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The rank of certain subfamilies of the elliptic curve $Y^2 = X^3 - X + T^2$*

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Abstract

Let E be the elliptic curve over $\mathbb{Q}(T)$ given by the equation

$$E : Y^2 = X^3 - X + T^2.$$

It is known that the torsion subgroup is trivial,

$$\mathrm{rank}_{\mathbb{C}(T)}(E) = 2 \quad \text{and} \quad \mathrm{rank}_{\mathbb{Q}(T)}(E) = 2.$$

We find a parametrization of rank ≥ 3 over the function field $\mathbb{Q}(a, i, s, n, k, l)$ where $s^2 = i^3 + a^2$. From this we get families of rank ≥ 3 and ≥ 4 over fields of rational functions in four variables and a family of rank ≥ 5 parametrized by an elliptic curve of positive rank. We also found a particular elliptic curve with rank ≥ 11.

Keywords: parametrization, elliptic surface, elliptic curve, function field, rank, family of elliptic curves, torsion

MSC: 11G05

1. Introduction

Let E be the elliptic curve over $\mathbb{Q}(T)$ given by the equation

$$Y^2 = X^3 - X + T^2.$$

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In [2, Theorem 1], Brown and Myers proved that if $t \geq 2$ is an integer, the elliptic curve $E_t : Y^2 = X^3 - X + t^2$ has rank at least 2 over $\mathbb{Q}$, with linearly independent points $(0, t)$ and $(-1, t)$. They also prove that there are infinitely many integer values of t for which the elliptic curve E_t over $\mathbb{Q}$ has rank at least 3. In [5], Eikenberg showed that the torsion subgroup is trivial, the rank of the group $E(\mathbb{Q}(T))$ equals 2 as does the rank of $E(\mathbb{C}(T))$, both groups have as generators the points $(0, T)$ and $(1, T)$. These results follow also from the more general result by Shioda (see [14, Theorem A_2]). Eikenberg gives quadratic polynomials $T(n) \in \mathbb{Q}[n]$ for which $E_{T(n)}(\mathbb{Q}(n))$ is of rank at least 3, [5, Theorem 4.2.1.]. He also shows that there are infinitely values of t for which E_t has rank at least 5.

In this paper we find a subfamily of E for which the rank over the function field $\mathbb{Q}(a, i, s, n, k, l)$ where $s^2 = i^3 + a^2$ is ≥ 3 and three independent points are listed. From this we get families of rank ≥ 3 and ≥ 4 over fields of rational functions in four variables and a family of rank ≥ 5 over an elliptic curve of positive rank. We also found a particular elliptic curve with rank ≥ 11.

In [16], an elliptic curve $Y^2 = X^3 - T^2 X + 1$ was analyzed in a similar way, and the results obtained contain some resemblances with the results of this paper.

2. Subfamilies of higher rank

We know that the elliptic curve E observed in this section and defined above, has rank 2 over $\mathbb{Q}(T)$ and $\mathbb{C}(T)$, with generators $(0, T)$ and $(-1, T)$. First we give two subfamilies which have generic rank ≥ 3 and we give the third independent point. By observing $T(n)$ which are polynomials in the variable n of degree 3 over $\mathbb{Q}$ with an additional point with first coordinate $X(n)$ which is a polynomial in the variable n of degree 2 over $\mathbb{Q}$ on the elliptic curve $Y^2 = X^3 - X + T(n)^2$ over $\mathbb{Q}(n)$ (see [13, Theorem 10.10]), we obtain the following.

Theorem 2.1.
For $T^{(1)}_{\pm}(a, i, s, n, k, l) =$

$$an^3 + (3ak + sl)n^2 + \left(3ak^2 + 2slk - al^2 \pm \frac{s}{i}\right) n - sl^3 - akl^2 + slk^2 \pm \frac{a}{i} l + ak^3 \pm \frac{s}{i} k,$$

the elliptic curve $Y^2 = X^3 - X + T^{(1)}_{\pm}(a, i, s, n, k, l)^2$ *has rank* ≥ 3 *over the function field* $\mathbb{Q}(a, i, s, n, k, l)$ *where* $s^2 = i^3 + a^2$*, with an additional independent point* $C^{(1)}_{\pm}(a, i, s, n, k, l)$ *with first coordinate*

