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On k-periodic binary recurrences

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Abstract

We apply a new approach, namely the fundamental theorem of homogeneous linear recursive sequences, to k-periodic binary recurrences which allows us to determine Binet's formula of the sequence if k is given. The method is illustrated in the cases $k = 2$ and $k = 3$ for arbitrary parameters. Thus we generalize and complete the results of Edson-Yayenie, and Yayenie linked to $k = 2$ hence they gave restrictions either on the coefficients or on the initial values. At the end of the paper we solve completely the constant sequence problem of 2-periodic sequences posed by Yayenie.

Keywords: linear recurrences, k-periodic binary recurrences

MSC: 11B39, 11D61

1. Introduction

Let a, b, c, d, and q_0, q_1 denote arbitrary complex numbers, and consider the following construction of the sequence (q_n). For $n \geq 2$, the terms q_n are defined by

$$q_n = \begin{cases} aq_{n-1} + bq_{n-2}, & \text{if } n \text{ is even;} \\ cq_{n-1} + dq_{n-2}, & \text{if } n \text{ is odd.} \end{cases} \tag{1.1}$$

The sequence (q_n) is called 2-periodic binary recurrence, and it was described first by Edson and Yayenie [2]. The authors discussed the specific case $q_0 = 0$, $q_1 = 1$ and $b = d = 1$, gave the generating function and Binet-type formula of

(q_n), further they proved several identities among the terms of (q_n). In the same paper the sequence (q_n) was investigated for arbitrary initial values q_0 and q_1, but $b = d = 1$ were still assumed.

Later Yayenie [6] took one more step by determining the Binet's formula for (q_n), where b and d were arbitrary numbers, but the initial values were fixed as $q_0 = 0$ and $q_1 = 1$.

The main tool in the papers [2, 6] is to work with the generating function. In this paper we suggest a new approach, namely to apply the fundamental theorem of homogeneous liner recurrences (see Theorem 1.1). This powerful method allows us to give the Binet's formula of (q_n) for any b and d and for arbitrary initial values. Moreover, we can also handle the case when the zeros of the quadratic polynomial

$$p_2(x) = x^2 - (ac + b + d)x + bd$$

coincide. Note, that $p_2(x)$ plays an important role in the aforesaid papers, but the sequence (q_n) has not been discussed yet when $p_2(x)$ has a zero with multiplicity 2. We will see that the application of the fundamental theorem of linear recurrences is very effective and it can even be used at k-periodic sequences generally. At the end of the paper we solve an open problem concerning constant subsequences (see 2.2.2 in [6]).

The k-periodic second order linear recurrence

$$q_n = \begin{cases} a_0 q_{n-1} + b_0 q_{n-2}, & \text{if } n \equiv 0 \pmod{k}; \\ a_1 q_{n-1} + b_1 q_{n-2}, & \text{if } n \equiv 1 \pmod{k}; \\ \vdots & \vdots \\ a_{k-1} q_{n-1} + b_{k-1} q_{n-2}, & \text{if } n \equiv k-1 \pmod{k}. \end{cases} \tag{1.2}$$

was introduced by Cooper in [1], where mainly the combinatorial interpretation of the coefficients A_k and B_k appearing in the recurrence relation $q_n = A_k q_{n-k} + B_k q_{n-2k}$ was discussed. Note that Lemma 4 of the work of Shallit [4] also describes an approach to compute the coefficients for q_n. Edson, Lewis and Yayenie [3] also studied the k-periodic extension, again with $q_0 = 0$, $q_1 = 1$ and with the restrictions $b_0 = b_1 = \cdots = b_{k-1} = 1$.

