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About the equation $B_m^{(a,b)} = f(x)$*

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Abstract

Let a, b be nonnegative coprime integers. We call an integer $an + b \in \mathbb{N}$ (denoted by $B_m^{(a,b)}$) an (a, b)-type balancing number if

$$(a + b) + (2a + b) + \cdots + (a(n - 1) + b) = (a(n + 1) + b) + \cdots + (a(n + r) + b)$$

for some $r \in \mathbb{N}$.

In this paper we consider and give numerical results for the equation $B_m^{(a,b)} = f(x)$ where $B_m^{(a,b)}$ is an (a, b)-type balancing number and $f(x)$ is a polynomial belonging to combinatorial numbers (that is binomial coefficients, power sums and products of consecutive integers).

Moreover we investigate the equation when an (a, b)-type balancing number with different parameters are equal to a Fibonacci or a Lucas number. In this case we use a parallel program to find the solutions of simultaneous Pell equations.

Keywords: balancing numbers, elliptic curves, Magma, combinatorial numbers, parallel algorithm

MSC: 11D25, 11D41, 11D45

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1. Introduction

A positive integer n is called a balancing number (see [2] and [4]) if

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$$

for some $r \in \mathbb{Z}^+$. Here r is called the balancer corresponding to the balancing number n. Denote by B_m the mth term of the sequence of balancing numbers. For example 6 and 35 are balancing numbers with balancers 2 and 14, respectively.

K. Liptai [5, 6] proved that there is no Fibonacci and Lucas balancing numbers. In these proofs the same method were used which is based on the result of Baker and Davenport (see [1]). Using an other way from L. Szalay [12] got the same result. This method used a program by Magma [9], but later G. Szekrényesi [13] made a parallel program which was faster than earlier one and arbitrarily large coefficients were used. This program used the fast algorithm for finding solutions of "small solutions" of Thue equations or inequalities. In this case we know about the integer solutions (x, y) that $|y| < 10^{500}$. Using this program we investigated the problem of existence of Fibonacci or Lucas numbers among balancing numbers (for details see [13]).

To prove one of our main results we need the following lemma of P. E. Ferguson (see [3]).

Lemma 1.1. *The only solutions of the equation*

$$x^2 - 5y^2 = \pm 4 \tag{1.1}$$

are $x = \pm L_n$, $y = \pm F_n$ $(n = 0, 1, 2, \ldots)$, where L_n and F_n are the n^{th} terms of the Lucas and Fibonacci sequences, respectively.

Later K. Liptai, F. Luca, Á. Pintér and L. Szalay [7] generalized the balancing numbers which are called (k, l)-power numerical center.

Let y, k, l be fixed positive integers with $y \geq 4$. A positive integer x $(x \leq y - 2)$ is called a (k, l)-power numerical center for y if

$$1^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l.$$

They [7] proved several effective and ineffective finiteness statements for (k, l)-power numerical center using Baker-type diophantine results and Bilu-Tichy theorem. There is another generalization of balancing numbers (see [8]).

Let $a > 0$ and $b \geq 0$ be coprime integers. We call an integer $an + b \in \mathbb{N}$ an (a, b)-type balancing number if

$$(a+b) + (2a+b) + \cdots + (a(n-1)+b) = (a(n+1)+b) + \cdots + (a(n+r)+b)$$

for some $n, r \in \mathbb{N}$. Here r is called the balancer corresponding to the balancing number $an + b$ denoted by $B_m^{(a,b)}$.

T. Kovács, K. Liptai and P. Olajos [8] got a simple proposition for (a, b)-type balancing numbers.

Lemma 1.2. *If* $B_m^{(a,b)}$ *is an* (a,b)*-type balancing number then the following equation*

$$z^2 - 8\left(B_m^{(a,b)}\right)^2 = a^2 - 4ab - 4b^2 \tag{1.2}$$

is valid for some $z \in \mathbb{Z}$.

In the case when $a = 2$ and $b = 1$ P. Olajos [10] proved that

$$B_{m+2}^{(2,1)} = 6 \cdot B_{m+1}^{(2,1)} - 1 \cdot B_m, \ (m \geq 1), \text{ where } B_1^{(2,1)} = 17, \ B_2^{(2,1)} = 99.$$

He also considered Fibonacci and Lucas numbers among $(2,1)$-type balancing numbers.

Let us consider the equation

$$B_m^{(a,b)} = f(x) \tag{1.3}$$

where $f(x)$ is a polynomial with integer coefficients.

