

# On unification of some weak separation properties\*

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## Abstract

In this paper, a new kind of sets called regular  $\psi$ -generalized closed (briefly  $r\psi g$ -closed) sets are introduced and studied in a topological space. Some of their properties are investigated. Finally, some unifications of some weak forms of almost regular, almost normal and mildly normal spaces have been given.

*Keywords:*  $\psi$ -open set,  $r\psi g$ -closed set, almost  $\psi$ -regular space, almost  $\psi$ -normal space, mildly  $\psi$ -normal space.

*MSC:* 54D15, 54A05, 54C08.

## 1. Introduction

The concept of generalized closed sets in a topological space was introduced by N. Levine [11]. After that, the concept of generalized closed sets has been investigated by many mathematicians. It is well known that separation axioms are one of the basic subjects of study in general topology and in several branches of mathematics. In 1973, Singal et al. introduced the concept of almost regular [25], almost normal

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[26] and mildly normal [27] spaces. Recently, Ekici, Noiri and Park [4, 5, 6, 19, 21, 22] continued the study of several weaker forms of separation axioms.

Throughout this paper  $(X, \tau)$  always means a topological space on which no separation axioms are assumed unless explicitly stated. The closure and interior of a set  $A (\subseteq X)$  are denoted by  $clA$  and  $intA$  respectively. A subset  $A$  is said to be regular open (resp. regular closed) if  $A = intclA$  (resp.  $A = clintA$ ). The collection of all regular open (regular closed) sets in a topological space  $(X, \tau)$  is denoted by  $RO(X)$  (resp.  $RC(X)$ ). The  $\delta$ -closure [28] of a subset  $A$  of  $X$  is denoted by  $cl_\delta A$  and is defined by  $cl_\delta A = \{x : A \cap U \neq \emptyset, \text{ for each } U \in RO(X) \text{ with } x \in U\}$ . Let  $(X, \tau)$  be a space and  $A \subseteq X$ . A point  $x \in X$  is called a condensation point of  $A$  if for each  $U \in \tau$  with  $x \in U$ , the set  $U \cap A$  is uncountable.  $A$  is called  $\omega$ -closed [8] if it contains all its condensation points. The complement of an  $\omega$ -closed set is called  $\omega$ -open. It is well known that the family of all  $\omega$ -open subsets of a space  $(X, \tau)$ , denoted by  $\tau_\omega$ , forms a topology on  $X$  finer than  $\tau$ . A subset  $A$  of a space  $X$  is said to be preopen [14] (resp. semi-open [10],  $\delta$ -preopen [23],  $\alpha$ -open [16],  $\beta$ -open [1]) if  $A \subseteq intclA$  (resp.  $A \subseteq clintA$ ,  $A \subseteq intcl_\delta A$ ,  $A \subseteq intclintA$ ,  $A \subseteq clintclA$ ). The family of all preopen (resp. semi-open,  $\delta$ -preopen,  $\alpha$ -open,  $\beta$ -open) sets in a space  $X$  is denoted by  $PO(X)$  (resp.  $SO(X)$ ,  $\delta$ - $PO(X)$ ,  $\alpha O(X)$ ,  $\beta O(X)$ ).

We now recall a few definitions and observe that many of the existing relevant definitions considered in various papers turn out to be special cases of the ones given below.

**Definition 1.1.** [3] Let  $(X, \tau)$  be a topological space. A mapping  $\psi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$  is called an operation on  $\mathcal{P}(X)$ , where  $\mathcal{P}(X)$  denotes as usual the power set of  $X$ , if for each  $A \in \mathcal{P}(X) \setminus \{\emptyset\}$ ,  $intA \subseteq \psi(A)$  and  $\psi(\emptyset) = \emptyset$ .

The set of all operations on a space  $X$  will be denoted by  $\mathcal{O}(X)$ .

*Observation 1.2.* It is easy to check that some examples of operations on a space  $X$  are the well known operators viz.  $int$ ,  $intcl$ ,  $intcl_\delta$ ,  $clint$ ,  $intclint$ ,  $clintcl$ .

**Definition 1.3.** [3] Let  $\psi$  be an operation on a space  $(X, \tau)$ . Then a subset  $A$  of  $X$  is called  $\psi$ -open if  $A \subseteq \psi(A)$ . Complements of  $\psi$ -open sets will be called  $\psi$ -closed sets. The family of all  $\psi$ -open (resp.  $\psi$ -closed) subsets of  $X$  is denoted by  $\psi\mathcal{O}(X)$  (resp.  $\psi\mathcal{C}(X)$ ).

*Observation 1.4.* It is clear that if  $\psi$  stands for any of the operators  $int$ ,  $intcl$ ,  $intcl_\delta$ ,  $clint$ ,  $intclint$ ,  $clintcl$ , then  $\psi$ -openness of a subset  $A$  of  $X$  coincides with respectively the openness, preopenness,  $\delta$ -preopenness, semi-openness,  $\alpha$ -openness and  $\beta$ -openness of  $A$  (see [13, 23, 10, 12, 1]).