$$X_{C^{(1)}_{\pm}}(a, i, s, n, k, l) = i(n + k)^2 - il^2.$$

Proof. For
$Y_{C^{(1)}_{\pm}}(a, i, s, n, k, l) = sn^3 + (al + 3ks)n^2 + \frac{2aikl \pm a - isl^2 + 3isk^2}{i} n + \frac{-iskl^2 \pm ak - ail^3 + isk^3 + aik^2 l \pm sl}{i}$,
we have

$$X_{C^{(1)}_{\pm}}(a, i, s, n, k, l)^3 - X_{C^{(1)}_{\pm}}(a, i, s, n, k, l) + T^{(1)}_{\pm}(a, i, s, n, k, l)^2 - Y_{C^{(1)}_{\pm}}(a, i, s, n, k, l)^2$$

$$= (-s^2 + i^3 + a^2) q_\pm(a, i, s, n, k, l) = 0,$$

where $q_\pm \in \mathbb{Q}(a, i, s, n, k, l)$. Here we work over the function field $\mathbb{Q}(a, i, s, n, k, l)$ where $s^2 = i^3 + a^2$.

For the positive case the specialization $(a, i, s, n, k, l) \mapsto (6, -3, 3, 1, 1, 1)$ gives $T_+^{(1)}(6, -3, 3, 1, 1, 1) = 41$, and on the curve $E_{T_+^{(1)}(6,-3,3,1,1,1)} : Y^2 = X^3 - X + 41^2$ there are three corresponding points $(0, 41)$, $(-1, 41)$, $(-9, 31)$ which are independent points of $E_{41}(\mathbb{Q})$. This shows that the points from the claim of the theorem are independent elements of the group

$$E_{T_+^{(1)}(a,i,s,n,k,l)}(\{\mathbb{Q}(a, i, s, n, k, l) : s^2 = i^3 + a^2\}).$$

The proof for $T_-^{(1)}$ is analogous, we used the same specialization. □

Now we will construct two subfamilies of generic rank ≥ 4 by intersecting the families we have obtained. We try to find the solution to the equation

$$T_\pm^{(1)}(a, i, s, n, k, l) = T_\pm^{(1)}\left(a, 2a\frac{a-s}{i^2}, a\frac{4a^2 - 4as + i^3}{i^3}, n, k_2, m\right),$$

where actually $(i_2, s_2) := \left(2a\frac{a-s}{i^2}, a\frac{4a^2-4as+i^3}{i^3}\right) = (i, s) + (0, a)$ on the elliptic curve $Y^2 = X^3 + a^2$. This gives a polynomial $P(n)$ in the variable n of degree two. Now we choose

