcd\left((\alpha^n - u_{i,n})\mathcal{O}_{\mathbb{K}}, (\alpha^m - v_{j,m})\mathcal{O}_{\mathbb{K}}\right). \end{aligned} \tag{2.11}$$

Passing to the norms in $\mathbb{K}$, we get

$$a^d = N_{\mathbb{K}/\mathbb{Q}}(a\mathcal{O}_{\mathbb{K}}) \leq \prod_{\substack{1\leq i\leq 2\\ 1\leq j\leq 2}} N_{\mathbb{K}/\mathbb{Q}}\left(\gcd\left((\alpha^n - u_{i,n})\mathcal{O}_{\mathbb{K}}, (\alpha^m - v_{j,m})\mathcal{O}_{\mathbb{K}}\right)\right). \tag{2.12}$$

For $i, j \in \{1,2\}$ put

$$I_{i,n,j,m} = \gcd\left(\left(\alpha^n - u_{i,n}\right)\mathcal{O}_{\mathbb{K}}, \left(\alpha^m - v_{j,m}\right)\mathcal{O}_{\mathbb{K}}\right). \tag{2.13}$$

In order to bound the norm of $I_{i,n,j,m}$ in $\mathbb{K}$, we use the following lemma.

Lemma 2.6. *When $a > 4M$, there exist coprime integers λ, ν satisfying $\max\{|\lambda|, |\nu|\} \le \sqrt{n}$ such that $|n\lambda + m\nu| \le 3\sqrt{n}$ and*

$$\alpha^{n\lambda+m\nu} - u_{i,n}^{\lambda} v_{j,m}^{\nu} \in I_{i,n,j,m}. \tag{2.14}$$

Proof. The existence of a pair of integers λ, ν not both zero such that the inequalities $\max\{|\lambda|, |\nu|\} \le \sqrt{n}$ and $|n\lambda + m\nu| \le 3\sqrt{n}$ hold follows from Lemma 2.6 for $(a, b, X) = (n, m, X)$. The condition $X \ge 3$ is fulfilled for our case by (iii) of Lemma 2.3. The fact that λ and ν can be chosen to be in fact coprime follows by replacing the pair (λ, ν) by $(\lambda/\gcd(\lambda, \nu), \nu/\gcd(\lambda, \nu))$. Finally, observing that

$$\alpha^n \equiv u_{i,n} \pmod{I_{i,n,j,m}} \qquad \text{and} \qquad \alpha^m \equiv v_{j,m} \pmod{I_{i,n,j,m}},$$

exponentiating the first of the above congruences to power λ, the second to power ν, and multiplying the resulting congruences, we get containment (2.14). □

In what follows, in this section we make the following assumption:

Assumption 2.7. *Assume that that pair (λ, ν) from the conclusion of Lemma 2.6 satisfies*

$$\alpha^{n\lambda+m\nu} - u_{i,n}^{\lambda} v_{j,m}^{\nu} \ne 0 \qquad \textit{for all} \qquad i, j \in \{1,2\}. \tag{2.15}$$

The main result of this section is the following.

Lemma 2.8. *Under the Assumption 2.7, when $a > 4M$, we have*

$$a \le 2^4 (3M)^{8\sqrt{n}}. \tag{2.16}$$

Proof. By congruence (2.14), we have

$$I_{i,n,j,m} \mid \left(\alpha^{n\lambda+m\nu} - u_{i,n}^{\lambda} v_{j,m}^{\nu}\right)\mathcal{O}_{\mathbb{K}},$$

and taking norms in $\mathbb{K}$ we get

$$N_{\mathbb{K}/\mathbb{Q}}(I_{i,n,j,m}) \mid N_{\mathbb{K}/\mathbb{Q}}\left((\alpha^{n\lambda+m\nu} - u_{i,n}^{\lambda} v_{j,m}^{\nu})\mathcal{O}_{\mathbb{K}}\right) = N_{\mathbb{K}/\mathbb{Q}}(\alpha^{n\lambda+m\nu} - u_{i,n}^{\lambda} v_{j,m}^{\nu}).$$

Since the number appearing on the right above is not zero by Assumption 2.7, we get

$$N_{\mathbb{K}/\mathbb{Q}}(I_{i,n,j,m}) \le N_{\mathbb{K}/\mathbb{Q}}\left(\alpha^{n\lambda+m\nu} - u_{i,n}^{\lambda} v_{j,m}^{\nu}\right),$$

therefore

$$N_{\mathbb{K}/\mathbb{Q}}(I_{i,n,j,m}) \le \prod_{s=1}^{d} \left|(\alpha^{(s)})^{n\lambda+m\nu} - (u_{i,n}^{(s)})^{\lambda} (v_{j,m}^{(s)})^{\nu}\right|.$$

Inequalities (2.9) and (2.10) together with the inequalities for λ and ν from the statement of Lemma 2.6 and the fact that $\alpha^{(s)} \in \{\alpha, \beta\}$ imply that

$$\left|(\alpha^{(s)})^{n\lambda+m\nu} - (u_{i,n}^{(s)})^{\lambda}(v_{j,m}^{(s)})^{\nu}\right| \leq |\alpha|^{3\sqrt{n}} + (3M)^{2\sqrt{n}} < 2(3M)^{2\sqrt{n}},$$

for $s = 1, \ldots, d$, where for the last inequality we used $(3M)^2 \geq 3^2 > \alpha^3$. Hence,

$$N_{\mathbb{K}/\mathbb{Q}}(I_{i,n,j,m}) \leq 2^d(3M)^{2d\sqrt{n}},$$

Thus, by inequality (2.12), we get

$$a^d \leq \prod_{\substack{1\leq i\leq 2\\ 1\leq j\leq 2}} N_{\mathbb{K}/\mathbb{Q}}(I_{i,n,j,m}) \leq 2^{4d}(3M)^{8d\sqrt{n}},$$

giving

$$a \leq 2^4(3M)^{8\sqrt{n}},$$

which is what we wanted to prove. □

Lemma 2.8 has the following consequence.

Lemma 2.9. *Under the Assumption 2.7, when $a > 4M$, we have*

$$n < (41\log(3M))^2. \tag{2.17}$$

Proof. Combining the inequality (2.16) of Lemma 2.8 for a with (ii) of Lemma 2.3 and inequality (2.2), we get

$$\alpha^{n/2-1} \leq \sqrt{F_n} < a \leq 2^4(3M)^{8\sqrt{n}}.$$

It gives

$$\frac{n}{2} - 1 < \frac{4\log 2}{\log\alpha} + \left(\frac{8\log(3M)}{\log\alpha}\right)\sqrt{n} < 5.8 + 16.7\log(3M)\sqrt{n},$$

or

$$n < \left(\frac{13.6}{\log(3M)\sqrt{n}} + 33.4\right)\log(3M)\sqrt{n} < (41\log M)\sqrt{n},$$

because $n \geq 3$. So

$$n < (41\log(3M))^2,$$

which is what we wanted to prove. □

From now on, we assume that

$$n \geq (41\log(3M))^2. \tag{2.18}$$

Lemma 2.2 tells us that if this the case, then also the inequality $a > 4M$ holds. In particular, for such values of n Assumption 2.7 cannot hold. This is the case we study next.

2.4. General remarks when Assumption 2.7 does not hold

From now on, we study the cases when Assumption 2.7 does not hold. In this case, there exist $i_0, j_0 \in \{1,2\}$ such that

$$\alpha^{n\lambda+m\nu} = u_{i_0,n}^{\lambda} v_{j_0,m}^{\nu}. \tag{2.19}$$

In particular

$$(\alpha^4)^{n\lambda+m\nu} = (u_{i_0,n}^4)^{\lambda}(v_{j_0,m}^4)^{\nu}. \tag{2.20}$$

Observe that if $u=0$, then

$$u_{i,n} = (-1)^i\sqrt{(-1)^n}, \qquad i \in \{1,2\},$$

therefore $u_{i_0,4}^4 = 1$. Similarly, if $v=0$, then $v_{j_0,m}^4 = 1$. If $u \neq 0$, then write

$$5u^2 + 4(-1)^n = d_{u,n}y_{u,n}^2,$$

where $d_{u,n}$ is a positive square free integer and $y_{u,n}$ is some positive integer. Observe that $d_{u,n}$ is coprime to 5 so $5d_{u,n}$ is square free. Observe further that $5u^2$ and $d_{u,n}y_{u,n}^2$ have the same parity and

$$u_{i,n}^2 = \frac{1}{2}\left(\frac{5u^2 + d_{u,n}y_{u,n}^2}{2} + (-1)^i\sqrt{5d_{u,n}}uy_{u,n}\right) \in \mathbb{Q}(\sqrt{5d_{u,n}}) = \mathbb{K}_{u,n}$$

for $i=1,2$. Moreover, $u_{1,n}^2$ is an algebraic integer and a unit in the quadratic field $\mathbb{K}_{u,n}$ the inverse of which is $u_{2,n}^2$. Similarly, if $v \neq 0$, we write

$$5v^2 + 4(-1)^m = d_{v,m}y_{v,m}^2,$$

where $d_{v,m}$ is some positive square free integer and $y_{v,m}$ is some positive integer. As in the case of $u_{i,n}^2$, we have

$$v_{j,m}^2 \in \mathbb{Q}(\sqrt{5d_{v,m}}) = \mathbb{K}_{v,m}$$

is a unit in the quadratic field $\mathbb{K}_{v,m}$. We continue with the following result.