**Definition 1.5.** [3] Let  $(X, \tau)$  be a topological space,  $\psi \in \mathcal{O}(X)$  and  $A \subseteq X$ . Then the intersection of all  $\psi$ -closed sets containing  $A$  is called the  $\psi$ -closure of  $A$ , denoted by  $\psi-clA$ ; alternately,  $\psi-clA$  is the smallest  $\psi$ -closed set containing  $A$ . The union of all  $\psi$ -open subsets of  $G$  is the  $\psi$ -interior of  $G$ , denoted by  $\psi-intG$ .

It is known from [9] that  $x \in \psi-clA$  iff  $A \cap U \neq \emptyset$ , for all  $U$  with  $x \in U \in \psi\mathcal{O}(X)$  and  $x \in \psi-intG$  iff  $\exists x \in U \in \psi\mathcal{O}(X)$  such that  $x \in U \subseteq G$ . In [9], it is also shown that  $X \setminus \psi-clG = \psi-int(X \setminus G)$ .

*Observation 1.6.* Obviously if one takes interior as the operation  $\psi$ , then  $\psi$ -closure becomes equivalent to the usual closure. Similarly,  $\psi$ -closure becomes  $pcl$ ,  $pcl_\delta$ ,  $scl$ ,  $\alpha-cl$ ,  $\beta-cl$ , if  $\psi$  is taken to stand for the operators  $intcl$ ,  $intcl_\delta$ ,  $clint$ ,  $intclint$  and  $clintcl$  respectively (see [13, 23, 10, 12, 1] for details).

**Definition 1.7.** A subset  $A$  of a space  $(X, \tau)$  is called

- (a) generalized closed (briefly,  $g$ -closed) [11] if  $clA \subseteq U$  whenever  $A \subseteq U$  and  $U \in \tau$ ;
- (b) regular generalized closed (briefly,  $rg$ -closed) [20] if  $clA \subseteq U$  whenever  $A \subseteq U \in RO(X)$ ;
- (c) generalized preregular closed [7] (briefly,  $gpr$ -closed), or preregular generalized closed [18] if  $pclA \subseteq U$  whenever  $A \subseteq U \in RO(X)$ ;
- (d)  $rag$ -closed [19] if  $cl_\alpha A \subseteq U$  whenever  $A \subseteq U \in RO(X)$ ;
- (e)  $g\delta pr$ -closed [6] if  $pcl_\delta A \subseteq U$  whenever  $A \subseteq U \in RO(X)$ ;
- (f)  $rg\omega$ -closed [2] if  $cl_\omega(A) \subseteq U$  whenever  $A \subseteq U \in RO(X)$ .

## 2. Properties of $r\psi g$ -closed sets

**Definition 2.1.** Let  $\psi$  be an operation on a topological space  $(X, \tau)$ . A subset  $A$  of  $X$  is called a regular  $\psi$ -generalized closed set or simply an  $r\psi g$ -closed set (resp.  $g\psi$ -closed set [24]) if  $\psi-cl(A) \subseteq U$  whenever  $A \subseteq U \in RO(X)$  (resp.  $A \subseteq U \in \tau$ ). The complement of an  $r\psi g$ -closed set (resp.  $g\psi$ -closed set) is called an  $r\psi g$ -open (resp.  $g\psi$ -open [24]) set.

*Remark 2.2.* Let  $\psi$  be an operation on a topological space  $(X, \tau)$ . Then we have the following relation between  $r\psi g$ -closed sets and other known sets :

$$\psi\text{-closed set} \Rightarrow g\psi\text{-closed set} \Rightarrow r\psi g\text{-closed set}$$

**Example 2.3.** Let  $X = \{a, b, c\}$  and  $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ . Then  $(X, \tau)$  is a topological space. Consider the mapping  $\psi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$  defined by  $\psi(A) = intA$  for all  $A \subseteq X$ . Then  $\psi \in O(X)$ . It can shown that  $\{a\}$  is a  $r\psi g$ -closed set which is not  $g\psi$ -closed.

*Remark 2.4.* Obviously if on a space  $(X, \tau)$  one takes the operation  $\psi = int$ , then  $r\psi g$ -closed sets become equivalent to  $rg$ -closed sets [7, 20]. Similarly,  $r\psi g$ -closed sets become  $gpr$ -closed sets [15, 18, 21],  $rag$ -closed sets [19],  $g\delta pr$ -closed sets [6],  $rg\omega$ -closed sets [2] if the role of  $\psi$  is taken to stand for  $intcl$ ,  $intclint$ ,  $intcl_\delta$ ,  $\tau_\omega-int$  respectively.

Some characterizations of some weak separation properties via  $g\gamma$ -closed set with the operation were studied in [4].