At the end of the first section we recall the fundamental theorem of linear recurrences. A homogeneous linear recurrence $(G_n)_{n=0}^{\infty}$ of order k $(k \geq 1, k \in \mathbb{N})$ is defined by the recursion

$$G_n = A_1 G_{n-1} + A_2 G_{n-2} + \cdots + A_k G_{n-k} \quad (n \geq k), \tag{1.3}$$

where the initial values $G_0, \ldots, G_{k-1}$ and the coefficients $A_1, \ldots, A_k$ are complex numbers, $A_k \neq 0$ and $|G_0| + \cdots + |G_{k-1}| > 0$. The characteristic polynomial of the sequence (G_n) is the polynomial

$$g(x) = x^k - A_1 x^{k-1} - \cdots - A_k.$$

Denote by $\alpha_1, \ldots, \alpha_t$ the distinct zeros of the characteristic polynomial $g(x)$, which can there be written in the form

$$g(x) = (x - \alpha_1)^{e_1} \cdots (x - \alpha_t)^{e_t}. \tag{1.4}$$

The following result (see e.g. [5]) plays a basic role in the theory of recurrence sequences, and here in our approach.

Theorem 1.1. *Let (G_n) be a sequence satisfying the relation (1.3) with $A_k \neq 0$, and $g(x)$ its characteristic polynomial with distinct roots $\alpha_1, \ldots, \alpha_t$. Let $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_t, A_1, \ldots, A_k, G_0, \ldots, G_{k-1})$ denote the extension of the field of rational numbers and let $g(x)$ be given in the form (1.4). Then there exist uniquely determined polynomials $g_i(x) \in K[x]$ of degree less than e_i $(i = 1, \ldots, t)$ such that*

$$G_n = g_1(n)\alpha_1^n + \cdots + g_t(n)\alpha_t^n \quad (n \geq 0).$$

2. *k*-periodic binary recurrences

Let $k \geq 2$ be an integer, further let q_0, q_1 and a_i, b_i, $i = 0, \ldots, k-1$ denote arbitrary complex numbers with $|q_0| + |q_1| \neq 0$ and $b_0 b_1 \cdots b_{k-1} \neq 0$.

Consider the sequence (q_n) defined by (1.2). By [1] it is known that the terms of (q_n) satisfy the recurrence relation

$$q_n = A_k q_{n-k} - (-1)^k b_0 b_1 \ldots b_{k-1} q_{n-2k} \tag{2.1}$$

of order $2k$, where the coefficient A_k is also described in [1]. Put $D = A_k^2 - 4(-1)^k b_0 b_2 \ldots b_{k-1}$, and let

$$p_k(x) = x^2 - A_k x + (-1)^k b_0 b_1 \cdots b_{k-1}$$

denote the polynomial determined by the characteristic polynomial $z^{2k} - A_k z^k + (-1)^k b_0 b_1 \cdots b_{k-1}$ of the recurrence (2.1) by the substitution $x = z^k$. The not necessarily distinct zeros of $p_k(x)$ are

$$\kappa = \frac{A_k + \sqrt{D}}{2} \quad \text{and} \quad \mu = \frac{A_k - \sqrt{D}}{2}.$$

At this point we would like to use Theorem 1.1, therefore we must distinguish two cases.

2.1. Case $D \neq 0$

If D is nonzero, then κ and μ are distinct. From Theorem 1, we deduce that there exist complex numbers κ_j and μ_j $(j = 1, \ldots, k)$ such that

$$q_n = \underbrace{\sum_{j=1}^{k} \kappa_j \varepsilon^{(j-1)n} \kappa^{n/k}}_{K_n} + \underbrace{\sum_{j=1}^{k} \mu_j \varepsilon^{(j-1)n} \kappa^{n/k}}_{M_n}, \tag{2.2}$$

where $\varepsilon = \exp(2\pi i/k)$ is a primitive root of unity of order k. If one claims to determine the coefficients κ_j and μ_j, it is sufficient to replace n by $0,\ 1,\ \dots,\ 2k-1$ in (2.2) and, after evaluating $q_2, \dots, q_{2k-1}$ by (1.2), to solve the system of $2k$ linear equations. Instead, we can shorten the calculations since, as we will see soon, only certain linear combinations of $\kappa_1, \dots, \kappa_k$ and $\mu_1, \dots, \mu_k$ are needed, respectively.