Lemma 2.10. *In case when $uv \neq 0$, and inequality* (2.18) *holds, it is not possible that $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{5d_{u,n}})$ and $\mathbb{Q}(\sqrt{5d_{v,m}})$ are three distinct quadratic fields.*

Proof. Assume that the three quadratic fields $\mathbb{Q}(\sqrt{5})$, $\mathbb{K}_{u,n}$ and $\mathbb{K}_{v,m}$ were distinct. Then $d_{u,n}$ and $d_{v,m}$ are distinct square free integers larger than 1 which are coprime to 5. By Galois theory, there is an automorphism of $\mathbb{Q}(\sqrt{5}, \sqrt{5d_{u,n}}, \sqrt{5d_{v,m}})$, let's call it σ, such that $\sigma(\sqrt{5}) = -\sqrt{5}$, $\sigma(\sqrt{d_{u,n}}) = -\sqrt{d_{u,n}}$ and $\sigma(\sqrt{d_{v,m}}) = -\sqrt{d_{v,m}}$. Observe that σ leaves both $\sqrt{5d_{u,n}}$ and $\sqrt{5d_{v,m}}$ invariant, therefore $\sigma(u_{i,n}^2) = u_{i,n}^2$ and $\sigma(v_{j,m}^2) = v_{j,m}^2$ for $i,j \in \{1,2\}$, while $\sigma(\alpha) = \beta$. Applying σ to the equation (2.20), we get

$$(\beta^4)^{\lambda m+\nu n} = (u_{i_0,n}^4)^{\lambda}(v_{j_0,m}^4)^{\nu}. \tag{2.21}$$

Multiplying relations (2.20) and (2.21), we get

$$1 = (u_{i_0,n}^2)^{4\lambda}(v_{j_0,m}^2)^{4\nu} \qquad \text{or} \qquad (u_{i_0,n}^2)^{4\lambda} = (v_{j_0,m}^2)^{-4\nu}.$$

Thus, $u_{i_0,n}^{4\lambda}$ is in $\mathbb{Q}(\sqrt{5d_{u,n}}) \cap \mathbb{Q}(\sqrt{5d_{v,m}}) = \mathbb{Q}$. Since $u_{i_0,n}^2$ is in fact a positive unit ditinct from 1 in $\mathbb{K}_{u,n}$, we get that $\lambda = 0$, and then also $\nu = 0$, which is not allowed. □

We now put

$$\mathbb{U} = \mathbb{Q}(\sqrt{5}, u_{1,n}^4, v_{1,m}^4).$$

If $u = 0$, then $u_{1,n}^4 = 1$, so that $\mathbb{U}$ has degree 2 or 4 over $\mathbb{Q}$. The same holds when $v = 0$. Finally, when $uv \neq 0$, then $u_{1,n}^4 \in \mathbb{Q}(\sqrt{5d_{u,n}})$ and $v_{1,m}^4 \in \mathbb{Q}(\sqrt{5d_{v,m}})$, so

$$\mathbb{U} \subseteq \mathbb{Q}(\sqrt{5}, \sqrt{5d_{u,n}}, \sqrt{5d_{v,m}}).$$

Lemma 2.9 implies that the field appearing in the right hand side of the above containment cannot have degree 8 over $\mathbb{Q}$. Hence, $\mathbb{U}$ must have degree 2 or 4 over $\mathbb{Q}$ in case $uv \neq 0$ as well.

We shall refer to the case when $[\mathbb{U} : \mathbb{Q}] = 4$ as the *rank two case*, and to the case when $[\mathbb{U} : \mathbb{Q}] = 2$ as the *rank one case.*

2.5. The rank two case

We start with the following result.

Lemma 2.11. *Assume that inequality* (2.18) *holds. Then in the rank two case, we have* $uv \neq 0$.

Proof. Assume, for example, that $u = 0$. Then, since we are in the rank two case, it follows that $d_{v,m} > 1$. Now equation (2.20) implies that

$$(\alpha^4)^{n\lambda n+m\nu} = (u_{i_0,n}^4)^{\lambda}(v_{j_0,m}^4)^{\nu} = (v_{j_0,m}^4)^{\nu}.$$

This shows that $(v_{j_0,m}^4)^{\nu} \in \mathbb{Q}(\sqrt{5}) \cap \mathbb{Q}(\sqrt{5d_{v,m}}) = \mathbb{Q}$. Since $v_{j_0,m}^2$ is in fact a unit of infinite order in $\mathbb{K}_{v,m}$, we get that $\nu = 0$, which implies that also $n\lambda + m\nu = 0$, therefore $n\lambda = 0$. Thus, $\lambda = \nu = 0$, which is not allowed. The same contradiction is obtained when $v = 0$. □

Lemma 2.12. *Assume that inequality* (2.18) *holds. Then in the rank two case, we have* $d_{u,n} = d_{v,m} > 1$.

Proof. If this were not so, then we would either have $d_{u,n} = 1$ and $d_{v,m} > 1$ or $d_{u,n} > 1$ and $d_{v,m} = 1$. Assume say that $d_{u,n} = 1$ and $d_{v,m} > 1$. Then $u_{i_0,n}^4 \in \mathbb{Q}(\sqrt{5})$. Relation (2.20) now shows that

$$(\alpha^4)^{n\lambda+m\nu}(u_{i_0,n}^{-4})^{\lambda} = (v_{j_0,m}^4)^{\nu}.$$

The above relation shows that $(v_{j_0,m}^4)^\nu \in \mathbb{Q}(\sqrt{5}) \cap \mathbb{Q}(\sqrt{5d_{v,m}}) = \mathbb{Q}$. This implies easily that $\nu = 0$. Now relation (2.20) shows that $(\alpha^4)^{n\lambda} = (u_{i_0,n}^4)^{n\lambda}$. Since λ and $\nu = 0$ are coprime, we get that $\lambda = 1$, and so $\alpha^{4n} = u_{i_0,n}^4$. This shows that $\alpha^n = \pm u_{i_0,n}$. In particular,

$$\alpha^n = |u_{i_0,n}| < 3M$$

(see inequality (2.9)), so that

$$n \leq \frac{\log(3M)}{\log \alpha} < 3\log(3M),$$

which contradicts inequality (2.18). □

Lemma 2.13. *Assume that inequality* (2.18) *holds. Then we cannot be in the rank two case.*

Proof. Assume that we are in the rank two case. By Lemma 2.12, we have $d_{u,n} = d_{v,m} > 1$. Put $D = d_{u,n}$. We then have the following relations

$$5u^2 - Dy_{u,n}^2 = 4(-1)^{n+1};$$
$$5v^2 - Dy_{v,m}^2 = 4(-1)^{m+1}.$$

By a result of Nagell (see Theorem 3 in [5]), we have $n \equiv m \pmod 2$. Further, put $\varepsilon = (-1)^{n+1}$ and let $(X, Y) = (a, b)$ be the minimal solution in positive integers of the Diophantine equation

$$5X^2 - DY^2 = 4\varepsilon. \tag{2.22}$$

Then all other positive integer solutions (X, Y) of the above equation (2.22) are of the form

$$\frac{\sqrt{5}X + \sqrt{D}Y}{2} = \left(\frac{\sqrt{5}a + \sqrt{D}b}{2}\right)^k$$

for some odd positive integer k. In particular, putting $\zeta = (\sqrt{5}a + \sqrt{D}b)/2$, we then have

$$\frac{\sqrt{5}|u| + \sqrt{D}y_{u,n}}{2} = \zeta^{k_u} \qquad \text{and} \qquad \frac{\sqrt{5}|v| + \sqrt{D}y_{v,m}}{2} = \zeta^{k_v}$$

for some odd positive integers k_u and k_v. We now see invoking (2.6) that

$$u_{i,n} = \operatorname{sign}(u)\left(\frac{\sqrt{5}|u| + (-1)^i \operatorname{sign}(u)\sqrt{D}y_{u,n}}{2}\right) = \operatorname{sign}(u)\zeta^{\eta_{i,u}k_u},$$

where $\eta_{i,u} = 1$ if $\operatorname{sign}(u) = (-1)^i$ and $\eta_{i,u} = -1$ if $\operatorname{sign}(u) = (-1)^{i+1}$. Similarly,

$$v_{j,m} = \operatorname{sign}(v)\zeta^{\eta_{j,v}k_v}$$

where $\eta_{j,v} \in \{\pm 1\}$. Going back to relation (2.19), we get

$$\alpha^{n\lambda+m\nu} = \text{sign}(u)^{\lambda}\text{sign}(v)^{\nu}\zeta^{\eta_{i_0,u}\lambda k_u+\eta_{j_0,v}\nu k_v}.$$