**Definition 2.5.** A subset  $A$  of a space  $(X, \tau)$  is said to be  $g\gamma$ -closed [4] if  $\gamma-cl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U \in \tau$ . The complement of  $g\gamma$ -closed set is said to be  $g\gamma$ -open [4].

The next two examples show that union (intersection) of two  $r\psi g$ -closed sets is not in general an  $r\psi g$ -closed set.

**Example 2.6.** (a) Let  $X = \{a, b, c, d, e\}$  and  $\tau = \{\emptyset, X, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$ . Then  $(X, \tau)$  is a topological space with  $RO(X) = \{\emptyset, X, \{a, b\}, \{c, d\}\}$ . Consider the mapping  $\psi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$  by  $\psi(A) = \text{int}clA$  for all  $A \subseteq X$ . Then  $\psi \in O(X)$ . It is easy to check that  $\{a\}$  and  $\{b\}$  are two  $r\psi g$ -closed sets but their union  $\{a, b\}$  is not  $r\psi g$ -closed.

(b) Consider the topological space  $(X, \tau)$ , where  $X = \{a, b, c\}$  and  $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ . Let  $\psi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$  be a map defined by  $\psi(A) = \text{int}A$  for all  $A \subseteq X$ . Then  $\psi \in O(X)$ . It is easy to check that  $\{a, b\}$  and  $\{a, c\}$  are two  $r\psi g$ -closed set but their intersection  $\{a\}$  is not a  $r\psi g$ -closed set.

**Theorem 2.7.** *Let  $\psi$  be an operation on a topological space  $(X, \tau)$ . Let  $A \subseteq X$  be an  $r\psi g$ -closed subset of  $X$ . Then  $\psi\text{-cl}(A) \setminus A$  does not contain any non-empty regular closed set.*

*Proof.* Let  $F$  be a regular closed subset of  $(X, \tau)$  such that  $F \subseteq \psi\text{-cl}(A) \setminus A$ . Then  $F \subseteq X \setminus A$  and hence  $A \subseteq X \setminus F \in RO(X)$ . Since  $A$  is  $r\psi g$ -closed,  $\psi\text{-cl}(A) \subseteq X \setminus F$  and hence  $F \subseteq X \setminus \psi\text{-cl}(A)$ . So  $F \subseteq \psi\text{-cl}(A) \cap (X \setminus \psi\text{-cl}(A)) = \emptyset$ .  $\square$

That the converse of the above theorem is false as shown by the next example.

**Example 2.8.** Let  $X = \{a, b, c, d\}$  and  $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ . Then  $(X, \tau)$  is a topological space. Consider the mapping  $\psi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$  defined by  $\psi(A) = \text{int}A$  for all  $A \subseteq X$ . Then  $\psi \in O(X)$ . Let  $A = \{a\}$ . Then  $\psi\text{-cl}(A) \setminus A = \{a, c, d\} \setminus \{a\} = \{c, d\}$  does not contain any non-empty regular closed set. But  $A$  is not  $r\psi g$ -closed.

**Theorem 2.9.** *Let  $\psi$  be an operation on a topological space  $(X, \tau)$ . Then a subset  $A$  is  $r\psi g$ -open iff  $F \subseteq \psi\text{-int}(A)$  whenever  $F$  is a regular closed subset such that  $F \subseteq A$ .*

*Proof.* Let  $A$  be an  $r\psi g$ -open subset of  $X$  and  $F$  be a regular closed subset of  $X$  such that  $F \subseteq A$ . Then  $X \setminus A$  is an  $r\psi g$ -closed set and  $X \setminus A \subseteq X \setminus F \in RO(X)$ . So  $\psi\text{-cl}(X \setminus A) = X \setminus \psi\text{-int}(A) \subseteq X \setminus F$ . Thus  $F \subseteq \psi\text{-int}(A)$ .

Conversely, let  $F \subseteq \psi\text{-int}(A)$  whenever  $F$  is regular closed such that  $F \subseteq A$ . Let  $X \setminus A \subseteq U$  where  $U \in RO(X)$ . Then  $X \setminus U \subseteq A$  and  $X \setminus U$  is regular closed. By the assumption,  $X \setminus U \subseteq \psi\text{-int}(A)$  and hence  $\psi\text{-cl}(X \setminus A) = X \setminus \psi\text{-int}(A) \subseteq U$ . Hence  $X \setminus A$  is  $r\psi g$ -closed and hence  $A$  is  $r\psi g$ -open.  $\square$

**Theorem 2.10.** *Let  $\psi$  be an operation on a topological space  $(X, \tau)$  and  $A$  be an  $r\psi g$ -closed subset of  $X$ . If  $B \subseteq X$  be such that  $A \subseteq B \subseteq \psi\text{-cl}(A)$ , then  $B$  is also an  $r\psi g$ -closed set.*