Now, by (2.2), for any non-negative integer t, we have $q_t = K_t + M_t$. Moreover,

$$q_{t+k} = \sum_{j=1}^{k} \kappa_j \varepsilon^{(j-1)(t+k)} \kappa^{(t+k)/k} + \sum_{j=1}^{k} \mu_j \varepsilon^{(j-1)(t+k)} \kappa^{(t+k)/k} = \kappa K_t + \mu M_t. \quad (2.3)$$

Since the determinant $\mu - \kappa$ of the system of two linear equations

$$\begin{cases} K_t + M_t = q_t \\ \kappa K_t + \mu M_t = q_{t+k} \end{cases} \quad (2.4)$$

is non-zero, therefore (2.4) possesses the unique solution

$$K_t = \frac{q_{t+k} - \mu q_t}{\kappa - \mu}, \quad M_t = -\frac{q_{t+k} - \kappa q_t}{\kappa - \mu}.$$

To give the explicit formula for the term of the sequence (q_n), we use the technique described in (2.3) for $n = sk + t$ and t with $0 \le t < k$. It is easy to see that $q_n = q_{sk+t} = \kappa^s K_t + \mu^s M_t$. Hence we proved the following theorem.

Theorem 2.1. *In the case $D \neq 0$, the n^{th} term of the sequence (q_n) satisfies*

$$q_n = \frac{q_{k+(n \bmod k)} - \mu q_{n \bmod k}}{\kappa - \mu} \kappa^{\lfloor n/k \rfloor} - \frac{q_{k+(n \bmod k)} - \kappa q_{n \bmod k}}{\kappa - \mu} \mu^{\lfloor n/k \rfloor}.$$

2.2. Case $D = 0$

If D is zero, then κ and μ coincide with $A_k/2$. By Theorem 1, there exist complex numbers u_j and v_j, $j = 1, \dots, k$ such that

$$q_n = \sum_{j=1}^{k} (u_j n + v_j) \varepsilon^{(j-1)n} \kappa^{n/k} = nU_n + V_n, \quad (2.5)$$

where

$$U_n = \sum_{j=1}^{k} u_j \varepsilon^{(j-1)n} \kappa^{n/k}, \quad V_n = \sum_{j=1}^{k} v_j \varepsilon^{(j-1)n} \kappa^{n/k}. \quad (2.6)$$

Then $q_t = tU_t + V_t$, together with (2.5) and (2.6) provides $q_{t+k} = \kappa((t+k)U_t + V_t)$. The unique solution of the system

$$\begin{cases} tU_t + V_t = q_t \\ \kappa(t+k)U_t + \kappa V_t = q_{t+k} \end{cases}$$

is

$$U_t = \frac{q_{t+k} - \kappa q_t}{\kappa k}, \quad V_t = -\frac{tq_{t+k} - (t+k)\kappa q_t}{\kappa k}.$$

Consequently, if $n = sk + t$ with $0 \le t < k$ then, clearly, $q_n = \kappa^s(U_t n + V_t)$, and by the notation

$$\omega = q_{t+k} - \kappa q_t, \quad \nu = tq_{t+k} - (t+k)\kappa q_t,$$

the following theorem holds.

Theorem 2.2. *If $D = 0$ then*

$$q_n = \frac{1}{k}\left(\omega n + \nu\right)\kappa^{\lfloor n/k \rfloor - 1},$$

where $\omega = q_{k+(n \bmod k)} - \kappa q_{n \bmod k}$ and $\nu = -(n \bmod k)\, q_{k+(n \bmod k)} + (k + (n \bmod k))\kappa q_{n \bmod k}$.