Since α and ζ are multiplicatively independent, we get that

$$n\lambda + m\nu = 0, \qquad \text{sign}(u)^{\lambda}\text{sign}(v)^{\nu} = 1, \qquad \eta_{i_0,u}\lambda k_u + \eta_{j_0,v}\nu k_v = 0.$$

From the left relation above we get that λ and ν have opposite signs. From the right relation above, we get that $\lambda/\nu = -\eta_{j_0,v}\eta_{i_0,u}k_v/k_u$, and since λ and ν are coprime, we get that they are both odd and that $\eta_{i_0,u} = \eta_{j_0,v}$. Finally, since λ and ν are both odd, from the middle relation above we get that $\text{sign}(u) = \text{sign}(v)$. Put $e = \gcd(k_u, k_v)$. Writing $k_u = e\ell_u$, $k_v = e\ell_v$, and putting $\delta = \text{sign}(u)$ and $\eta = \eta_{i_0,u}$, we get that

$$u_{i_0,n} = \delta(\zeta^{\eta e})^{\ell_u} = (\delta\zeta^{\eta e})^{\ell_u} \qquad \text{and} \qquad v_{j_0,m} = \delta(\zeta^{\eta e})^{\ell_v} = (\delta\zeta^{\eta e})^{\ell_v}.$$

Writing $\zeta_1 = \delta\zeta^{\eta e}$, we get that

$$u_{i_0,n} = \zeta_1^{\ell_u} \qquad \text{and} \qquad v_{j_0,m} = \zeta_1^{\ell_v}.$$

Further, $\ell_u/\ell_v = k_u/k_v = -\nu/\lambda = n/m$, so that if we put $k = \gcd(m,n)$, then $n = \ell_u k$ and $m = \ell_v k$. Since $u_{1,n}u_{2,n} = \varepsilon = v_{1,m}v_{2,m}$, it follows that if i_1 and j_1 are such that $\{i_0, i_1\} = \{j_0, j_1\} = \{1, 2\}$, then

$$u_{i_1,n} = \varepsilon\zeta_1^{-\ell_u} = \zeta_2^{\ell_u} \qquad \text{and} \qquad v_{j_1,m} = \varepsilon\zeta_1^{-\ell_v} = \zeta_2^{\ell_v},$$

where $\zeta_2 = \varepsilon\zeta_1^{-1}$. Thus,

$$\begin{aligned} \alpha^n - u_{i_0,n} &= (\alpha^k)^{\ell_u} - \zeta_1^{\ell_u}; \\ \alpha^n - u_{i_1,n} &= (\alpha^k)^{\ell_u} - \zeta_2^{\ell_u}; \\ \alpha^m - v_{j_0,m} &= (\alpha^k)^{\ell_v} - \zeta_1^{\ell_v}; \\ \alpha^m - v_{j_1,m} &= (\alpha^k)^{\ell_v} - \zeta_2^{\ell_v}. \end{aligned}$$

Since ℓ_u and ℓ_v are coprime, it follows that

$$I_{i_0,n,j_0,m} = \gcd\left(\left((\alpha^k)^{\ell_u} - \zeta_1^{\ell_u}\right)\mathcal{O}_{\mathbb{K}}, \left((\alpha^k)^{\ell_v} - \zeta_1^{\ell_v}\right)\mathcal{O}_{\mathbb{K}}\right) = (\alpha^k - \zeta_1)\mathcal{O}_{\mathbb{K}}. \quad (2.23)$$

Similarly,

$$I_{i_1,n,j_1,m} = \gcd\left(\left((\alpha^k)^{\ell_u} - \zeta_2^{\ell_u}\right)\mathcal{O}_{\mathbb{K}}, \left((\alpha^k)^{\ell_v} - \zeta_2^{\ell_v}\right)\mathcal{O}_{\mathbb{K}}\right) = (\alpha^k - \zeta_2)\mathcal{O}_{\mathbb{K}}. \quad (2.24)$$

As for $I_{i_0,n,j_1,m}$, we have

$$(\alpha^k)^{\ell_u} \equiv \zeta_1^{\ell_u} \pmod{I_{i_0,n,j_1,m}} \qquad \text{and} \qquad (\alpha^k)^{\ell_v} \equiv \zeta_2^{\ell_v} \pmod{I_{i_0,n,j_1,m}}.$$

Exponentiating the first congruence above to ℓ_v and the second to ℓ_u, and comparing the resulting congruences, we get

$$\zeta_1^{\ell_u\ell_v} \equiv \zeta_2^{\ell_u\ell_v} \pmod{I_{i_0,n,j_1,m}}$$

so that

$$I_{i_0,n,j_1,m} \mid (\zeta_1^{2\ell_u\ell_v} - \varepsilon)\mathcal{O}_{\mathbb{K}}, \tag{2.25}$$

and the principal ideal on the right above is not zero. Similarly,

$$I_{i_1,n,j_0,m} \mid (\zeta_2^{2\ell_u\ell_v} - \varepsilon)\mathcal{O}_{\mathbb{K}}. \tag{2.26}$$

Hence, divisibility relation (2.11) together with relations (2.23)–(2.26) now implies

$$a \mid (\alpha^k - \zeta_1)(\alpha^k - \zeta_2)(\zeta_1^{2\ell_u\ell_v} - \varepsilon)(\zeta_2^{2\ell_u\ell_v} - \varepsilon).$$

Taking norms in $\mathbb{K}$, we get that

$$a^d \le |N_{\mathbb{K}/\mathbb{Q}}(\alpha^k - \zeta_1)||N_{\mathbb{K}/\mathbb{Q}}(\alpha^k - \zeta_2)||N_{\mathbb{K}/\mathbb{Q}}(\zeta_1^{2\ell_u\ell_v} - \varepsilon)||N_{\mathbb{K}/\mathbb{Q}}(\zeta_2^{2\ell_u\ell_v} - \varepsilon)|. \tag{2.27}$$

Since

$$u_{i_0,n}^{(s)} = (\zeta_1^{(s)})^{\ell_u}$$

and $\ell_u \ge 1$, it follows, by (2.9), that

$$|\zeta_1^{(s)}| < 3M.$$

Similarly, $|\zeta_2^{(s)}| < 3M$. Furthermore,

$$\zeta \ge \frac{\sqrt{5}+\sqrt{3}}{2} > \alpha.$$

Since

$$\zeta^{e\ell_u} = |u_{i,n}| \qquad \text{for some} \qquad i \in \{1,2\},$$

we get that

$$\ell_u \le e\ell_u \le \frac{\log(3M)}{\log\alpha} < 2.1\log(3M).$$

Similarly, $\ell_v \le 2.1\log(3M)$. It now follows that

$$|(\alpha^{(s)})^k - \zeta_1^{(s)}| \le \alpha^k + 3M \le 6M\alpha^k \qquad \text{for all} \qquad s = 1,\dots,d.$$

Similarly,

$$|(\alpha^{(s)})^k - \zeta_2^{(s)}| \le \alpha^k + 3M \le 6M\alpha^k \qquad \text{for all} \qquad s = 1,\dots,d.$$