*Proof.* Let  $A$  be an  $r\psi g$ -closed set and  $B \subseteq U \in RO(X)$ . Then  $A \subseteq U \in RO(X)$  and hence  $\psi\text{-cl}(A) \subseteq U$ . Thus by monotonicity and idempotent property of  $\psi\text{-cl}$  we have  $\psi\text{-cl}(B) \subseteq U$ , showing  $B$  to be  $r\psi g$ -closed.  $\square$

**Theorem 2.11.** *Let  $(X, \tau)$  be a topological space and  $\psi$  be an operation on  $X$ . If  $A$  is an  $r\psi g$ -closed subset of  $X$ , then  $\psi\text{-cl}(A) \setminus A$  is  $r\psi g$ -open.*

*Proof.* Let  $A$  be an  $r\psi g$ -closed subset of  $(X, \tau)$  and  $F$  be a regular closed subset such that  $F \subseteq \psi\text{-cl}(A) \setminus A$ , so by Theorem 2.7,  $F = \emptyset$  and thus  $F \subseteq \psi\text{-int}(\psi\text{-cl}(A) \setminus A)$ . So by Theorem 2.9,  $\psi\text{-cl}(A) \setminus A$  is  $r\psi g$ -open.  $\square$

**Example 2.12.** Consider Example 2.8 once again. If we take  $A = \{a\}$  then  $\psi\text{-cl}(A) \setminus A = \{c, d\}$  is  $r\psi g$ -open but  $A$  is not  $r\psi g$ -closed.

**Definition 2.13.** Let  $\psi$  be an operation on a topological space  $(X, \tau)$ . Then  $(X, \tau)$  is said to be  $r\psi g\text{-}T_{1/2}$  if every  $r\psi g$ -closed set in  $(X, \tau)$  is  $\psi$ -closed.

**Theorem 2.14.** Let  $\psi$  be an operation on a topological space  $(X, \tau)$ . Then the following are equivalent :

- (i)  $(X, \tau)$  is  $r\psi g\text{-}T_{1/2}$ .
- (ii) Every singleton is either regular closed or  $\psi$ -open.

*Proof.* (i)  $\Rightarrow$  (ii) : Suppose  $\{x\}$  is not regular closed for some  $x \in X$ . Then  $X \setminus \{x\}$  is not regular open and hence  $X$  is the only regular open set containing  $X \setminus \{x\}$ . Thus  $X \setminus \{x\}$  is  $r\psi g$ -closed. Hence  $X \setminus \{x\}$  is  $\psi$ -closed (by (i)). Thus  $\{x\}$  is  $\psi$ -open.

(ii)  $\Rightarrow$  (i) : Let  $A$  be any  $r\psi g$ -closed subset of  $(X, \tau)$  and  $x \in \psi\text{-cl}(A)$ . We have to show that  $x \in A$ . If  $\{x\}$  is regular closed and  $x \notin A$ , then  $x \in \psi\text{-cl}(A) \setminus A$ . Thus  $\psi\text{-cl}(A) \setminus A$  contains a non-empty regular closed set  $\{x\}$ , a contradiction to Theorem 2.7. So  $x \in A$ . Again if  $\{x\}$  is  $\psi$ -open, then since  $x \in \psi\text{-cl}(A)$ , it follows that  $x \in A$ . So in both the cases  $x \in A$ . Thus  $A$  is  $\psi$ -closed.  $\square$

*Remark 2.15.* Let  $\psi$  be an operation on a space  $(X, \tau)$ . Then every  $r\psi g\text{-}T_{1/2}$  space reduces to preregular  $T_{1/2}$  [7] (resp.  $\delta p$ -regular  $T_{1/2}$  [6],  $rg\omega\text{-}T_{1/2}$  [2]) if one takes  $\psi$  to be  $PO(X)$  (resp.  $\delta PO(X)$ ,  $\tau_\omega$ ).

**Theorem 2.16.** Let  $\psi$  be an operation on a topological space  $(X, \tau)$ . Then the following are equivalent :

- (i) Every regular open set of  $X$  is  $\psi$ -closed.
- (ii) Every subset of  $X$  is  $r\psi g$ -closed.

*Proof.* (i)  $\Rightarrow$  (ii) : Let  $A \subseteq U \in RO(X)$ . Then by (i)  $U$  is  $\psi$ -closed and so  $\psi\text{-cl}(A) \subseteq \psi\text{-cl}(U) = U$ . Thus  $A$  is  $r\psi g$ -closed.

(ii)  $\Rightarrow$  (i) : Let  $U \in RO(X)$ . Then by (ii),  $U$  is  $r\psi g$ -closed and hence  $\psi\text{-cl}(U) \subseteq U$ , showing  $U$  to be a  $\psi$ -closed set.  $\square$

**Theorem 2.17.** Let  $\psi$  be an operation on a topological space  $(X, \tau)$ . If  $A$  be  $r\psi g$ -open then  $U = X$  whenever  $U$  is regular open and  $\psi\text{-int}(A) \cup (X \setminus A) \subseteq U$ .