They [8] proved finiteness results for equation (1.3) in the cases when $f(x)$ is a monic polynomial or perfect power. The authors proved another finiteness result also when $f(x)$ is equal to a combinatorial number.

For all $k, x \in \mathbb{N}$ let

$$\begin{aligned} S_k(x) &= 1^k + 2^k + \cdots + (x-1)^k, \\ T_k(x) &= -1^k + 2^k - \cdots + (-1)^{x-1}(x-1)^k, \\ \Pi_k(x) &= x(x+1)\dots(x+k-1). \end{aligned}$$

Lemma 1.3. *Let* $k \geq 2$ *and* $f(x)$ *be one of the polynomials* $\binom{x}{k}$, $\Pi_k(x)$, $S_{k-1}(x)$, $T_k(x)$. *Then the solutions of equation* (1.3) *satisfy* $\max(m, |x|) < c_1(a,b,k)$, *where* $c_1(a,b,k)$ *is an effectively computable constant depending only on* a, b *and* k.

In this paper they also considered all solutions $(x,y) \in \mathbb{Z}^2$ of equation

$$B_m^{(a,b)} = f(x)$$

when $a^2 - 4ab - 4b^2 = 1$ and $f(x) \in \{\binom{x}{2}, \binom{x}{3}, \binom{x}{4}\}$, $\Pi_2(x)$, $\Pi_3(x)$, $\Pi_4(x)$, $S_1(x)$, $S_2(x)$, $S_3(x)$, $S_5(x)$. For more details see [8].

Later Sz. Tengely [14] proved that the equation

$$B_m = x(x+1)(x+2)(x+3)(x+4)$$

has no solution. The author combined Baker's method and the so-called Mordell-Weil sieve to obtain all solutions.

The authors [8] partially solved equation (1.3), because they considered only the cases when $a^2 - 4ab - 4b^2 = 1$. In the following chapter we discuss this problem with certain conditions and not only for the cases above.

2. Numerical results

2.1. Results by MAGMA

By Lemma 1.3 we know that there are only finite number of solutions of equation (1.3). In the cases when $a \in [1,9]$, $b \in [0,7]$, $a \geq b$ and $\gcd(a,b) = 1$ we get the following result:

Theorem 2.1. *Let $2 \leq k \leq 4$ and $f(x)$ be one of the polynomials $\binom{x}{k}$, $\Pi_k(x)$, $S_{k-1}(x)$and $a \in [1,9]$, $b \in [0,7]$ where $a \geq b$ and $\gcd(a,b) = 1$. Then the solutions ($B_m^{(a,b)}$, x) of equation (1.3) are in the following table:*

a	b	$B_m^{(a,b)}$	$f(x)$	x	k
1	0	1	$\binom{x}{k}$	2	2
1	0	1	$\binom{x}{k}$	3	3
1	0	1	$\binom{x}{k}$	4	4
1	0	6	$\binom{x}{k}$	4	2
1	0	35	$\binom{x}{k}$	7	3
1	0	35	$\binom{x}{k}$	7	4
1	1	4	$\binom{x}{k}$	4	3
1	0	1	$S_{k-1}(x)$	2	2
1	0	6	$S_{k-1}(x)$	4	2
1	0	1	$S_{k-1}(x)$	2	3
1	0	204	$S_{k-1}(x)$	9	3
1	0	1	$S_{k-1}(x)$	2	4
1	0	6	$\Pi_k(x)$	2	2
7	5	600	$\Pi_k(x)$	24	2
1	0	6	$\Pi_k(x)$	1	3

Remark 2.2. We mention that in the case $k = 1$ we get infinitely many solutions for equation (1.3) since in this case the equation is a Pell-equation.

2.2. Results by a parallel program

In this subsection we consider the cases when $B_m^{(a,b)} = F_l$ or $B_m^{(a,b)} = L_p$ where F_l and L_p are Fibonacci and Lucas numbers, respectively. Let us consider the equation (1.1) and (1.2). In the first case above we get the following simultaneous Pell equations:

$$5x^2 - y^2 = \pm 4, \tag{2.1}$$

$$8x^2 - z^2 = -1(a^2 - 4ab - 4b^2), \tag{2.2}$$

where $x = B_m^{(a,b)} = F_l$. In the second case we have to solve the following:

$$x^2 - 5y^2 = \pm 4, \tag{2.3}$$

$$8x^2 - z^2 = -1(a^2 - 4ab - 4b^2), \tag{2.4}$$

where $x = B_m^{(a,b)} = L_p$. We use the parallel program from G. Szekrényesi to get the solutions of the equation systems above. So our numerical results is the following theorem.