Finally,

$$|(\zeta_1^{(s)})^{2\ell_u\ell_v} - \varepsilon| \le (|(\zeta_1^{(s)})^{\ell_u}|)^{2\ell_v} + 1 = |u_{i_0,n}^{(s)}|^{2\ell_v} + 1 < 2(3M)^{4.2\log(3M)},$$

for all $s = 1, \dots, d$ and a similar inequality holds with ζ_1 replaced by ζ_2. We thus get that

$$|N_{\mathbb{K}/\mathbb{Q}}(\alpha^k - \zeta_i)| < (6M)^d \alpha^{dk}, \qquad |N_{\mathbb{K}/\mathbb{Q}}(\zeta_i^{2\ell_u \ell_v} - \varepsilon)| < 2^d (3M)^{4.2d \log(3M)}$$

for $i = 1, 2$, which together with (2.27) gives

$$a^d < (6M)^{2d} \alpha^{2dk} 2^{2d} (3M)^{8.4d \log(3M)},$$

or

$$a < 16(3M)^{2+8.4 \log(3M)} \alpha^{2k}. \tag{2.28}$$

Observe that $k = n/\ell_u = m/\ell_v$, and $n > m$ (by (i) of Lemma 2.3) and $\ell_u > \ell_v$ are odd and coprime. Thus, $\ell_u \geq 3$. If $\ell_u = 3$, then $\ell_v = 1$, so $m = n/3$. If this is the case, then

$$a \leq ac = F_m - v \leq F_m + M < F_m + a/2$$

(because $a > 4M$), therefore $a < 2F_m = 2F_{n/3}$. With (ii) of Lemma 2.3 and inequality (2.2), we get

$$\alpha^{n/2-1} < \sqrt{F_n} < a < 2F_{n/3} < 2\alpha^{n/3-1},$$

therefore

$$n < \frac{6 \log 2}{\log \alpha}, \qquad \text{so} \qquad n \leq 4,$$

a contradiction. Thus, we conclude that it is not possible that $\ell_u = 3$. Thus, $\ell_u \geq 5$. Hence, $k \leq n/5$. Inequality (2.28) together with (ii) of Lemma 2.3 and (2.2) give

$$\alpha^{n/2-1} < \sqrt{F_n} < a < 16(3M)^{2+8.4 \log(3M)} \alpha^{2n/5}.$$

Then

$$\begin{aligned} \frac{n}{10} &< 1 + \frac{\log 16}{\log \alpha} + \left(\frac{2 + 8.4 \log(3M)}{\log \alpha} \right) \log(3M) \\ &< 7.8 + 2.1(2 + 8.4 \log(3M)) \log(3M) \\ &< 7.8 + 22(\log(3M))^2, \end{aligned}$$

so

$$n < 78 + 220(\log(3M))^2 < 300(\log(3M))^2,$$

which contradicts inequality (2.18). □

In particular, if inequality (2.18) holds, then we are in the rank one case.

2.6. The rank one case

Lemma 2.14. *Assume that* (2.18) *holds. We have* $u = \pm F_t$ *and* $v = \pm F_s$ *for some nonnegative integers* t, s *which are either zero or satisfy* $n \equiv t \pmod 2$ *and* $m \equiv s \pmod 2$.

Proof. Since we are in the rank one case, it follows that $u^2_{i_0,n} \in \mathbb{Q}(\sqrt{5})$. So, if $u \neq 0$, it follows that $d_{u,n} = 1$, so that $5u^2 + 4(-1)^n = y^2_{u,n}$. In particular, $y^2_{u,n} - 5u^2 = 4(-1)^n$. It is well–known that if (X, Y) are positive integers such that $Y^2 - 5X^2 = 4(-1)^k$ for some integer k, then $X = F_t$ for some nonnegative integer $t \equiv k \pmod 2$ (and the value of Y is L_k). In particular, $|u| = F_t$ for some integer t which is congruent to n modulo 2. The statement about v can be proved in the same way. $\square$

We now have

$$ab = F_n - u = F_n - \operatorname{sign}(u)F_t = F_{(n-t_1)/2}L_{(n+t_1)/2},$$

where $t_1 = \varepsilon_{u,t,n}t$ and $\varepsilon_{u,t,n} \in \{\pm 1\}$ depends on the sign of u as well as on the residue classes of n and t modulo 4. Similarly, we have

$$ac = F_m - v = F_m - \operatorname{sign}(v)F_s = F_{(m-s_1)/2}L_{(m+s_1)/2},$$

and $s_1 = \varepsilon_{v,m,s}s$ for some $\varepsilon_{v,m,s} \in \{\pm 1\}$. Observe also that either $t = 0$, or $t \geq 1$ and

$$\alpha^{t-2} \leq F_t \leq M,$$

so that

$$t \leq 2 + \frac{\log M}{\log \alpha} < 2 + 2.1 \log M < 2.1 \log(3M). \tag{2.29}$$

The same inequality holds with t replaced by $|t_1|$, s, $|s_1|$. Note also that

$$n \pm t_1 \geq n - t > (41 \log(3M))^2 - 2.1 \log(3M) > 0.$$

Lemma 2.15. *One of the following holds:*

(i) $n - t_1 = m - s_1$;

(ii) $n + t_1 = m + s_1$;

(iii) $s = 0$, $m = (n - t_1)/2$ *and* $b = L_{(n+t_1)/2}c$.

Proof. As a warm up, we start with the case when $t = 0$. Then

$$\begin{aligned} a &\leq \gcd(ab, ac) = \gcd(F_n, F_{(m-s_1)/2}L_{(m+s_1)/2}) \\ &\leq \gcd(F_n, F_{(m-s_1)/2}) \gcd(F_n, L_{(m+s_1)/2}) \\ &\leq F_{\gcd(n,(m-s_1)/2)}L_{\gcd(n,(m+s_1)/2)}. \end{aligned}$$

In the above argument, we used the fact that $\gcd(F_p, F_q) = F_{\gcd(p,q)}$ and that $\gcd(F_p, L_q) \le L_{\gcd(p,q)}$ for positive integers p and q. Put

$$\gcd(n,(m-t_1)/2) = n/d_1 \qquad \text{and} \qquad \gcd(n,(m+t_1)/2) = n/d_2.$$

If $d_1 = 1$, then $n \mid (m-t_1)/2$, therefore $n - t_1 > m - t_1 \ge 2n$, or

$$n \le -t_1 \le t < 2.1\log(3M),$$

contradicting inequality (2.18). A similar inequality holds if $d_2 = 1$. So, from now on, we assume that $\min\{d_1, d_2\} \ge 2$. If $\min\{d_1, d_2\} \ge 10$, we then have

$$\alpha^{n/2-1} < \sqrt{F_n} < a \le F_{n/d_1} L_{n/d_2} \le \alpha^{n/d_1 + n/d_2} \le \alpha^{n/5},$$

giving $n/2 - 1 < n/5$, so $n \le 3$, a contradiction.

So, we may assume that $\min\{d_1, d_2\} \le 9$. Assume that $\max\{d_1, d_2\} \le 9$. Write $n/d_1 = (m - s_1)/d_3$ and $n/d_2 = (m + s_1)/d_4$. If $d_3 \ge d_1 + 1$, we then get

$$m - s_1 = \frac{d_3 n}{d_1} \ge n + \frac{n}{d_1} > m + \frac{n}{d_1},$$

so

$$n < -d_1 s_1 \le d_1 s \le 9 \times 2.1\log(3M) < 20\log(3M),$$

contradicting inequality (2.18). Thus, $\max\{d_1, d_2\} \ge 10$. If $\min\{d_1, d_2\} \ge 3$, we then get that

$$\alpha^{n/2-1} < \sqrt{F_n} < a \le F_{n/d_1} L_{n/d_2} \le \alpha^{n/d_1 + n/d_2} \le \alpha^{n/3 + n/10},$$

giving $n < 15$, which is impossible. Thus, $\min\{d_1, d_2\} = 2$ giving

$$\text{either} \qquad n/2 = \gcd(n,(m-s_1)/2), \qquad \text{or} \qquad n/s = \gcd(n,(m+s_1)/2).$$

Thus, either $n/2 = (m - s_1)/2d_3$, or $n/2 = (m + s_1)/2d_4$ for some divisors d_3 or d_4 of $(m - s_1)/2$ and $(m + s_1)/2$, respectively. If we are in the first case and $d_3 > 1$, then

$$m - s_1 = d_3 n \ge 2n > m + n$$

giving $n < -s_1 \le s < 2.1\log(3M)$, a contradiction. The same inequality is obtained if $n/2 = (m + s_1)/2d_4$ for some divisor $d_4 > 1$ of $(m + s_1)/2$. The last case is $n/2 = (m - s_1)/2$ (or $n = m - s_1$), or $n/2 = (m + s_1)/2$ (or $n = m + s_1$), which is (ii) for the particular case when $t = 0$.