*Proof.* Let  $U \in RO(X)$  and  $\psi\text{-int}(A) \cup (X \setminus A) \subseteq U$  for an  $r\psi g$ -open set  $A$ . Then  $X \setminus U \subseteq [X \setminus \psi\text{-int}(A)] \cap A$ , i.e.,  $X \setminus U \subseteq \psi\text{-cl}(X \setminus A) \setminus (X \setminus A)$ . Since  $X \setminus A$  is  $r\psi g$ -closed by Theorem 2.7,  $X \setminus U = \emptyset$  and hence  $U = X$ .  $\square$

The converse of the theorem above is not always true as shown by the following example.

**Example 2.18.** Let  $X = \{a, b, c, d\}$  and  $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ . Consider the mapping  $\psi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$  defined by  $\psi(A) = \text{intcl}A$  for all  $A \subseteq X$ . Then  $\psi \in \mathcal{O}(X)$ . Let  $A = \{b, c, d\}$ . It can be easily verify that  $X$  is the only regular open set containing  $\psi\text{-int}(A) \cup (X \setminus A)$  but  $A$  is not a  $r\psi g$ -open set in  $X$ .

### 3. Weak separation properties

**Definition 3.1.** Let  $(X, \tau)$  be a topological space and  $\psi$  be an operation on  $X$ . Then  $(X, \tau)$  is said to be almost  $\psi$ -regular if for each regular closed set  $F$  of  $X$  and each  $x \notin F$  there exist disjoint  $\psi$ -open sets  $U$  and  $V$  such that  $x \in U$ ,  $F \subseteq V$ .

*Remark 3.2.* Let  $\psi$  be an operation on a space  $(X, \tau)$ . Then every almost  $\psi$ -regular space reduces to an almost regular [25] (resp. almost  $p$ -regular [13], almost  $\delta p$ -regular [6], almost  $\alpha$ -regular) space if one takes  $\psi$  to be  $\text{int}$  (resp.  $\text{intcl}$ ,  $\text{intcl}_\delta$ ,  $\text{intclint}$ ).

**Theorem 3.3.** Let  $\psi$  be an operation on a topological space  $(X, \tau)$ . Then the following statements are equivalent :

- (i)  $X$  is almost  $\psi$ -regular.
- (ii) For each  $x \in X$  and each  $U \in RO(X)$  with  $x \in U$  there exists  $V \in \psi O(X)$  such that  $x \in V \subseteq \psi\text{-cl}(V) \subseteq U$ .
- (iii) For each regular closed set  $F$  of  $X$ ,  $\bigcap \{\psi\text{-cl}(V) : F \subseteq V \in \psi O(X)\} = F$ .
- (iv) For each  $A \subseteq X$  and each  $U \in RO(X)$  with  $A \cap U \neq \emptyset$ , there exists  $V \in \psi O(X)$  such that  $A \cap V \neq \emptyset$  and  $\psi\text{-cl}(V) \subseteq U$ .
- (v) For each non-empty subset  $A$  of  $X$  and each regular closed subset  $F$  of  $X$  with  $A \cap F = \emptyset$ , there exist  $V, W \in \psi O(X)$  such that  $A \cap V \neq \emptyset$ ,  $F \subseteq W$  and  $W \cap V = \emptyset$ .
- (vi) For each regular closed set  $F$  and  $x \notin F$ , there exist  $U \in \psi O(X)$  and an  $r\psi g$ -open set  $V$  such that  $x \in U$ ,  $F \subseteq V$  and  $U \cap V = \emptyset$ .
- (vii) For each  $A \subseteq X$  and each regular closed set  $F$  with  $A \cap F = \emptyset$ , there exist  $U \in \psi O(X)$  and an  $r\psi g$ -open set  $V$  such that  $A \cap U \neq \emptyset$ ,  $F \subseteq V$  and  $U \cap V = \emptyset$ .

*Proof.* (i)  $\Rightarrow$  (ii) : Let  $U \in RO(X)$  with  $x \in U$ . Then  $x \notin X \setminus U \in RC(X)$ . Thus by (i), there exist disjoint  $G, V \in \psi O(X)$  such that  $x \in V$ ,  $X \setminus U \subseteq G$ . So,  $x \in V \subseteq \psi\text{-cl}(V) \subseteq \psi\text{-cl}(X \setminus G) = X \setminus G \subseteq U$ .

(ii)  $\Rightarrow$  (iii) : Let  $X \setminus F \in RO(X)$  and  $x \in X \setminus F$ . Then by (ii), there exists  $U \in \psi O(X)$  such that  $x \in U \subseteq \psi\text{-cl}(U) \subseteq X \setminus F$ . So  $F \subseteq X \setminus \psi\text{-cl}(U) = V$  (say)  $\in \psi O(X)$  and  $U \cap V = \emptyset$ . Then  $x \notin \psi\text{-cl}(V)$ . Thus  $F \supseteq \bigcap \{\psi\text{-cl}(V) : F \subseteq V \in \psi O(X)\}$ .