$$k_2 := \frac{1}{3}\frac{-4a^3m + 4a^2ms - ami^3 + 3aki^3 + sli^3}{i^3a}$$

so that the coefficient of the polynomial $P(n)$ of the term n^2 is zero. Now that we have $P(n)$ a linear polynomial in n we can choose $n_\pm(a, i, s, k, l, m) := (256a^{10}m^3 - 1024a^9m^3s + (-288m^2ki^3 + 192m^3i^3 + 1536m^3s^2)a^8 + (864m^2ski^3 - 96m^2sli^3 - 1024m^3s^3 - 576m^3si^3)a^7 + (256m^3s^4 - 144m^2i^6k \mp 144i^5m - 96m^3i^6 + 288m^2s^2li^3 + 576m^3s^2i^3 - 864m^2s^2ki^3)a^6 + (288m^2i^6ks + 192m^3i^6s \pm 288i^5ms - 192m^3s^3i^3 + 288m^2s^3ki^3 - 48m^2i^6sl - 288m^2s^3li^3)a^5 + (96m^2s^4li^3 \pm 108i^8k \mp 144i^5s^2m - 32m^3i^9 \mp 54li^8 \mp 72i^8m + 54kl^2i^9 + 96m^2i^6s^2l - 144m^2s^2i^6k - 96m^3s^2i^6 - 72m^2i^9k)a^4 + (72m^2i^9ks - 54kl^2i^9s + 54sl^3i^9 \pm 72i^8sm \pm 90li^8s + 32m^3i^9s \mp 162ski^8 - 24m^2i^9sl - 48m^2s^3i^6l)a^3 + (\pm 54s^2ki^8 \pm 18i^{11}m \mp 36i^8s^2l - 54s^2l^3i^9 \pm 27i^{11}k + 18ki^9s^2l^2 + 24m^2i^9s^2l)a^2 + (2s^3l^3i^9 - 18ki^9s^3l^2 \pm 9i^{11}sl)a - 2s^4l^3i^9)/(9ai^3(32a^7m^2 - 96a^6m^2s + (16m^2i^3 + 96m^2s^2)a^5 + (-32m^2i^3s - 32m^2s^3)a^4 + (\mp 12i^5 + 16m^2s^2i^3 - 6l^2i^6 + 8m^2i^6)a^3 + (6l^2i^6s - 8m^2i^6s \pm 18i^5s)a^2 + (\mp 3i^8 \mp 6s^2i^5 - 2s^2l^2i^6)a + 2s^3l^2i^6))$ such that

$$T_\pm^{(1)}(a, i, s, n_\pm(a, i, s, k, l, m), k, l) =$$

$$= T_\pm^{(1)}\left(a, 2a\frac{a-s}{i^2}, a\frac{4a^2 - 4as + i^3}{i^3}, n_\pm(a, i, s, k, l, m), \frac{1}{3}\frac{-4a^3m + 4a^2ms - ami^3 + 3aki^3 + sli^3}{i^3a}, m\right).$$

Proposition 2.2. *Let*

$$S^{(1)}_{\pm}(a,i,s,k,l,m) := T^{(1)}_{\pm}(a,i,s,n_{\pm}(a,i,s,k,l,m),k,l),$$

where $n_{\pm}$ is given above and $T^{(1)}_{\pm}$ is as in Theorem 2.1. The elliptic curve

$$Y^2 = X^3 - X + S^{(1)}_{\pm}(a,i,s,k,l,m)^2$$

over the function field $\mathbb{Q}(a,i,s,k,l,m)$ where $s^2 = i^3 + a^2$ has rank ≥ 4 with four independent points, the two generators $(0, S^{(1)}_{\pm}(a,i,s,k,l,m))$, $(-1, S^{(1)}_{\pm}(a,i,s,k,l,m))$ mentioned in the introduction, and two additional points

$$A^{(1)}{}_{\pm}(a,i,s,k,l,m) := C^{(1)}_{\pm}(a,i,s,n_{\pm}(a,i,s,k,l,m),k,l)$$

and

$$B^{(1)}{}_{\pm}(a,i,s,k,l,m) :=$$

$$C^{(1)}_{\pm}\left(a, 2a\frac{a-s}{i^2}, a\frac{4a^2-4as+i^3}{i^3}, n_{\pm}(a,i,s,k,l,m), \frac{1}{3}\frac{-4a^3m+4a^2ms-ami^3+3aki^3+sli^3}{i^3a}, m\right)$$

(notation for $C^{(1)}_{\pm}$ from Theorem 2.1).

Proof. With the specialization $(a,i,s,k,l,m) \mapsto (6,-3,3,1,1,1)$ we prove that the above listed four points on the elliptic curve (over $\mathbb{Q}(a,i,s,k,l,m)$ where $s^2 = i^3 + a^2$) are independent, since the specialization gives the elliptic curve

$$E_{S^{(1)}_{+}(6,-3,3,1,1,1)} : Y^2 = X^3 - X + \left(-\frac{5647}{13122}\right)^2$$

with the corresponding four independent points with first coordinates 0, -1, $-\frac{805}{972}$, $\frac{7084}{729}$.

