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# Cofinite derivations in rings

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**Abstract**

A derivation $d : R \to R$ is called cofinite if its image $\operatorname{Im} d$ is a subgroup of finite index in the additive group $R^+$ of an associative ring $R$. We characterize left Artinian (respectively semiprime) rings with all non-zero inner derivations to be cofinite.

*Keywords:* Derivation, Artinian ring, semiprime ring

*MSC:* 16W25, 16P20, 16N60

## 1. Introduction

Throughout this paper $R$ will always be an associative ring with identity. A derivation $d : R \to R$ is said to be *cofinite* if its image $\operatorname{Im} d$ is a subgroup of finite index in the additive group $R^+$ of $R$. Obviously, in a finite ring every derivation is cofinite. As noted in [3], only a few results are known concerning images of derivations.

We study properties of rings with cofinite non-zero derivations and prove the following

*Proposition* 1.1. Let $R$ be a left Artinian ring. Then every non-zero inner derivation of $R$ is cofinite if and only if it satisfies one of the following conditions:

(1) $R$ is finite ring;

(2) $R$ is a commutative ring;

(3) $R = F \oplus D$ is a ring direct sum of a finite commutative ring $F$ and a skew field $D$ with cofinite non-zero inner derivations.

Recall that a ring $R$ with 1 is called *semiprime* if it does not contains non-zero nilpotent ideals. A ring $R$ with an identity in which every non-zero ideal has a finite index is called *residually finite* (see [2] and [10]).

**Theorem 1.2.** *Let $R$ be a semiprime ring. Then all non-zero inner derivations are cofinite in $R$ if and only if it satisfies one of the following conditions:*

(1) *$R$ is finite ring;*

(2) *$R$ is a commutative ring;*

(3) *$R = F \oplus B$ is a ring direct sum, where $F$ is a finite commutative semiprime ring and $B$ is a residually finite domain generated by all commutators $xa-ax$, where $a, x \in B$.*

Throughout this paper for any ring $R$, $Z(R)$ will always denote the center, $Z_0 = Z_0(R)$ the ideal generated by all central ideals of $R$, $N(R)$ the set of all nilpotent elements of $R$, Der$R$ the set of all derivations of $R$, $\operatorname{Im} d = d(R)$ the image and $\operatorname{Ker} d$ the kernel of $d \in \operatorname{Der} R$, $U(R)$ the unit group of $R$, $|R : I|$ the index of a subring $I$ in the additive group $R^+$, $\partial_x(a) = xa - ax = [x, a]$ the commutator of $a, x \in R$ and $C(R)$ the commutator ideal of $R$ (i.e., generated by all $[x, a]$). If $|R : I| < \infty$, then we say that $I$ has a finite index in $R$.

Any unexplained terminology is standard as in [6], [4], [5], [8] and [11].

# 2. Some examples

We begin with some examples of derivations in associative rings.

**Example 2.1.** Let $D$ be an infinite (skew) field,

$$A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \ X = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in M_2(D).$$

Then we obtain that

$$\partial_A(X) = AX - XA = \begin{pmatrix} ax - xa & ay \\ -za & 0 \end{pmatrix},$$

and so the image $\operatorname{Im} \partial_A$ has an infinite index in $M_2(D)^+$.

Recall that a ring $R$ having no non-zero derivations is called *differentially trivial* [1].

**Example 2.2.** Let $F[X]$ be a commutative polynomial ring over a differentially trivial field $F$. Assume that $d$ is any derivation of $F[X]$. Then for every polynomial

$$f = \sum_{i=0}^{n} a_i X^{n-i} \in F[X]$$

we have

$$d(f) = (\sum_{i=0}^{n-1}(n-i)a_iX^{n-i-1})d(X) \in d(X)F[X],$$

where $d(X)$ is some element from $F[X]$. This means that the image $\operatorname{Im} d \subseteq d(X)F[X]$.