(iii)  $\Rightarrow$  (iv) : Let  $A$  be a subset of  $X$  and  $U \in RO(X)$  be such that  $A \cap U \neq \emptyset$ . Let  $x \in A \cap U$ . Then  $x \notin X \setminus U$ . Hence by (iii), there exists  $W \in \psi O(X)$  such that  $X \setminus U \subseteq W$  and  $x \notin \psi\text{-cl}(W)$ . Put  $V = X \setminus \psi\text{-cl}(W)$ . Then  $V \in \psi O(X)$  contains  $x$  and hence  $A \cap V \neq \emptyset$ . Now  $V \subseteq X \setminus W$ , so  $\psi\text{-cl}(V) \subseteq X \setminus W \subseteq U$ .

(iv)  $\Rightarrow$  (v) : Let  $F$  be a set as in the hypothesis of (v). Then  $X \setminus F \in RO(X)$  with  $A \cap (X \setminus F) \neq \emptyset$  and hence by (iv), there exists  $V \in \psi O(X)$  such that  $A \cap V \neq \emptyset$

and  $\psi\text{-cl}(V) \subseteq X \setminus F$ . If we put  $W = X \setminus \psi\text{-cl}(V)$ , then  $W \in \psi O(X)$ ,  $F \subseteq W$  and  $W \cap V = \emptyset$ .

(v)  $\Rightarrow$  (i) : Let  $F$  be a regular closed set such that  $x \notin F$ . Then  $F \cap \{x\} = \emptyset$ . Thus by (v), there exist  $U, V \in \psi O(X)$  such that  $x \in U$ ,  $F \subseteq V$  and  $U \cap V = \emptyset$ .

(i)  $\Rightarrow$  (vi) : Trivial in view of Remark 2.2.

(vi)  $\Rightarrow$  (vii) : Let  $A \subseteq X$  and  $F$  be a regular closed set with  $A \cap F = \emptyset$ . Then for  $a \in A$ ,  $a \notin F$  and hence by (vi), there exist  $U \in \psi O(X)$  and an  $r\psi g$ -open set  $V$  such that  $a \in U$ ,  $F \subseteq V$  and  $U \cap V = \emptyset$ . So  $A \cap U \neq \emptyset$ ,  $F \subseteq V$  and  $U \cap V = \emptyset$ .

(vii)  $\Rightarrow$  (i) : Let  $x \notin F$  where  $F$  is regular closed in  $X$ . Since  $\{x\} \cap F = \emptyset$ , by (vii) there exist  $U \in \psi O(X)$  and an  $r\psi g$ -open set  $W$  such that  $x \in U$ ,  $F \subseteq W$  and  $U \cap W = \emptyset$ . Then  $F \subseteq \psi\text{-int}(W) = V$  (say)  $\in \psi O(X)$  (by Theorem 2.9) and hence  $V \cap U = \emptyset$ .  $\square$

**Definition 3.4.** A topological space  $X$  is said to be almost  $\delta p$ -regular [6] if for each regular closed set  $A$  of  $X$  and each point  $x \in X \setminus A$ , there exist disjoint  $\delta$ -preopen sets  $U$  and  $V$  such that  $x \in U$  and  $A \subseteq V$ .

Almost  $\delta p$ -regular spaces were introduced and studied in [6].

*Remark 3.5.* If in a topological space  $(X, \tau)$  we take  $\psi = \text{intclint}$ , then an almost  $\psi$ -regular space reduces to an almost regular space [19].

**Definition 3.6.** Let  $\psi$  be an operation on a topological space  $(X, \tau)$ . Then  $(X, \tau)$  is said to be almost  $\psi$ -normal if for each closed set  $A$  and each regular closed set  $B$  of  $X$  such that  $A \cap B = \emptyset$ , there exist disjoint  $\psi$ -open sets  $U$  and  $V$  such that  $A \subseteq U$  and  $B \subseteq V$ .

*Remark 3.7.* Let  $\psi$  be an operation on a space  $(X, \tau)$ . Then an almost  $\psi$ -normal space reduces to an almost normal [26] (resp. almost  $p$ -normal [15, 21], almost  $\delta p$ -normal [5, 6], almost  $\alpha$ -normal) space if one takes  $\psi$  to be  $\text{int}$  (resp.  $\text{intcl}$ ,  $\text{intcl}_\delta$ ,  $\text{intclint}$ ).

We note that in a topological space  $(X, \tau)$  with an operation  $\psi$  on  $X$ ,  $A$  is  $g\psi$ -open iff  $F \subseteq \psi\text{-int}(A)$  whenever  $F \subseteq A$  and  $F$  is closed.