Assume next that $st \ne 0$. In this case,

$$\begin{aligned}
a &\le \gcd(ab, ac) = \gcd(F_{(n-t_1)/2} L_{(n+t_1)/2}, F_{(m-s_1)/2} L_{(m+s_1)/2}) \\
&\le \gcd(F_{(n-t_1)/2}, F_{(m-s_1)/2}) \gcd(F_{(n-t_1)/2}, L_{(m+s_1)/2}) \\
&\times \gcd(L_{(n+t_1)/2}, F_{(m-s_1)/2}) \gcd(L_{(n+t_1)/2}, L_{(m+s_1)/2}) \\
&\le F_{\gcd((n-t_1)/2,(m-s_1)/2)} L_{\gcd((n-t_1)/2,(m+s_1)/2)}
\end{aligned}$$

$$\times L_{\gcd((n+t_1)/2,(m-s_1)/2)} L_{\gcd((n+t_1)/2,(m+s_1)/2)}. \tag{2.30}$$

Write

$$\gcd\left(\frac{n-t_1}{2}, \frac{m-s_1}{2}\right) = \frac{n-t_1}{2d_1};$$
$$\gcd\left(\frac{n-t_1}{2}, \frac{m+s_1}{2}\right) = \frac{n-t_1}{2d_2};$$
$$\gcd\left(\frac{n+t_1}{2}, \frac{m-s_1}{2}\right) = \frac{n+t_1}{2d_3};$$
$$\gcd\left(\frac{n+t_1}{2}, \frac{m+s_1}{2}\right) = \frac{n+t_1}{2d_4}$$

for some positive integers d_1, d_2, d_3, d_4. Assume that $\min\{d_1, d_2, d_3, d_4\} \geq 10$. Then

$$\alpha^{n/2-1} < \sqrt{F_n} < a \leq F_{(n-t_1)/2d_1} L_{(n-t_1)/2d_2} L_{(n+t_1)/2d_3} L_{(n+t_1)/2d_4}$$
$$< \alpha^{(n-t_1)/2d_1+(n-t_1)/2d_2+(n+t_1)/2d_3+(n+t_1)/2d_4+2} \leq \alpha^{(n+t)/5+2},$$

giving

$$n < \frac{10}{3}\left(3 + \frac{t}{5}\right) < 10 + \frac{4.2}{3}\log(3M) < 12\log(3M),$$

contradicting inequality (2.18). Suppose $\min\{d_1, d_2, d_3, d_4\} \leq 9$. Assume that there exist $i \neq j$ such that both $d_i \leq 9$ and $d_j \leq 9$. Just to fix ideas, we assume that $i = 1$, $j = 3$. Put

$$\frac{n-t_1}{2d_1} = \frac{m-s_1}{2d_5}, \qquad \text{and} \qquad \frac{n+t_1}{2d_3} = \frac{m-s_1}{2d_7}. \tag{2.31}$$

Assume say that $d_5 \geq d_1 + 1$. Then

$$m - s_1 = \frac{d_5(n-t_1)}{d_1} \geq n - t_1 + \frac{n-t_1}{d_1} > m - t_1 + \frac{n-t_1}{d_1},$$

so

$$n \leq t_1 + d_1(t_1 - s_1) \leq t + 9(s+t) < 20\max\{s,t\} < 42\log(3M),$$

contradicting inequality (2.18). A similar contradiction is obtained if one supposes that $d_7 \geq d_3 + 1$. Thus, we may assume that $d_5 \leq d_1 \leq 9$ and $d_7 \leq d_3 \leq 9$. Equations (2.31) give

$$d_5 n - d_1 m = d_5 t_1 - d_1 s_1;$$
$$d_7 n - d_3 m = -d_7 t_1 - d_3 s_1.$$

One checks that the above system has a unique solution (m, n), and the same is true for the other values of $i \neq j$ in $\{1, 2, 3, 4\}$, not only for $(i, j) = (1, 3)$. We solve the system by Cramer's rule getting

$$\begin{vmatrix} d_5 & -d_1 \\ d_7 & -d_3 \end{vmatrix} n = \begin{vmatrix} d_5 t_1 - d_1 s_1 & -d_1 \\ -d_7 t_1 - d_3 s_1 & -d_3 \end{vmatrix}.$$

Thus, using Hadamard's inequality,

$$\begin{aligned} n &\le \begin{vmatrix} d_5t_1 - d_1s_1 & -d_1 \\ -d_7t_1 - d_3s_1 & -d_3 \end{vmatrix} \\ &\le \sqrt{d_1^2 + d_3^2} \times \sqrt{(d_5t_1 - d_1s_1)^2 + (d_7t_1 + d_3s_1)^2} \\ &\le 9\sqrt{2} \times 9 \times 2 \times \sqrt{2} \max\{s, t\} < 700 \log(3M), \end{aligned}$$

which contradicts inequality (2.18). So, we may assume that there exists at most one $i \in \{1, 2, 3, 4\}$ such that $d_i \le 9$. If $d_i \ge 2$, then

$$\begin{aligned} \alpha^{n/2-1} &< \sqrt{F_n} < a \le F_{(n-t_1)/2d_1} L_{(n-t_1)/2d_2} L_{(n+t_1)/2d_3} L_{(n+t_1)/2d_4} \\ &\le \alpha^{(n-t_1)/2d_1 + (n-t_1)/2d_2 + (n+t_1)/2d_3 + (n+t_1)/2d_4 + 2} \\ &\le \alpha^{(n+t)/4 + 3(n+t)/20 + 2}, \end{aligned}$$

which gives

$$\frac{n}{10} < 3 + \frac{2}{5}t, \quad \text{therefore} \quad n < 30 + 4t < 30 + 8.4 \log(3M) < 40 \log(3M),$$

which contradicts inequality (2.18). Thus, it remains to consider the case $d_i = 1$. Say $i = 1$. We then get $(n - t_1)/2 \mid (m - s_1)/2$. If $(m - s_1)/2$ is a proper multiple of $(n - t_1)/2$, we then get that

$$(m - s_1)/2 \ge 2 \times (n - t_1)/2 = n - t_1 > m/2 + n/2 - t_1,$$

giving

$$n \le 2t_1 - s_1 \le 2t + s \le 6.3 \log(3M),$$

which contradicts inequality (2.18). Thus, it remains the consider $n - t_1 = m - s_1$. This was when $d_i = 1$ and $i = 1$. For $i = 2, 3, 4$, we get that $n - t_1 = m + s_1$, $n + t_1 = m - s_1$, $n + t_1 = m + s_1$, respectively. Let us see that not all four possibilities occur.

Suppose say that $n - t_1 = m + s_1$. Then, as we have seen,

$$\gcd((n - t_1)/2, (m - s_1)/2) = \gcd((n - t_1)/2, (n - t_1)/2 - s_1) \mid s_1 \mid s,$$

$$\gcd((n + t_1)/2, (m + s_1)/2) = \gcd((n + t_1)/2, (n - t_1)/2) \mid t_1 \mid t,$$

and

$$\gcd((n + t_1)/2, (m - s_1)/2) = \gcd((n + t_1)/2, (n - t_1)/2 - s_1) \mid t_1 + s_1.$$

Observe that $s_1 + t_1 \ne 0$, for if $s_1 + t_1 = 0$, then since also $n - t_1 = m + s_1$, or $n = m + (s_1 + t_1) = m + 0$, we would get that $n = m$, a contradiction. Divisibilities (2.30) show that

$$a \le F_{\gcd((n-t_1)/2,(m-s_1)/2)} \gcd(F_{(n-t_1)/2}, L_{(m+s_1)/2}) L_{\gcd((n+t_1)/2,(m-s_1)/2)}$$

$$\times L_{\gcd((n+t_1)/2,(m+s_1)/2)} \le F_s \times 2 \times L_{t+s} \times L_t,$$

where we used the fact that $\gcd(F_k, L_k) \mid 2$ for all positive integers k with $k = (n-t_1)/2 = (m+s_1)/2$. Thus,

$$a \le 2\alpha^{2s+2t+1} < \alpha^{3+8.4\log(3M)}.$$

Since also $a > \sqrt{F_n} > \alpha^{n/2-1}$, we get

$$\frac{n}{2} - 1 < 3 + 8.4\log(3M), \qquad \text{therefore} \qquad n < 25\log(3M),$$

contradicting inequality (2.18). A similar argument applies when $n + t_1 = m - s_1$. Hence, we either have $n - t_1 = m - s_1$, or $m + t_1 = n + s_1$, which is (i).