*a*) Let $F$ be a field of characteristic 0. If we have

$$g = \left(\sum_{i=0}^{m} b_iX^{m-i}\right) \cdot d(X) \in d(X)F[X],$$

then the following system

$$\begin{cases} (1+m)d_0 & = b_0, \\ md_1 & = b_1, \\ & \vdots \\ 2d_{m-1} & = b_{m-1}, \\ d_m & = b_m, \end{cases}$$

has a solution in $F$, i.e., there exists such polynomial

$$h = \sum_{i=0}^{m+1} d_iX^{m+1-i} \in F[X],$$

that $d(h) = g$. This gives that $\operatorname{Im} d = d(X)F[X]$. If $d$ is non-zero, then the additive quotient group

$$G = F[X]/d(X)F[X]$$

is infinite and every non-zero derivation $d$ of a commutative Noetherian ring $F[X]$ is not cofinite.

*b*) Now assume that $F$ has a prime characteristic $p$ and $d(X) = X$. If $X^{p^l} - X^{p^s} \in \operatorname{Im} d$ for some positive integer $l, s$, where $l > s$, then

$$X^{p^l} - X^{p^s} = d(t)$$

for some polynomial $t = d_0X^m + d_1X^{m-1} + \cdots + d_{m-1}X + d_m \in F[X]$ and consequently

$$X^{p^l} - X^{p^s} = md_0X^m + (m-1)d_1X^{m-1} + \cdots + 2d_{m-1}X^2 + d_{m-1}X.$$

Let $k$ be the smallest non-negative integer such that

$$(m-k)d_k \neq 0.$$

Then $p^l = m - k$, a contradiction. This means that $|F[X] : \operatorname{Im} d| = \infty$.

**Example 2.3.** Let

$$\mathbb{H} = \{\alpha + \beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k} \mid \alpha, \beta, \gamma, \delta \in \mathbb{R},$$
$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,\ \mathbf{ij} = -\mathbf{ji} = \mathbf{k},\ \mathbf{jk} = -\mathbf{kj} = \mathbf{i},\ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}\}$$

be the skew field of quaternions over the field $\mathbb{R}$ of real numbers. Then

$$\partial_i(\mathbb{H}) = \{\gamma\mathbf{j} + \delta\mathbf{k} \mid \gamma, \delta \in \mathbb{R}\}$$

and so the index $|\mathbb{H} : \operatorname{Im}\partial_i|$ is infinite. Hence the inner derivation $\partial_i$ is not cofinite in $\mathbb{H}$.

**Example 2.4.** Let $D = F(y)$ be the rational functions field in a variable $y$ over a field $F$ and $\sigma : D \to D$ be an automorphism of the $F$-algebra $D$ such that

$$\sigma(y) = y + 1.$$

By

$$R = D((X;\sigma)) = \{\sum_{i=n}^{\infty} a_i X^i \mid a_i \in D \text{ for all } i \geq n,\ n \in \mathbb{Z}\}$$

we denote the ring of skew Laurent power series with a multiplication induced by the rule

$$(aX^k)(bX^l) = a\sigma^k(b)X^{k+l}$$

for any elements $a, b \in D$. Then we compute the commutator

$$\begin{aligned}\left[\sum_{i=n}^{\infty} a_i X^i, y\right] &= \sum_{i=n}^{\infty} a_i X^i y - y \sum_{i=n}^{\infty} a_i X^i \\ &= \sum_{i=n}^{\infty} a_i \sigma^i(y) X^i - \sum_{i=n}^{\infty} a_i y X^i \\ &= \sum_{i=n}^{\infty} a_i(\sigma^i(y) - y) X^i = \sum_{i=n}^{\infty} i a_i X^i.\end{aligned}$$

If now

$$f = \sum_{i=n}^{\infty} b_i X^i \in R,$$

then there exist elements $a_i \in D$ such that

$$b_i = i a_i$$

for any $i \geq n$. This implies that the image $\operatorname{Im}\partial_y = R$ and $\partial_y$ is a cofinite derivation of $R$.