Note, that the application of Theorems 2.1 and 2.2 results a more precise formula for the term q_n if k is fixed. In the next two sections, we go into details in the cases $k = 2$ and $k = 3$. We derive Theorem 5 in [2] as a corollary of Theorem 2.1 with $k = 2$.

3. The 2-periodic binary recurrences

Suppose that $bd \neq 0$ and $|q_0| + |q_1| \neq 0$ hold in (1.1). It is known, that the terms of the recurrence (q_n) satisfy the recurrence relation

$$q_n = (ac + b + d)q_{n-2} - bdq_{n-4}, \quad n \ge 4$$

of order four, where the initial values are, obviously, q_0, q_1, $q_2 = aq_1 + bq_0$ and $q_3 = (ac + d)q_1 + bcq_0$. Put $D = (ac + b + d)^2 - 4bd$. Thus the zeros of the polynomial $p_2(x) = x^2 - (ac + b + d)x + bd$ are

$$\kappa = \frac{ac + b + d + \sqrt{D}}{2} \qquad \text{and} \qquad \mu = \frac{ac + b + d - \sqrt{D}}{2}.$$

3.1. Case $D \neq 0$

First assume that n is even, i.d., $t = (n \bmod 2) = 0$ holds in Theorem 2.1. Thus we obtain

$$q_n = \frac{q_2 - \mu q_0}{\kappa - \mu}\,\kappa^{\lfloor n/2 \rfloor} - \frac{q_2 - \kappa q_0}{\kappa - \mu}\,\mu^{\lfloor n/2 \rfloor}.$$

Clearly, $q_2 - \mu q_0 = aq_1 + (b - \mu)q_0$, further $q_2 - \kappa q_0 = aq_1 + (b - \kappa)q_0$.

Suppose now, that n is odd, i.d., $t = 1$. Now Theorem 2.1 results

$$q_n = \frac{q_3 - \mu q_1}{\kappa - \mu}\,\kappa^{\lfloor n/2 \rfloor} - \frac{q_3 - \kappa q_1}{\kappa - \mu}\,\mu^{\lfloor n/2 \rfloor}.$$

Obviously, $q_3 - \mu q_1 = (ac + d - \mu)q_1 + (bc)q_0 = (\kappa - b)q_1 + (bc)q_0$, similarly $q_3 - \kappa q_1 = (\mu - b)q_1 + (bc)q_0$.

To join the even and odd cases together, we introduce

$$e_\kappa = a^{1-\xi(n)}(\kappa - b)^{\xi(n)}q_1 + (b - \mu)^{1-\xi(n)}(bc)^{\xi(n)}q_0$$

and

$$e_\mu = a^{1-\xi(n)}(\mu - b)^{\xi(n)}q_1 + (b - \kappa)^{1-\xi(n)}(bc)^{\xi(n)}q_0,$$

where $\xi(n) = (n \bmod 2)$ is the parity function. Thus

$$q_n = \frac{e_\kappa\, \kappa^{\lfloor n/2\rfloor} - e_\mu\, \mu^{\lfloor n/2\rfloor}}{\kappa - \mu}. \tag{3.1}$$

Observe that (3.1) returns with the explicit formula given in Theorem 5 of [2] if $b = d = 1$ and $q_0 = 0$, $q_1 = 1$. Indeed, now $e_\kappa = a^{1-\xi(n)}(\kappa - 1)^{\xi(n)}$, $e_\mu = a^{1-\xi(n)}(\mu - 1)^{\xi(n)}$, which together with $ac\kappa = (\kappa - 1)^2$ and $ac\mu = (\mu - 1)^2$ provide

$$q_n = \frac{a^{1-\xi(n)}}{(ac)^{\lfloor n/2\rfloor}} \frac{(\kappa - 1)^n - (\mu - 1)^n}{(\kappa - 1) - (\mu - 1)}. \tag{3.2}$$

Clearly, by $\alpha = \kappa - 1$ and $\beta = \mu - 1$, (3.2) coincides with the statement of Theorem 5 in [2].