**Theorem 3.8.** Let  $\psi$  be an operation on a topological space  $(X, \tau)$ . Then the following statements are equivalent :

- (i)  $X$  is almost  $\psi$ -normal.
- (ii) For each closed set  $A$  and regular closed set  $B$  of  $X$  such that  $A \cap B = \emptyset$ , there exist disjoint  $g\psi$ -open sets  $U$  and  $V$  of  $X$  such that  $A \subseteq U$  and  $B \subseteq V$ .
- (iii) For each closed set  $A$  and each regular open set  $B$  containing  $A$ , there exists a  $g\psi$ -open set  $V$  of  $X$  such that  $A \subseteq V \subseteq \psi\text{-cl}(V) \subseteq B$ .
- (iv) For each  $rg$ -closed set  $A$  and each regular open set  $B$  containing  $A$ , there exists a  $g\psi$ -open set  $V$  of  $X$  such that  $clA \subseteq V \subseteq \psi\text{-cl}(V) \subseteq B$ .
- (v) For each  $rg$ -closed set  $A$  and each regular open set  $B$  containing  $A$ , there exists a  $\psi$ -open set  $V$  of  $X$  such that  $clA \subseteq V \subseteq \psi\text{-cl}(V) \subseteq B$ .
- (vi) For each  $g$ -closed set  $A$  and each regular open set  $B$  containing  $A$ , there exists a  $\psi$ -open set  $V$  such that  $cl(A) \subseteq V \subseteq \psi\text{-cl}(V) \subseteq B$ .

(vii) For each  $g$ -closed set  $A$  and each regular open set  $B$  containing  $A$ , there exists a  $g\psi$ -open set  $V$  such that  $cl(A) \subseteq V \subseteq \psi-cl(V) \subseteq B$ .

*Proof.* (i)  $\Rightarrow$  (ii) : Obvious by Remark 2.2.

(ii)  $\Rightarrow$  (iii) : Let  $A$  be a closed set and  $B$  be a regular open set containing  $A$ . Then  $A \cap (X \setminus B) = \emptyset$ , where  $A$  is closed and  $X \setminus B$  is regular closed. So by (ii) there exist disjoint  $g\psi$ -open sets  $V$  and  $W$  such that  $A \subseteq V$  and  $X \setminus B \subseteq W$ . Thus by Remark 2.2 and Theorem 2.9,  $X \setminus B \subseteq \psi-int(W)$  and  $V \cap \psi-int(W) = \emptyset$ . Hence  $\psi-cl(V) \cap \psi-int(W) = \emptyset$  and hence  $A \subseteq V \subseteq \psi-cl(V) \subseteq X \setminus \psi-int(W) \subseteq B$ .

(iii)  $\Rightarrow$  (iv) : Let  $A$  be  $rg$ -closed and  $B$  be a regular open set containing  $A$ . Then  $clA \subseteq B$ . The rest follows from (iii).

(iv)  $\Rightarrow$  (v) : This follows from (iv) and the fact that a subset  $A$  is  $g\psi$ -open iff  $F \subseteq \psi-int(A)$  whenever  $F \subseteq A$  and  $F$  is closed.

(v)  $\Rightarrow$  (vi) : Follows from (v) and the fact that every  $g$ -closed set is an  $rg$ -closed set.

(vi)  $\Rightarrow$  (vii) : Trivial by Remark 2.2.

(vii)  $\Rightarrow$  (i) : Let  $A$  be any closed set and  $B$  be a regular closed set such that  $A \cap B = \emptyset$ . Then  $X \setminus B$  is a regular open set containing  $A$  where  $A$  is  $g$ -closed (as every closed set is  $g$ -closed). So there exists a  $g\psi$ -open set  $G$  of  $X$  such that  $clA \subseteq G \subseteq \psi-cl(G) \subseteq X \setminus B$ . Put  $U = \psi-int(G)$  and  $V = X \setminus \psi-cl(G)$ . Then  $U$  and  $V$  are two disjoint  $\psi$ -open subsets of  $X$  such that  $clA \subseteq U$  (as  $G$  is  $g\psi$ -open), i.e.,  $A \subseteq U$  and  $B \subseteq V$ . Hence  $X$  is almost  $\psi$ -normal.  $\square$

*Remark 3.9.* If in a topological space  $(X, \tau)$  if we take  $\psi = intclint$ , then an almost  $\psi$ -normal space reduces to an almost normal space that follows from the next theorem.