Theorem 2.3. *If $a \in [1,9]$, $b \in [0,7]$, $a \geq b$ and $\gcd(a,b) = 1$ we get the following "small solutions" of equations $B_m^{(a,b)} = F_l$ or $B_m^{(a,b)} = L_p$ detailed in the next tables (that is there is an upper bound for integer unknowns in Thue inequalities which is equal to 10^{500}):*

a	b	m	r	$B_m^{(a,b)} = F_l$	l
1	0	1	0	1	1 or 2
7	1	228	94	1597	17

a	b	m	r	$B_m^{(a,b)} = L_p$	p
1	0	1	0	1	1
1	1	3	1	4	1
1	1	10	4	11	6

3. Proofs

3.1. Proof of Theorem 2.1

Consider the equation (1.3) when $f(x)$ one of polynomials $\binom{x}{2}$, $\binom{x}{3}$ and $\binom{x}{4}$. Using the transformations $X = 2x - 1$, $X = (x-1)^2$, $X = x^2 - 3x + 1$ respectively to the polynomials above then we get the following by Lemma 1.2:

$$\begin{aligned}(2^2 z)^2 &= 2X^4 - 4X^2 + 2 - 16C(a,b),\\ (6z)^2 &= X^3 - 4X^2 + 4X - 36C(a,b),\\ \left(2^2 3z\right)^2 &= 2X^4 - 4X^2 + 2 - 144C(a,b),\end{aligned}$$

where $C(a,b)$ denotes the quantity $-(a^2 - 4ab - 4b^2)$.

These types of equations are solvable by MAGMA (`IntegralQuarticPoints` and `IntegralPoints`), so after testing them we get the solutions above.

Let us consider two example of using MAGMA commands. In the first example set the parameters as the following: $k = 2$, $a = 2$, $b = 1$. In this case we have to use the transformations above that is we have to solve the equation

$$(2^2 z)^2 = 2X^4 - 4X^2 - 126.$$

The suitable command is `IntegralQuarticPoints([2,0,-4,0,-126])`. We get the solutions `(5,32),(3,0)` for $(X, 2^2 z)$. Using these results we know that no solutions for $B_m^{(2,1)}$, because 3 and 1 are not $(2,1)$-type balancing numbers. We used the property that $B_m^{(2,1)} \geq 17$.

Let us consider the second one. In this case let parameters $k = 3$, $a = 2$, $b = 1$. Our equation is the following:

$$(6z)^2 = X^3 - 4X^2 + 4X - 288.$$

Using the commands `IntegralPoints(EllipticCurve([0,-4,0,4,-288]))` we get the solution $(X, 6z) = (8, 0)$, that is there is no $(2,1)$-type balancing number with the main property above.

Now let $f(x)$ be equal to $S_{k-1}(x)$. If $k = 2$ then $S_1(x) = \binom{x}{2}$ that is we get the solutions.

Using the transformations $X = 2(2x-1)^2$, $X = \binom{x}{2}$ respectively to the equation (1.2) when $f(x) = S_2(x)$ and $f(x) = S_3(x)$ we get

$$\begin{aligned}\left(2^3 3z\right)^2 &= X^3 - 4X^2 + 4X - 576C(a,b),\\ z^2 &= 8X^4 - C(a,b).\end{aligned}$$

By MAGMA we get the solutions by the commands `IntegralQuarticPoints` and `IntegralPoints` above when $f(x) = S_{k-1}(x)$.

At last let $f(x) = \Pi_2(x)$, $\Pi_3(x)$, $\Pi_4(x)$ and by using the transformations $X = 2x + 1$, $X = 2(x + 1)^2$ and $x^2 + 3x + 1$ we get the following from (1.2)

$$\begin{aligned}(2z)^2 &= 2X^4 - 4X^2 + 2 - 4C(a,b),\\ z^2 &= X^3 - 4X^2 + 4X - C(a,b),\\ z^2 &= 8X^4 - 16X^2 + 8 - C(a,b).\end{aligned}$$

By MAGMA we get the solutions above. We have to mention that in the case $a = 2$, $b = 1$ of $\Pi_4(x)$ we get a singular equation, because $C(2,1) = 8$ and the curve $z^2 = 8X^4 - 16X^2$ is singular. Theres is no problem, because $\Pi_4(x)$ is even, but all $B_m^{(2,1)}$ are odd that is there is no solution of the equation (1.3).