3.2. Case $D = 0$

Note, that neither [2] nor [6] worked this subcase out. Observe, that $D = 0$ is possible, for example, let $b = rs^2$, $d = rt^2$, further $a = r$ and $c = 4st - s^2 - t^2$. Clearly, $\kappa = \mu = (ac + b + d)/2$.

Assume first that n is even, or equivalently $t = 0$. Then $\omega = q_2 - \kappa q_0 = aq_1 + (b - \kappa)q_0$, while $\nu = 2\kappa q_0$.

Supposing $t = 1$, it gives $\omega = q_3 - \kappa q_1 = (ac + d - \kappa)q_1 + (bc)q_0 = (\kappa - b)q_1 + (bc)q_0$ and $\nu = -(q_3 - 3\kappa q_1) = (\kappa + b)q_1 - (bc)q_0$.

Henceforward,

$$q_n = \frac{1}{2}(\omega n + \nu)\kappa^{\lfloor n/2\rfloor - 1}$$

describes the general case, where $\omega = a^{1-\xi(n)}(\kappa - b)^{\xi(n)}q_1 + (b - \kappa)^{1-\xi(n)}(bc)^{\xi(n)}q_0$ and $\nu = \xi(n)(\kappa + b)q_1 + (-1)^{\xi(n)}(2\kappa)^{1-\xi(n)}(bc)^{\xi(n)}q_0$.

4. The 3-periodic binary recurrences

This section follows the structure of the previous one. Let a, b, c, d, e, f and q_0, q_1 are arbitrary complex numbers with $bdf \neq 0$ and $|q_0| + |q_1| \neq 0$. For $n \geq 2$, the terms of the sequence (q_n) are defined by

$$q_n = \begin{cases} aq_{n-1} + bq_{n-2}, & \text{if } n \equiv 0 \pmod 3; \\ cq_{n-1} + dq_{n-2}, & \text{if } n \equiv 1 \pmod 3; \\ eq_{n-1} + fq_{n-2}, & \text{if } n \equiv 2 \pmod 3). \end{cases}$$

It is known, that recurrence (q_n) satisfies the recurrence relation

$$q_n = (ace + bc + de + af)\, q_{n-3} + bdf q_{n-6}$$

of order six, where the initial values are

$$\begin{aligned} q_0, q_1, q_2 &= eq_1 + fq_0, \\ q_3 &= (ae + b)\, q_1 + afq_0, \\ q_4 &= (ace + bc + de)\, q_1 + (acf + df)\, q_0, \\ q_5 &= \left(ace^2 + bce + de^2 + aef + bf\right) q_1 + \left(acef + def + af^2\right) q_0. \end{aligned}$$

Put $D = (ace + bc + de + af)^2 + 4bdf$. Thus, the roots of the polynomial

$$p_3(x) = x^2 - (ace + bc + de + af)\, x - bdf$$

are

$$\kappa = \frac{(ace + bc + de + af) + \sqrt{D}}{2} \quad \text{and} \quad \mu = \frac{(ace + bc + de + af) - \sqrt{D}}{2}.$$

In the sequel, we need the sequence (a_n) defined by $a_n = 1$ if 3 divides n, and $a_n = 0$ otherwise.