**Lemma 2.5.** *Let $R = F[X, Y]$ be a commutative polynomial ring in two variables $X$ and $Y$ over a field $F$. Then $R$ has a non-zero derivation that is not confinite.*

*Proof.* Let us $f = \sum \alpha_{ij} X^i Y^j \in R$ and $d : R \to R$ be a derivation defined by the rules

$$\begin{aligned} d(X) &= X, \\ d(Y) &= 0, \\ d(f) &= \sum i\alpha_{ij} X^{i-1} Y^j d(X). \end{aligned}$$

It is clear that $\operatorname{Im} d \subseteq XR$ and $|R : XR| = \infty$. □

In the same way we can prove the following

**Lemma 2.6.** *Let $R = F[\{X_\alpha\}_{\alpha \in \Lambda}]$ be a commutative polynomial ring in variables $\{X_\alpha\}_{\alpha \in \Lambda}$ over a field $F$. If $\operatorname{card} \Lambda \geq 2$, then $R$ has a non-zero derivation that is not confinite.*

# 3. Cofinite inner derivations

**Lemma 3.1.** *If every non-zero inner derivation of a ring $R$ is cofinite, then for each ideal $I$ of $R$ it holds that $I \subseteq Z(R)$ or $|R : I| < \infty$.*

*Proof.* Indeed, if $I$ is a non-zero ideal of $R$ and $0 \neq a \in I$, then the image $\operatorname{Im} \partial_a \subseteq I$. □

*Remark* 3.2. If $\delta$ is a cofinite derivation of an infinite ring $R$, then $|R : \operatorname{Ker} \delta| = \infty$.

In fact, if the kernel $\operatorname{Ker} \delta = \{a \in R \mid \delta(a) = 0\}$ has a finite index in $R$, in view of the group isomorphism

$$R^+ / \operatorname{Ker} \delta \cong \operatorname{Im} \delta,$$

we conclude that $\operatorname{Im} \delta$ is a finite group.

**Lemma 3.3.** *If $I$ is a central ideal of a ring $R$, then $C(R)I = (0)$.*

*Proof.* For any elements $t, r \in R$ and $i \in I$ we have

$$(rt)i = r(ti) = (ti)r = t(ir) = t(ri) = (tr)i,$$

and therefore

$$(rt - tr)i = 0.$$

Hence $C(R)I = (0)$. □

**Lemma 3.4.** *Let $R$ be a non-simple ring with all non-zero inner derivations to be cofinite. If all ideals of $R$ are central, then $R$ is commutative or finite.*

*Proof.* *a*) If a ring $R$ is not local, then $R = M_1 + M_2 \subseteq Z(R)$ for any two different maximal ideals $M_1$ and $M_2$ of $R$.

*b*) Suppose that $R$ is a local ring and $J(R) \neq (0)$, where $J(R)$ is the Jacobson ideal of $R$. Then $J(R)C(R) = (0)$, $C(R) \neq R$ and, consequently,

$$C(R)^2 = (0).$$

If we assume that $R$ is not commutative, then

$$(0) \neq C(R) < R,$$

and so there exists an element $x \in R \setminus Z(R)$ such that

$$\{0\} \neq \operatorname{Im} \partial_x \subseteq C(R).$$

Then $|R : C(R)| < \infty$. Since $C(R) \subseteq Z(R)$, we deduce that the index $|R : Z(R)|$ is finite. By Proposition 1 of [7], the commutator ideal $C(R)$ is finite and $R$ is also finite. □