**Theorem 3.10.** *A topological space  $(X, \tau)$  is almost normal if and only if it is almost  $\alpha$ -normal.*

*Proof.* One part of the Theorem is obvious as  $\tau \subseteq \tau_\alpha$ . We shall only show that if  $X$  is almost  $\alpha$ -normal then it is normal. Let  $A$  be a closed set and  $B$  be a regular closed set such that  $A \cap B = \emptyset$ . Then by  $\alpha$ -normality of  $X$ , there exist two disjoint  $\alpha$ -open sets  $G$  and  $H$  such that  $A \subseteq G$  and  $B \subseteq H$ . Let  $U = int(cl(int(G)))$  and  $V = int(cl(int(H)))$ . Then  $U$  and  $V$  are two open subsets of  $X$  such that  $A \subseteq U$ ,  $B \subseteq V$  and  $U \cap V = \emptyset$ . Thus  $X$  is almost normal.  $\square$

**Definition 3.11.** Let  $\psi$  be an operation on a topological space  $(X, \tau)$ . Then  $(X, \tau)$  is said to be mildly  $\psi$ -normal if for every pair of disjoint regular closed sets  $A$  and  $B$  of  $X$ , there exist two disjoint  $\psi$ -open sets  $U$  and  $V$  such that  $A \subseteq U$  and  $B \subseteq V$ .

*Remark 3.12.* Let  $\psi$  be an operation on a space  $(X, \tau)$ . Then a mildly  $\psi$ -normal space reduces to a mildly normal [27, 17] (resp. mildly  $p$ -normal [15, 21], mildly  $\delta p$ -normal [5, 6], mildly  $\alpha$ -normal) if one takes  $\psi$  to be  $int$  (resp.  $intcl$ ,  $intcl_\delta$ ,  $intclint$ ).

**Theorem 3.13.** Let  $\psi$  be an operation on a topological space  $(X, \tau)$ . Then the following are equivalent :

(i)  $X$  is mildly  $\psi$ -normal.

(ii) For any disjoint  $L, K \in RC(X)$ , there exist  $g\psi$ -open sets  $U$  and  $V$  such that  $L \subseteq U$  and  $K \subseteq V$ .

(iii) For  $L, K \in RC(X)$  with  $L \cap K = \emptyset$ , there exist disjoint  $r\psi g$ -open sets  $U$  and  $V$  such that  $L \subseteq U$  and  $K \subseteq V$ .

(iv) For any  $L \in RC(X)$  and any  $V \in RO(X)$  with  $L \subseteq V$ , there exists an  $r\psi g$ -open set  $U$  of  $X$  such that  $L \subseteq U \subseteq \psi-cl(U) \subseteq V$ .

(v) For any  $L \in RC(X)$  and any  $V \in RO(X)$  with  $L \subseteq V$ , there exists a  $\psi$ -open set  $U$  of  $X$  such that  $L \subseteq U \subseteq \psi-cl(U) \subseteq V$ .

*Proof.* (i)  $\Rightarrow$  (ii) : Follows from Remark 2.2.

(ii)  $\Rightarrow$  (iii) : Follows from Remark 2.2.

(iii)  $\Rightarrow$  (iv) : Let  $L \in RC(X)$  and  $V \in RO(X)$  be such that  $L \subseteq V$ . Then by (iii) there exist disjoint  $r\psi g$ -open sets  $U$  and  $W$  such that  $L \subseteq U$  and  $X \setminus V \subseteq W$ . Thus by Theorem 2.9,  $X \setminus V \subseteq \psi-int(W)$  and  $U \cap \psi-int(W) = \emptyset$ . So  $\psi-cl(U) \cap \psi-int(W) = \emptyset$  and hence  $L \subseteq U \subseteq \psi-cl(U) \subseteq X \setminus \psi-int(W) \subseteq V$ .

(iv)  $\Rightarrow$  (v) : Let  $L \in RC(X)$  and  $V \in RO(X)$  be such that  $L \subseteq V$ . Thus by (iv) there exists an  $r\psi g$ -open set  $G$  of  $X$  such that  $L \subseteq G \subseteq \psi-cl(G) \subseteq V$ . Since  $L \in RC(X)$ , by Theorem 2.9,  $L \subseteq \psi-int(G) = U$  (say). Hence  $U \in \psi O(X)$  and  $L \subseteq U \subseteq \psi-cl(U) \subseteq \psi-cl(G) \subseteq V$ .

(v)  $\Rightarrow$  (i) : Let  $L, K \in RC(X)$  be such that  $L \cap K = \emptyset$ . Then  $X \setminus K \in RO(X)$  with  $L \subseteq X \setminus K$ . Thus by (v) there exists a  $\psi$ -open set  $U$  of  $X$  such that  $L \subseteq U \subseteq \psi-cl(U) \subseteq X \setminus K$ . Put  $V = X \setminus \psi-cl(U)$ . Then  $U$  and  $V$  are disjoint  $\psi$ -open sets such that  $L \subseteq U$  and  $K \subseteq V$ .  $\square$

By the similar arguments as shown in Theorem 3.10 we have

*Remark 3.14.* In a topological space  $(X, \tau)$  if we take  $\psi = intclint$ , then a mildly  $\psi$ -normal space and mildly normal space are identical.

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