4.1. Case $D \neq 0$

The consequence of Theorem 2.1 is the nice formula

$$q_n = \frac{e_\kappa \kappa^{\lfloor n/3 \rfloor} - e_\mu \mu^{\lfloor n/3 \rfloor}}{\kappa - \mu},$$

where

$$\begin{aligned} e_\kappa = {} & (ae + b)^{a_n} (\kappa - af)^{a_{n+2}} (e\kappa + fb)^{a_{n+1}} q_1 \\ & + (af - \mu)^{a_n} (f(ac + d))^{a_{n+2}} (f(\kappa - bc))^{a_{n+1}} q_0, \end{aligned}$$

and

$$\begin{aligned} e_\mu = {} & (ae + b)^{a_n} (\mu - af)^{a_{n+2}} (e\mu + fb)^{a_{n+1}} q_1 \\ & + (af - \kappa)^{a_n} (f(ac + d))^{a_{n+2}} (f(\mu - bc))^{a_{n+1}} q_0. \end{aligned}$$

Indeed, for $t = 0, 1, 2$

$$q_{t+3} - \mu q_t = \begin{cases} (ae + b)q_1 + (af - \mu)q_0, & \text{if } t = 0; \\ (\kappa - af)q_1 + (ac + d)fq_0, & \text{if } t = 1; \\ (e\kappa + fb)q_1 + (\kappa - bc)fq_0, & \text{if } t = 2, \end{cases} \tag{4.1}$$

and $q_{t+3} - \kappa q_t$ can similarly be obtained from (4.1) by switching κ and μ.

4.2. Case $D = 0$

When $t = 0$ we obtain $\omega = (ae + b)q_1 + (af - \kappa)q_0$, $\nu = 3\kappa q_0$. Secondly, $t = 1$ yields $\omega = (\kappa - af)q_1 + (ac + d)fq_0$ and $\nu = (2\kappa + af)q_1 - (ac + d)fq_0$. Finally, $\omega = (\kappa e + bf)q_1 + (\kappa - bc)fq_0$ and $\nu = (\kappa e - 2bf)q_1 + (\kappa + 2bc)fq_0$ when $t = 2$.

So, we obtain

$$q_n = \frac{1}{3}(\omega n + \nu)\kappa^{\lfloor n/3\rfloor - 1},$$

where

$$\begin{aligned} w = {} & (ae + b)^{a_n}(\kappa - af)^{a_{n+2}}(\kappa e - bf)^{a_{n+1}} q_1 \\ & + (af - \kappa)^{a_n}((ac + d)f)^{a_{n+1}}((\kappa - bc)f)^{a_{n+2}} q_0 \end{aligned}$$

and

$$\begin{aligned} \nu = {} & (1 - a_n)(2\kappa + af)^{a_{n+2}}(\kappa e - 2bf)^{a_{n+1}} q_1 \\ & + (3\kappa)^{a_n}(-(ac + d)f)^{a_{n+2}}((\kappa + 2bc)f)^{a_{n+1}} q_0. \end{aligned}$$

5. Constant subsequences in 2-periodic binary recurrences

In the last section we solve the problem posed in 2.2.2 of [6]. There, after pointing on few examples, the author claim a general sufficiency condition for the sequence (1.1) to be constant from a term q_ν (actually, $\nu = 1$ was asked in [6]). The forthcoming theorem describes the complete answer.

Theorem 5.1. *The sequence (q_n) takes the constant value $q \in \mathbb{C}$ from the ν^{th} terms ($\nu \geq 0$) if and only if one of the following cases holds.*

1. *$q_0 = q_1 = 0$, further a, b, c, d are arbitrary, ($\nu = 0$, $q = 0$),*
2. *$q_0 = q_1 = q \neq 0$, $a + b = 1$, $c + d = 1$, ($\nu = 0$, $q \neq 0$),*
3. *$q_0 \neq 0$ is arbitrary, $q_1 = 0$, $b = 0$, moreover a, c, d are arbitrary, ($\nu = 1$, $q = 0$),*
4. *$q_0 \neq q$ is arbitrary and $q_1 = q$ with $q \neq 0$, and $a = 1$, $b = 0$, $c + d = 1$, ($\nu = 1$, $q \neq 0$),*
5. *q_0 and $q_1 \neq 0$ are arbitrary, b, c are arbitrary, $a = -bq_0/q_1$, $d = 0$, ($\nu = 2$, $q = 0$),*
6. *q_0 and $q_1 \neq q$ are arbitrary with $q_1 \neq q_0$ and $q = aq_1 + bq_0$, where $a + b = 1$, $a \neq 1$, $c = 1$, $d = 0$, ($\nu = 2$, $q \neq 0$),*
7. *q_0 and $q_1 \neq 0$ are arbitrary, $a \neq 0$ and c are arbitrary, $b = 0$, $d = -ac$, ($\nu = 3$, $q = 0$),*