**Lemma 3.5.** *If $N(R) \subseteq Z(R)$, then every idempotent is central in a ring $R$.*

*Proof.* If $d \in \operatorname{Der} R$ and $e = e^2 \in R$, then we obtain $d(e) = d(e)e + ed(e)$, and this implies that

$$ed(e)e = 0 \text{ and } d(e)e, ed(e) \in N(R).$$

Then $ed(e) = e^2 d(e) = ed(e)e = 0$ and $d(e)e = 0$. As a consequence, $d(e) = 0$ and so $e \in Z(R)$. □

**Lemma 3.6.** *Let $R$ be a ring with all non-zero inner derivations to be cofinite. Then one of the following conditions holds:*

(1) *$R$ is a finite ring;*

(2) *$R$ is a commutative ring;*

(3) *$R$ contains a finite central ideal $Z_0$ such that $R/Z_0$ is an infinite residually finite ring (and, consequently, $R/Z_0$ is a prime ring with the ascending chain condition on ideals).*

*Proof.* Assume that $R$ is an infinite ring which is not commutative and its every non-zero inner derivation is cofinite. Then $|R : C(R)| < \infty$ and every non-zero ideal of the quotient ring $B = R/Z_0$ has a finite index. If $B$ is finite (or respectively $C(R) \subseteq Z_0$), then $|R : Z(R)| < \infty$ and, by Proposition 1 of [7], the commutator ideal $C(R)$ is finite. From this it follows that a ring $R$ is finite, a contradiction. Hence $B$ is an infinite ring and $C(R)$ is not contained in $Z_0$. Since $Z_0 C(R) = (0)$, we deduce that $Z_0$ is finite. By Corollary 2.2 and Theorem 2.3 from [2], $B$ is a prime ring with the ascending chain condition on ideals. □

Let $D(R)$ be the subgroup of $R^+$ generated by all subgroups $d(R)$, where $d \in \operatorname{Der} R$.

**Corollary 3.7.** *Let $R$ be an infinite ring that is not commutative and with all non-zero derivations (respectively inner derivations) to be cofinite. Then either $R$ is a prime ring with the ascending chain condition on ideals or $Z_0$ is non-zero finite, $Z_0 D(R) = (0)$, $D(R) \cap U(R) = \emptyset$ and $D(R)$ is a subgroup of finite index in $R^+$ (respectively $Z_0 C(R) = (0)$, $C(R) \cap U(R) = \emptyset$ and $|R : C(R)| < \infty$).*

*Proof.* We have $Z_0 \neq R$, $Z_0 C(R) = (0)$ and the quotient $R/Z_0$ is an infinite prime ring with the ascending chain condition on ideals by Corollary 2.2 and Theorem 2.3 from [2]. By Lemma 3.6, $Z_0$ is finite. Assume that $Z_0 \neq (0)$. If $d$ is a non-zero derivation of $R$, then $Z_0 d(R) \subseteq Z_0$ and so $Z_0 d(R) = (0)$.

If we assume that $A = \operatorname{ann}_l d(R)$ is infinite, then $A/Z_0$ is an infinite left ideal of $B$ with a non-zero annihilator, a contradiction with Lemma 2.1.1 from [6]. This gives that $A$ is finite and, consequently, $A = Z_0$.

Finally, if $u \in D(R) \cap U(R)$, then $Z_0 = uZ_0 = (0)$, a contradiction. □

**Corollary 3.8.** *Let $R$ be a ring that is not prime. If $R$ contains an infinite subfield, then it has a non-zero derivation that is not cofinite.*

*Proof of Proposition 1.1.* (⇐) It is clear.