3.2. Proof of Theorem 2.3

Let us consider first the case when $B_m^{(a,b)} = F_l$. We have to solve the simultaneous Pell equations by the parallel program (G. Szekrényesi [13]) or by MAGMA (L. Szalay [12]). We used the parallel one to determine the "small" (less then 10^{500}) solutions of system of the equations (2.1) and (2.2). It means that this program besides others could not find all solutions.

Generally the parallel program have been faster than others (e.g Maple, Magma or Kant). It uses the fast algorithm for finding the "small" integer solutions of Thue inequalities in parallel way by the method from Pethő and Schulenberg [11]. The program also containes a solver for simultaneous Pell equations, which is based on the algorithm of L. Szalay [12]. The program could use arbitrarily large coefficients which is inpossible in others.

The results detailed in the next table. I have to mention that the sign $+$ or $-$ in the table below denotes the correct sign of the right hand side of the first equation of our Pell system. Denote the expression $-(a^2 - 4ab - 4b^2)$ by $C(a,b)$ again.

a	b	$C(a,b)$	$+(x,y,z)$	$-(x,y,z)$
1	0	-1	(1,1,3)	(0,2,1); (1,3,3)
1	1	7	(2,4,5); (1,1,1)	(1,3,1)
2	1	8	(1,1,0)	(1,3,0); (3,7,8)
3	1	7	(2,4,5); (1,1,1)	(1,3,1)
3	2	31	(2,4,1); (233,521,659); (5,11,13)	–
4	1	4	(1,1,2); (5,11,14)	(1,3,2)
4	3	68	–	(3,7,2)
5	1	-1	(1,1,3)	(0,2,1); (1,3,3)
5	2	31	(2,4,1); (233,521,659); (5,11,13)	–
5	3	71	–	(8,18,21); (3,7,1)
5	4	119	(5,11,9)	–
6	1	-8	(1,1,4)	(1,3,4)
6	5	184	(5,11,4)	–
7	1	-17	(2,4,7);(13,29,37);(1,1,5);(1597,3571,4517)	(8,18,23); (1,3,5)
7	2	23	(2,4,3)	(3,7,7)
7	3	71	–	(8,18,21); (3,7,1)
7	4	127	(13,29,35)	–
7	5	191	(5,11,3)	–
7	6	263	(13,29,33)	–
8	1	-28	(1,1,6)	(3,7,10); (1,3,6)
8	3	68	–	(3,7,2)
8	5	196	(13,29,34); (5,11,2)	–
8	7	356	–	–
9	1	-41	(1,1,7)	(1,3,7)
9	2	7	(2,4,5); (1,1,1)	(1,3,1)
9	4	127	(13,29,35)	–
9	5	199	(5,11,1)	–
9	7	367	(89,199,251)	–

Using tha data from this table, we can get the solutions of Theorem 2.3 for Fibonacci balancing numbers.

Consider now the cases of Lucas numbers that is the equations (2.3) and (2.4). We get the following solutions detailed in the table below:

a	b	$C(a,b)$	$+(x,y,z)$	$-(x,y,z)$
1	0	-1	–	(1,1,3)
1	1	7	(2,0,5)	(4,2,11); (11,5,31); (1,1,1)
2	1	8	(3,1,8)	(1,1,0)
3	1	7	(2,0,5)	(4,2,11); (11,5,31); (1,1,1)
3	2	31	(2,0,1); (7,3,19)	–
4	1	4	–	(29,13,82); (1,1,2)
4	3	68	(3,1,2); (7,3,18)	(11,5,30)
5	1	-1	–	(1,1,3)
5	2	31	(2,0,1); (7,3,19)	–
5	3	71	(3,1,1)	–
5	4	119	–	(4,2,3)
6	1	-8	(7,3,20)	(1,1,4)
6	5	184	–	(11,5,28)
7	1	-17	(2,0,7); (47,21,133)	(76,34,215); (1,1,5)
7	2	23	(3,1,7); (2,0,3)	–
7	3	71	(3,1,1)	–
7	4	127	–	(4,2,1); (11,5,29)
7	5	191	(18,8,49)	–
7	6	263	–	–
8	1	-28	(3,1,10)	(1,1,6)
8	3	68	(3,1,2); (7,3,18)	(11,5,30)
8	5	196	(7,3,14)	–
8	7	356	(7,3,6)	–
9	1	-41	–	(4,2,13); (1,1,7)
9	2	7	(2,0,5)	(4,2,11); (11,5,31); (1,1,1)
9	4	127	–	(4,2,1); (11,5,29)
9	5	199	–	–
9	7	367	(7,3,5)	–

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