8. *q_0 and $q_1 \neq cq_0$ are arbitrary, where $a \neq 0$ and $c \neq 0$ are arbitrary, $b = -ac$, $d = 0$, ($\nu = 4$, $q = 0$).*

Proof. Obviously, each of the conditions appearing in Theorem 5.1 is sufficient. We are going to show that one of them is necessary. Suppose that the sequence (q_n) takes the constant value $q \in \mathbb{C}$ from the ν^{th} terms.

I. First assume that $\nu \geq 5$ is an integer. We introduce the notation $(u, v) = (a, b)$ and $(\breve{u}, \breve{v}) = (c, d)$ if ν is odd, while $(u, v) = (c, d)$ and $(\breve{u}, \breve{v}) = (a, b)$ if ν is even. Then the equations

$$\begin{aligned} q_{\nu-3} &= uq_{\nu-4} + vq_{\nu-5} & q_{\nu-2} &= \breve{u}q_{\nu-3} + \breve{v}q_{\nu-4} \\ q_{\nu-1} &= uq_{\nu-2} + vq_{\nu-3} & q &= \breve{u}q_{\nu-1} + \breve{v}q_{\nu-2} \\ q &= uq + vq_{\nu-1} & q &= \breve{u}q + \breve{v}q \\ q &= uq + vq & & \end{aligned}$$

hold, where $q \neq q_{\nu-1}$. The last two equations in the left column imply $v(q_{\nu-1} - q) = 0$. Therefore $v = 0$ follows, and it simplifies the whole left column.

If $q \neq 0$ then $u = 1$ and $\breve{u} + \breve{v} = 1$ fulfill. Hence $q_{\nu-1} = q_{\nu-2}$, consequently $q = \breve{u}q_{\nu-1} + \breve{v}q_{\nu-2}$ leads to $q = q_{\nu-1}$ and we arrived at a contradiction.

Consider now the case $q = 0$. Thus $q_{\nu-1} \neq 0$, and then we have the system

$$\begin{aligned} q_{\nu-3} &= uq_{\nu-4} & q_{\nu-2} &= \breve{u}q_{\nu-3} + \breve{v}q_{\nu-4} \\ q_{\nu-1} &= uq_{\nu-2} & 0 &= \breve{u}q_{\nu-1} + \breve{v}q_{\nu-2} \end{aligned}$$

to examine. Clearly, $uq_{\nu-2} \neq 0$. The equalities in the second row provide $0 = u\breve{u}q_{\nu-2} + \breve{v}q_{\nu-2}$, subsequently $(u\breve{u} + \breve{v})q_{\nu-2} = 0$, and then $u\breve{u} + \breve{v} = 0$. Insert it to $q_{\nu-2} = u\breve{u}q_{\nu-4} + \breve{v}q_{\nu-4}$ (coming from the first row), and we obtain $q_{\nu-2} = 0$, which is impossible.

Hence, we have shown that if the constant subsequence of (q_n) starts at the term q_ν, then necessarily $\nu \leq 4$.

II. In the second place we assume that $\nu \leq 4$ and distinguish five cases. Note, that for the subscript $k \geq \nu$ the equalities $q_{k+2} = aq_{k+1} + bq_k$, $q_{k+2} = cq_{k+1} + dq_k$ simplify to

$$q = aq + bq, \quad q = cq + dq, \tag{5.1}$$

respectively.