(⇒) Assume that $R$ is an infinite ring which is not commutative and its every non-zero inner derivation is cofinite. Then $Z_0 \neq R$ and $R/Z_0$ is an infinite prime ring by Lemma 3.6. Then $J(R) \subseteq Z_0$. Then

$$R/Z_0 = \sum_{i=1}^{m}{}^{\oplus} M_{n_i}(D_i)$$

is a ring direct sum of finitely many full matrix rings $M_{n_i}(D_i)$ over skew fields $D_i$ ($i = 1, \dots, m$) and so by applying Example 2.1 and Remark 3.2, we have that $R/Z_0 = F_1 \oplus D_1$ is a ring direct sum of a finite commutative ring $F_1$ and an infinite skew field $D_1$ that is not commutative. As a consequence of Proposition 1 from [8, §3.6] and Lemma 3.5,

$$R = F \oplus D$$

is a ring direct sum of a finite ring $F$ and an infinite ring $D$. Then $F = Z_0$. □

# 4. Semiprime rings with cofinite inner derivations

**Lemma 4.1.** *Let $R$ be a prime ring. If $R$ contains a non-zero proper commutative ideal $I$, then $R$ is commutative.*

*Proof.* Assume that $C(R) \neq (0)$. Then for any elements $u \in R$ and $a, b \in I$ we have

$$abu = a(bu) = (bu)a = b(ua) = uab$$

and so $ab \in Z(R)$. This gives that

$$I^2 \subseteq Z(R)$$

and therefore
$$I^2C(R) = (0).$$
Since $I^2 \neq (0)$, we obtain a contradiction with Lemma 2.1.1 of [6]. Hence $R$ is commutative. □

**Lemma 4.2.** *Let $R$ be a reduced ring (i.e. $R$ has no non-zero nilpotent elements). If $R$ contains a non-zero proper commutative ideal $I$ such that the quotient ring $R/I$ is commutative, then $R$ is commutative.*

*Proof.* Obviously, $C(R) \leq I$ and $I^2 \neq (0)$. If $C(R) \neq (0)$, then, as in the proof of Lemma 4.1,
$$C(R)^3 \leq I^2C(R) = (0)$$
and thus $C(R) = (0)$. □

**Lemma 4.3.** *If a ring $R$ contains an infinite commutative ideal $I$, then $R$ is commutative or it has a non-zero derivation that is not cofinite.*

*Proof.* Suppose that $R$ is not commutative. If all non-zero derivations are cofinite in $R$, then $B = R/Z_0$ is a prime ring by Lemma 3.6 and $C(B) \neq (0)$. Therefore $I^2C(R) \subseteq Z_0$ and, consequently, $I \subseteq Z_0$, a contradiction. □

*Proof of Theorem 1.2.* ($\Leftarrow$) It is obviously.

($\Rightarrow$) Suppose that $R$ is an infinite ring which is not commutative and its every non-zero inner derivation is cofinite. Then $B = R/Z_0$ is a prime ring satisfying the ascending chain condition on ideals.

Assume that $B$ is not a domain. By Proposition 2.2.14 of [11],
$$\operatorname{ann}_l b = \operatorname{ann}_r b = \operatorname{ann} b$$
is a two-sided ideal for any $b \in B$, and by Lemma 2.3.2 from [11], each maximal right annihilator in $B$ has the form $\operatorname{ann}_r a$ for some $0 \neq a \in B$. Then $\operatorname{ann}_r a$ is a prime ideal. Since $|B : \operatorname{ann}_r a|$ is finite, left and right ideals $Ba$, $aB$ are finite and this gives a contradiction. Hence $B$ is a domain.

Now assume that $Z_0 \neq (0)$. In view of Corollary to Proposition 5 from [8, §3.5] we conclude that $Z_0$ is not nilpotent. As a consequence of Lemma 3 from [9] and Lemma 3.5,
$$R = Z_0 \oplus B_1$$
is a ring direct sum with a ring $B_1$ isomorphic to $B$. □

*Remark* 4.4. If $R$ is a ring with all non-zero inner derivations to be cofinite and $R/Z_0$ is an infinite simple ring, then $R = Z_0 \oplus B$ is a ring direct sum of a finite central ideal $Z_0$ and a simple non-commutative ring $B$.

*Problem* 4.5. Characterize domains and, in particular, skew fields with all non-zero derivations (respectively inner derivations) to be cofinite.

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