Finally, let's us discuss the case $s = 0$. We follow the previous program. We have

$$\begin{aligned} a &\le \gcd(ab, ac) = \gcd(F_{(n-t_1)/2}L_{(n+t_1)/2}, F_m) \\ &\le \gcd(F_{(n-t_1)/2}, F_m)\gcd(L_{(n+t_1)/2}, F_m) \\ &\le F_{\gcd((n-t_1)/2,m)}L_{\gcd((n+t_1)/2,m)}. \end{aligned}$$

As in previous arguments, put

$$\gcd((n-t_1)/2, m) = (n-t_1)/2d_1, \quad \text{and} \quad \gcd((n+t_1)/2, m) = (n+t_1)/2d_2.$$

If $\min\{d_1, d_2\} \ge 5$, we have

$$\alpha^{n/2-1} < a \le F_{(n-t_1)/2d_1}L_{(n+t_1)/2d_2} \le \alpha^{(n-t_1)/2d_1+(n+t_1)/2d_2} \le \alpha^{(n+t)/5},$$

so that

$$n < \frac{10}{3}\left(1 + \frac{t}{5}\right) < 4 + \frac{4.2}{3}\log(3M) < 6\log(3M),$$

contradicting inequality (2.18). Assume now that both $d_1 \le 4$ and $d_2 \le 4$. Put d_3 and d_4 such that $m/d_3 = (n-t_1)/2d_1$ and $m/d_4 = (n+t_1)/2d_2$. If $d_3 \ge 2d_1 + 1$, we then have

$$m = \frac{d_3}{2d_1}(n - t_1) \ge n - t_1 + \frac{n-t_1}{2d_1} > m - t_1 + \frac{n-t_1}{2d_1},$$

so

$$n \le (2d_1+1)t_1 \le (2d_1+1)t \le 9 \times 2.1\log(3M) < 20\log(3M),$$

contradicting inequality (2.18). A similar contradiction is obtained if we assume that $d_4 \ge 2d_2 + 1$. Thus, $d_3 \le 2d_1 \le 8$ and $d_4 \le 2d_2 \le 8$. We then get

$$\frac{n+t_1}{n-t_1} = \frac{d_2d_3}{d_1d_4},$$

so that

$$n(d_1d_4 - d_2d_3) = -t_1(d_1d_4 + d_2d_3).$$

Therefore

$$n \le t(d_1d_4 + d_2d_3) \le 64 \times 2.1 \log(3M) < 400 \log(3M),$$

contradicting inequality (2.18). Assume $\min\{d_1, d_2\} \le 4$ and $\max\{d_1, d_2\} \ge 5$. If $\min\{d_1, d_2\} \ge 2$, we then get

$$\alpha^{n/2-1} < a < \alpha^{(n-t_1)/2d_1+(n+t_1)/2d_2} \le \alpha^{(n+t)(1/4+1/10)},$$

giving

$$n < \frac{20}{3}\left(1 + \frac{7}{20}t\right) < 7 + \frac{7}{3} \times 2.1 \log(3M) < 12 \log(3M),$$

which contradicts inequality (2.18). So, the last possibility is $\min\{d_1, d_2\} = 1$. Hence, we either have $\gcd((n-t_1)/2, m) = (n-t_1)/2$, or $\gcd((n+t_1)/2, m) = (n+t_1)/2$. In particular, $m = \delta(n-t_1)/2$, or $m = \delta(n+t_1)/2$ for some positive integer δ. If $\delta \ge 3$, we get

$$n > m \ge \frac{3(n \pm t_1)}{2} \ge \frac{3(n-t)}{2},$$

giving $n < 3t < 10 \log(3M)$, a contradiction. If $\delta = 2$, we get that $m = n - t_1$ or $m = n + t_1$, which is (i) because $s = 0$. Suppose now that $\delta = 1$. Then either $m = (n-t_1)/2$, or $m = (n+t_1)/2$. Assume that $m = (n+t_1)/2$. Then

$$\begin{aligned} a &\le \gcd(ab, ac) = \gcd(F_{(n-t_1)/2}L_{(n+t_1)/2}, F_{(n+t_1)/2}) \\ &\le \gcd(F_{(n-t_1)/2}, F_{(n+t_1)/2}) \gcd(L_{(n+t_1)/2}, F_{(n+t_1)/2}) \le 2F_t, \end{aligned}$$

so we get that

$$\alpha^{n/2-1} \le 2F_t < \alpha^{t+1}, \qquad \text{therefore} \qquad n < 4 + 2t < 10 \log(3M),$$

a contradiction. Finally, in case $m = (n-t_1)/2$, we then have

$$ab = F_{(n-t_1)/2}L_{(n+t_1)/2}, \qquad ac = F_m = F_{(n-t_1)/2},$$

therefore

$$ab = (ac)L_{(n+t_1)/2}, \qquad \text{so} \qquad b = L_{(n+t_1)/2}c,$$

which is (iii). □

We can now give a lower bound for b.

Lemma 2.16. *Assume that inequality* (2.18) *holds. Then*

$$b > \alpha^{n/2-14\log(3M)}. \tag{2.32}$$

Proof. If we are in case (iii) of Lemma 2.15, then

$$b \geq L_{(n+t_1)/2} \geq \alpha^{n/2-t/2-1} \geq \alpha^{n/2-1-1.05\log(3M)} \geq \alpha^{n/2-3\log(3M)}.$$

Assume next that $n - t_1 = m - s_1$ and $st \neq 0$. Then

$$\gcd((n-t_1)/2, (m+s_1)/2) = \gcd((n-t_1)/2, (n-t_1)/2 + s_1) \mid s_1 \mid s,$$

$$\gcd((n+t_1)/2, (m-s_1)/2) = \gcd((n+t_1)/2, (n-t_1)/2) \mid t_1 \mid t,$$

and

$$\gcd((n+t_1)/2, (m+s_1)/2) = \gcd((n+t_1)/2, (n-t_1)/2 + s_1) \mid t_1 - s_1.$$

Observe that $t_1 - s_1 \neq 0$ since if $t_1 - s_1 = 0$, then $n - m = t_1 - s_1 = 0$, so $n = m$, which is impossible. Now relation (2.30) shows that

$$\begin{aligned} a &\leq F_{(n-t_1)/2} L_s L_t L_{t+s} \leq \alpha^{(n+t)/2+2s+t+2} \\ &\leq \alpha^{n/2+2+3.5\max\{s,t\}} < \alpha^{n/2+10\log(3M)}. \end{aligned} \tag{2.33}$$

Since $|u| \leq M < a$, it follows that

$$\alpha^{n-2} < F_n = ab + u \leq ab + |u| \leq ab + M < 2ab < 2b\alpha^{n/2+10\log(3M)},$$

giving

$$b > 2^{-1}\alpha^{n/2-2-10\log(3M)} > \alpha^{n/2-4-10\log(3M)} > \alpha^{n/2-14\log(3M)},$$

which is the desired inequality. A similar argument applies when $n + t_1 = m + s_1$ and $st \neq 0$.

Assume next that $t = 0$. Then $n = m - s_1$ or $n = m + s_1$. Assume say that $n = m - s_1$. Then

$$\begin{aligned} a &\leq \gcd(F_n, F_{(m-s_1)/2} L_{(m+s_1)/2}) \leq F_{\gcd(n,(m-s_1)/2)} L_{\gcd(n,(m+s_1)/2} \\ &= F_{n/2} L_{\gcd(n,n/2+s_1)} \leq F_{n/2} L_s, \end{aligned}$$

so

$$a \leq \alpha^{n/2+s} \leq \alpha^{n/2+2.1\log(3M)},$$

which is an inequality better than (2.33). In turn, we get that inequality (2.32) holds. A similar argument applies when $t = 0$ and $n = m + s_1$, and also when $s = 0$ and either $m = n - t_1$ or $m = n + t_1$. We give no further details here. □

We now write

$$b \leq \gcd(ab, bc) = \gcd(F_n - u, F_\ell - w).$$

Write, as we did in Section 2.2,

$$F_\ell - w = \frac{\alpha^{-\ell}}{\sqrt{5}} \left(\alpha^\ell - w_{1,\ell}\right) \left(\alpha^\ell - w_{2,\ell}\right), \tag{2.34}$$

where

$$w_{k,\ell} = \frac{\sqrt{5}w + (-1)^k\sqrt{5w^2+4(-1)^\ell}}{2}, \qquad k \in \{1,2\}. \tag{2.35}$$