$\nu = 0$. If $q = 0$ then $q_0 = q_1 = 0$ and, trivially, all the coefficients a, b, c and d are arbitrary. If $q \neq 0$ then $q_0 = q_1 = q$ and (5.1) must hold. Consequently, $a + b = 1$ and $c + d = 1$ follow.

$\nu = 1$. Here $q_0 \neq q$. Further, $q = aq + bq_0$, together with the first equality of (5.1) provides $b(q_0 - q) = 0$. Thus $b = 0$.

Clearly, $q = 0$ satisfies both (5.1) and $q = aq + bq_0$ without further restrictions on a, b and c.

If q is non-zero, then (5.1) and $b = 0$ imply $a = 1$ and $c + d = 1$.

$\nu = 2$. Besides (5.1), we also have

$$q = aq_1 + bq_0, \quad q = cq + dq_1 \tag{5.2}$$

with $q_1 \neq q$. The last equality and the second property of (5.1) give $d = 0$ via $d(q_1 - q) = 0$.

Assume first $q = 0$. Then, except $0 = aq_1 + bq_0$, all the equalities in (5.1) and (5.2) are fulfilled. Since $q_1 \neq 0$, we can write $a = -bq_0/q_1$. Obviously b and c are arbitrary.

If $q \neq 0$ then $c = 1$ and $a + b = 1$ follow. The value of the constant q is $aq_1 + bq_0$. Observe, that $a \neq 1$ otherwise $b = 0$, and then $q_1 = q$ would come.

$\nu = 3$. Now $q_2 \neq q$. The conditions $q_2 = aq_1 + bq_0$, $q = cq_2 + dq_1$, $q = aq + bq_2$ and (5.1) are valid. Thus $b(q_2 - q)$ vanish, i.e. $b = 0$. Hence we obtain the system

$$\begin{array}{rclcrcl} q_2 & = & aq_1 & \qquad & q & = & cq_2 + dq_1 \\ q & = & aq & & q & = & cq + dq \end{array}$$

Suppose first that $q = 0$. Then $q_2 = aq_1$ and $0 = cq_2 + dq_1$ provide $0 = (ac + d)q_1$. Since $q_1 = 0$ would give $q_2 = 0$ therefore $ac + d$ must be zero, so $d = -ac$. Also $a \neq 0$ holds, otherwise $q_2 = 0$ leads to a contradiction. Clearly, c is arbitrary.

Assume now that q is non-zero. Thus, from the last system above, we conclude $a = 1$, $c + d = 1$ and $q_2 = q_1$. Hence, the remaining equation $q = cq_2 + dq_1$ becomes $q = cq_2 + (1 - c)q_2$, and we arrived at a contradiction by $q \neq q_2$. Subsequently, $q \neq 0$ does not provide a constant sequence from the third term.

$\nu = 4$. The technique we apply resembles us to the previous cases. Here $q_3 \neq q$. We have $q_2 = aq_1 + bq_0$, $q_3 = cq_2 + dq_1$, $q = aq_3 + bq_2$, $q = cq + dq_3$ and (5.1). Similarly, $d(q_3 - q)$ implies $d = 0$. Thus

$$\begin{array}{rclcrcl} q_2 & = & aq_1 + bq_0 & \qquad & q_3 & = & cq_2 \\ q & = & aq_3 + bq_2 & & q & = & cq \\ q & = & aq + bq & & & & \end{array}$$

If $q = 0$ then $q_3 = cq_2 \neq 0$, further $0 = aq_3 + bq_2$ and $q_3 = cq_2$ yield $ac + b = 0$. Clearly, $c \neq 0$. Moreover $a \neq 0$ holds, otherwise $b = 0$ and $q_2 = 0$ and $q_3 = 0$ follow. Finally, $q_1 \neq cq_0$ since $q_2 \neq 0$.

The assertion $q \neq 0$, similarly to the case $\nu = 3$, leads to a contradiction. $\square$

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