As for the numbers $u_{i,n}$ and $v_{j,m}$ (see inequalities (2.9) and (2.10)), we also have that $w_{k,\ell}$ and all its conjugates $w_{k,\ell}^{(s)}$ satisfy

$$|w_{k,\ell}^{(s)}| < 3M.$$

We put $= \mathbb{Q}(\sqrt{5}, u_{1,n}, w_{1,\ell})$, and use the argument from the beginning of Section 2.3, in particular an analog of inequality (2.11) to say that

$$\begin{aligned} b\mathcal{O} &\mid \gcd\left((\alpha^n - u_{1,n})(\alpha^n - u_{2,n})\mathcal{O}, \left(\alpha^\ell - w_{1,\ell}\right)\left(\alpha^\ell - w_{2,\ell}\right)\mathcal{O}\right) \\ &\mid \prod_{\substack{1\le i\le 2\\ 1\le k\le 2}} \gcd\left((\alpha^n - u_{i,n})\mathcal{O}, \left(\alpha^\ell - w_{k,\ell}\right)\mathcal{O}\right). \end{aligned} \tag{2.36}$$

Put

$$I_{i,n,k,\ell} = \gcd\left((\alpha^n - u_{i,n})\mathcal{O}, (\alpha^\ell - w_{k,\ell})\mathcal{O}\right), \qquad i,k \in \{1,2\}.$$

Using Lemma 2.6, we construct coprime integers λ', ν' satisfying the inequalities $\max\{|\lambda'|, |\nu'|\} \le \sqrt{n}$, $|n\lambda' + \ell\nu'| \le 3\sqrt{n}$ and furthermore

$$\alpha^{n\lambda'+\ell\nu'} - u_{i,n}^{\lambda'} w_{k,\ell}^{\nu'} \in I_{i,n,k,\ell}.$$

As in Section 2.3, we make the following assumption.

Assumption 2.17. *Assume that the pair* (λ', ν') *satisfies*

$$\alpha^{n\lambda'+\ell\nu'} - u_{i,n}^{\lambda'} w_{k,\ell}^{\nu'} \neq 0 \qquad \textit{for all} \qquad i,k \in \{1,2\}.$$

Then the argument of Lemma 2.8 shows that

$$b \le 2^4 (3M)^{8\sqrt{n}}.$$

Combined with Lemma 2.16, we get that

$$\alpha^{n/2 - 14\log(3M)} < 2^4 (3M)^{8\sqrt{n}},$$

therefore

$$n/2 - 14\log(3M) < \frac{\log(16)}{\log\alpha} + \left(\frac{8\log(3M)}{\log\alpha}\right)\sqrt{n} < 5.8 + 16.7\log(3M)\sqrt{n},$$

so

$$n < \left(\frac{11.6}{\sqrt{n}} + \frac{28\log(3M)}{\sqrt{n}} + 16.7\log(3M)\right)\sqrt{n}.$$

Since n satisfies inequality (2.18), we have that $\sqrt{n} > 41\log(3M)$, therefore

$$\frac{11.6}{\sqrt{n}} < 2 \qquad \text{and} \qquad \frac{28\log(3M)}{\sqrt{n}} < 1.$$

Hence, we get that

$$\sqrt{n} < 3 + 16.7\log(3M) < 20\log(3M),$$

contradicting inequality (2.18). The conclusion is:

Lemma 2.18. *If inequality* (2.18) *holds, then Assumption 2.17 cannot hold.*

Thus, there exist $i_1, k_1 \in \{1,2\}$ such that

$$\alpha^{n\lambda'+\ell\nu'} = u_{i_1,n}^{\lambda'} w_{k_1,\ell}^{\nu'}.$$

Since we already know that $u_{i_1,n}^2 \in \mathbb{Q}(\sqrt{5})$ (because we are in the rank one case), it follows that $w_{k_1,\ell}^{2\nu'} \in \mathbb{Q}(\sqrt{5})$. In particular, either $w = 0$, or $w \neq 0$ but $5w^2 + 4(-1)^\ell = y_{w,\ell}^2$ holds for some positive integer ℓ. In particular, $w = \pm F_r$ for some nonnegative integer r which is either 0 or is congruent to ℓ modulo 2. Thus

$$bc = F_\ell - w = F_{(\ell-r_1)/2} L_{(\ell+r_1)/2}$$

where $r_1 = \pm r$. Since $|w| \leq M$, we also have $r < 2.1\log(3M)$.

We now show that both m and ℓ are large.

Lemma 2.19. *Assume that inequality* (2.18) *holds. Then*

$$\min\{\ell, m\} > n/2 - 17\log(3M). \tag{2.37}$$

Proof. Since $b > \alpha^{n/2-14\log(3M)}$ by Lemma 2.16, and since n satisfies inequality (2.18), it follows that $b > 2M$. Indeed, this last inequality is implied by

$$\alpha^{n/2-14\log(3M)} > 2M,$$

or

$$n/2 - 14\log(3M) > \frac{\log 2M}{\log\alpha},$$

which in turn is implied by

$$n/2 - 14\log(3M) > 2.1\log(3M),$$

which in turn is implied by $n > 33\log(3M)$, which holds when n satisfies inequality (2.18). Hence,

$$\begin{aligned}\alpha^{\ell-1} &> F_\ell = bc + w \geq bc - M \geq b - M > b/2\\ &\geq 2^{-1}\alpha^{n/2-14\log(3M)} > \alpha^{n/2-2-14\log(3M)},\end{aligned}$$

giving

$$\ell - 1 > n/2 - 2 - 14\log(3M), \qquad \text{or} \qquad \ell > n/2 - 17\log(3M).$$

The same argument works for m. □

We now return to Lemma 2.15 and get the following result.

Lemma 2.20. *If inequality* (2.18) *holds, then part (iii) of Lemma 2.15 cannot hold.*

Proof. Assume that (iii) of Lemma 2.15 holds. Then

$$bc = L_{(n+t_1)/2}c^2 = F_{(\ell-r_1)/2}L_{(\ell+r_1)/2}.$$

Since n satisfies inequality (2.18), we have that

$$(n+t_1)/2 > (n-t)/2 > ((41\log(3M))^2 - 2.1\log(3M))/2 > 12,$$

therefore $L_{(n+t_1)/2}$ has a primitive prime factor p. Its order of appearance in the Fibonacci sequence is $n+t_1$. Since $p \mid F_{(\ell-r_1)/2}L_{(\ell+r_1)/2}$, it follows that either $(\ell-r_1)/2$ is a multiple of $n+t_1$, or $\ell+r_1$ is a multiple of $n+t_1$. But obviously

$$(\ell+r_1)/2 < (n+r)/2 < n-t \le n+t_1,$$

where the middle inequality holds because it is equivalent to $n > 2r+t$, which is implied by (2.18) since then

$$n > (41\log(3M))^2 > 6.3\log(3M) > r+2t.$$

Thus, the only possibility is that $\ell+r_1$ is a multiple of $n+t_1$. Since

$$2(n+t_1) \ge 2n-2t > n+r > \ell+r \ge \ell+r_1,$$

it follows that the only possibility is that $\ell+r_1 = n+t_1$. Hence,

$$L_{(n+t_1)/2}c^2 = F_{(\ell-r_1)/2}L_{(\ell+r_1)/2} = F_{(\ell-r_1)/2}L_{(n+t_1)/2},$$

giving $F_{(\ell-r_1)/2} = c^2$. Since the largest square in the Fibonacci sequence is $F_{12} = 12^2$ (see [1] for a more general result), we get that $(\ell-r_1)/2 \le 12$, so

$$\ell \le 24 + r_1 \le 24 + r < 30\log(3M). \tag{2.38}$$

However, this last inequality contradicts the inequality (2.37) because n satisfies inequality (2.18). This shows that indeed part (iii) of Lemma 2.15 cannot happen. □

We now revisit the argument of Lemma 2.15 and prove in exactly the same way the following result.

Lemma 2.21. *Assume that inequality* (2.18) *holds. Then one of the following holds:*

(i) $n - t_1 = \ell - r_1$;

(ii) $n + t_1 = \ell + r_1$.

Proof. We follow the proof of Lemma 2.15. The relevant inequality here is, instead of (2.30),

$$b \le \gcd(ab, bc) = \gcd(F_{(n-t_1)/2}L_{(n+t_1)/2}, F_{(\ell-r_1)/2}L_{(\ell+r_1)/2}). \tag{2.39}$$

In the proof of Lemma 2.15 we used the lower bound $a > \alpha^{n/2-1}$, whereas here we use the lower bound $b > \alpha^{n/2-14\log(3M)}$ given by Lemma 2.16. We only go through a couple scenarios which have not been contemplated in the proof of Lemma 2.15.

One of them is when $u = w = 0$. Then

$$\alpha^{n/2-14\log(3M)} < b = \gcd(F_n, F_\ell) = F_{\gcd(n,\ell)}.$$

Clearly, $\gcd(n, \ell) = n/d_1$ for some divisor $d_1 > 1$ of n because $\ell < n$. If $d_1 \ge 3$, we get

$$\alpha^{n/2-14\log(3M)} < F_{n/d_1} < \alpha^{n/d_1} \le \alpha^{n/3},$$

or $n < 84\log(3M)$, contradicting inequality (2.18). Hence, $\gcd(n, \ell) = n/2$, and the only possibility is $\ell = n/2$. But then

$$bc = F_{n/2}, \qquad ab = F_n = F_{n/2}L_{n/2}, \qquad \text{giving} \qquad a = L_{n/2}c.$$

Hence,

$$F_{(m-s_1)/2}L_{(m+s_1)/2} = ac = L_{n/2}c^2.$$

Since n is large, $L_{n/2}$ has primitive divisors whose order of appearance in the Fibonacci sequence is exactly n. We deduce that n divides either $(m - s_1)/2$ or $m + s_1$. Since we have $(m - s_1)/2 \le (m + s)/2 < (n + s)/2 < n$ and $m + s_1 \le m + s < n + s < 2n$ whenever n satisfies inequality (2.18), we conclude that the only possibility is that $m + s_1 = n$. Thus, we get the equations $L_{n/2}c^2 = F_{(m-s_1)/2}L_{(m+s_1)/2} = F_{(m-s_1)/2}L_{n/2}$, so $F_{(m+s_1)/2} = c^2$, giving $(m + s_1)/2 \le 12$. This gives

$$m \le 24 - s_1 \le 24 + s < 24 + 2.1\log(3M),$$

which contradicts inequality (2.37) of Lemma 2.19 when n satisfies inequality (2.18).

This shows that we cannot have u and w be simultaneously zero.

Next we follow along the proof of Lemma 2.15 replacing (m, s, s_1) by (ℓ, r, r_1). Everything works out until we arrive at the analogue of (iii) of Lemma 2.15, which for us is $w = r = 0$, $\ell = (n - t_1)/2$ and $a = L_{(n+t_1)/2}c$. But in this case

$$L_{(n+t_1)/2}c^2 = ac = F_{(m-s_1)/2}L_{(m+s_1)/2}.$$

Using again the information that $(n + t_1)/2$ is large and $L_{(n+t_1)/2}$ has primitive prime divisors, we conclude that the only possible scenario is $m + s_1 = n + t_1$, leading to $F_{(m-s_1)/2} = c^2$, which gives that $(m - s_1)/2$ is small, contradicting inequality (2.37). We give no further details. □

We can now give a lower bound for c.

Lemma 2.22. *Assume that inequality* (2.18) *holds. Then*

$$c > \alpha^{n/2-31\log(3M)}. \tag{2.40}$$

Proof. This is very similar to the proof of Lemma 2.16. Assume, for example, that $n - t_1 = \ell - r_1$ and $tr \neq 0$. Then

$$\gcd((n-t_1)/2, (\ell+r_1)/2) = \gcd((n-t_1)/2, (n-t_1)/2 + r_1) \mid r_1 \mid r,$$

$$\gcd((n+t_1)/2, (\ell-r_1)/2) = \gcd((n+t_1)/2, (n-t_1)/2) \mid t_1 \mid t,$$

and

$$\gcd((n+t_1)/2, (\ell+r_1)/2) = \gcd((n+t_1)/2, (n-t_1)/2 + r_1) \mid t_1 - r_1.$$

Observe that $t_1 - r_1 \neq 0$ since if $t_1 - r_1 = 0$, then $n - \ell = t_1 - r_1 = 0$, so $n = \ell$, which is impossible. Now relation (2.39) implies that

$$\begin{aligned} b &\le F_{(n-t_1)/2} L_r L_t L_{t+r} \le \alpha^{(n+t)/2+2r+t+2} \\ &\le \alpha^{n/2+2+3.5\max\{r,t\}} < \alpha^{n/2+10\log(3M)}. \end{aligned} \tag{2.41}$$

Since $|w| \le M < b$, it follows, by inequality (2.37), that

$$\alpha^{n/2-17\log(3M)-2} \le \alpha^{\ell-2} \le F_\ell = bc + w \le bc + M < 2bc < 2c\alpha^{n/2+10\log(3M)},$$

giving

$$c > 2^{-1}\alpha^{n/2-2-27\log(3M)} > \alpha^{n/2-4-27\log(3M)} > \alpha^{n/2-31\log(3M)},$$

which is the desired inequality. A similar argument applies when $n + t_1 = \ell + r_1$ and $tr \neq 0$.

A similar proof works when either $t = 0$ or $r = 0$ providing better lower bounds for c. We give no further details here. □

We now revisit the argument of Lemma 2.15 and prove in exactly the same way the following result.

Lemma 2.23. *Assume that inequality* (2.18) *holds. Then one of the following holds:*

(i) $m - s_1 = \ell - r_1$*;*

(ii) $m + s_1 = \ell + r_1$*.*

Proof. This is entirely similar with the proof of Lemma 2.15, except that we use the relation

$$c \le \gcd(ac, bc) = \gcd(F_{(m-s_1)/2} L_{(m+s_1)/2}, F_{(\ell-r_1)/2} L_{(\ell+r_1)/2})$$

and the lower bound (2.40) on c. We give no further details. □

Finally, we prove the following result.

Lemma 2.24. *Inequality* (2.18) *does not hold.*

Proof. From Lemmas 2.15, 2.21 and 2.23, one gets easily that either $n-t_1=m-s_1=\ell-r_1$ or $n+t_1=m+s_1=\ell+r_1$. Assume say that $N=n-t_1=m-s_1=\ell+r_1$. Then

$$ab=F_N L_{N+2t_1},\qquad ac=F_N L_{N+2s_1},\qquad bc=F_N L_{N+2r_1}.$$

If U and V denote any two of the numbers $N, N+2r_1, N+2s_1, N+2t_1$, then $U/2<V<2U$ because n satisfies inequality (2.18). Also, all the above four numbers exceed 12. Using again the primitive divisor theorem, we conclude that $N+2r_1$ is one of the numbers $\{N, N+2s_1, N+2t_1\}$, so $r_1\in\{0,s_1,t_1\}$. But if $r_1=s_1$, then since also $\ell-r_1=m-s_1$, we get $m=\ell$, so $ac=F_{(m-s_1)/2}L_{(m+s_1)/2}=F_{(\ell-r_1)/2}L_{(\ell+r_1)/2}=bc$, contradicting the fact that $a>b>c\geq 1$. Thus, $r_1=0$. Similarly, we get $s_1=t_1=0$, therefore $n=m=\ell$, which is not allowed. A similar argument works when $n+t_1=m+s_1=\ell+r_1$. □

Proof of Theorem 1.1. We are now ready to finish the proof of Theorem 1.1. Indeed,

$$2a\leq ab=F_n+u\leq F_n+M.$$

So, either $a\leq M$, or $a>M$ in which case $a\leq 2a-M\leq F_n<\alpha^n$ giving

$$\frac{\log a}{\log\alpha}<n<(41\log(3M))^2.$$

The above inequality implies that

$$\log M>41^{-1}\sqrt{2}\sqrt{\log a}>0.034\sqrt{\log a}. \tag{2.42}$$

In case $a\leq M$, we get $\log M\geq\log a>0.034\sqrt{\log a}$ because $a\geq 3$ so $\log a>1$. Hence, inequality (2.42) always holds, showing that $M>\exp(0.034\sqrt{\log a})$, which is what we wanted to prove. □

3. The proof of Corollary 1.2

The condition $a<\exp(415.62)$ (coming directly from Theorem 1.1) implies $n\leq 1730$ via the inequalities $\alpha^{n-2}<F_n<a^2$. It is easy to see that $n\geq 8$ entails $n>m$, moreover from $n\geq 8$ and $m\geq 7$ we conclude $m\geq\ell$. These make it possible to apply a computer search for checking all the candidates (n,m,ℓ). Obviously $n\geq 5$ must be fulfilled, therefore we can verify individually the cases $5\leq n\leq 7$. Totally 222 solutions to the system (2.4) have been found in (a,b,c,u,v,w,n,m,ℓ), the largest a is occurring in

$$(a,b,c,u,v,w,n,m,\ell)=(235,11,1,-1,-2,2,18,13,8).$$

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