The proof for $S^{(1)}_{-}$ is analogous, by picking an adequate specialization. □

Remark 2.3. The variety (from Theorem 2.1)

$$s^2 = i^3 + a^2$$

can be observed as an elliptic curve $Y^2 = X^3 + T^2$ over the field $\mathbb{Q}(T)$. In [12, Corollary 8] it is shown that the torsion subgroup of $s^2 = i^3 + a^2$ over $\mathbb{Q}(a)$ is equal $\{O, (0,a), (0,-a)\}$. This elliptic curve has rank 0 over $\mathbb{Q}(a)$. For more details see [6, p. 112]. Points on the variety $s^2 = i^3 + a^2$ from Theorem 2.1 can easily be obtained, for example $(a,i,s) = (6,-3,3)$ is a point on the variety. For $a = 0$ we have $i = u^2$ and $s = u^3$, in this case $T^{(1)}_{\pm}(0,u^2,u^3,n,k,l)$ in Theorem 2.1 is a quadratic polynomial in n. We also have parametrizations of this variety [3, Section 14.2]:

$$\begin{cases} a(t) = 2t^3 - 1, \\ i(t) = 2t, \\ s(t) = 2t^3 + 1, \end{cases}$$

For this parametrization Theorem 2.1 and Proposition 2.2 transform into:

Corollary 2.4.

(i) *Let* $T_{\pm}^{(2)}(t,n,k,l) := T_{\pm}^{(1)}(2t^3-1,2t,2t^3+1,n,k,l) = ((4t^4-2t)n^3+((4l+12k)t^4+(2l-6k)t)n^2+((-4l^2+8lk+12k^2)t^4\pm 2t^3+(4lk-6k^2+2l^2)t\pm 1)n+(-4kl^2-4l^3+4k^3+4lk^2)t^4\pm(2k+2l)t^3+(2lk^2-2l^3+2kl^2-2k^3)t\pm(k-l))/(2t)$. *The elliptic curve* $Y^2=X^3-X+T_{\pm}^{(2)}(t,n,k,l)^2$ *over* $\mathbb{Q}(t,n,k,l)$ *has rank* ≥ 3 *and three independent points have first coordinates* $(0,T_{\pm}^{(2)}(t,n,k,l))$, $(-1,T_{\pm}^{(2)}(t,n,k,l))$, $C_{\pm}^{(1)}(2t^3-1,2t,2t^3+1,n,k,l)$. *Notation for* $T_{\pm}^{(1)}$ *and* $C_{\pm}^{(1)}$ *as in Theorem 2.1.*

(ii) *Let*

$$S_{\pm}^{(2)}(t,k,l,m) := S_{\pm}^{(1)}(2t^3-1,2t,2t^3+1,k,l,m).$$

Then the elliptic curve $Y^2=X^3-X+S_{\pm}^{(2)}(t,k,l,m)^2$ *over the function field* $\mathbb{Q}(t,n,k,l)$ *is of rank* ≥ 4, *with four independent points* $(0,S_{\pm}^{(2)}(t,k,l,m))$, $(-1,S_{\pm}^{(2)}(t,k,l,m))$, $A_{\pm}^{(1)}(2t^3-1,2t,2t^3+1,k,l,m)$, $B_{\pm}^{(1)}(2t^3-1,2t,2t^3+1,k,l,m)$. *Here the notation is from Proposition 2.2.*

Proof.

(i) For the specialization $(t,n,k,l)\mapsto(1,2,1,1)$ on the curve

$$E_{T_{+}^{(2)}(1,2,1,1)}: Y^2=X^3-X+53^2$$

the corresponding points with first coordinates 0, -1, 16 are independent, so the claim of the corollary is true. The proof for $T_{-}^{(2)}$ is analogous, by picking an adequate specialization.

(ii) The specialization $(t,k,l,m)\mapsto(2,1,1,1)$ gives the elliptic curve

$$E_{S_{+}^{(2)}(2,1,1,1)}: Y^2=X^3-X+\left(-\frac{49050562229}{10497600}\right)^2$$

over $\mathbb{Q}$ for which the four listed points with first coordinates 0, -1, $\frac{14863849}{72900}$, $-\frac{48719569}{311040}$ are independent. This proves that for the elliptic curve $Y^2=X^3-X+S_{+}^{(2)}(t,k,l,m)^2$ over the field $\mathbb{Q}(t,k,l,m)$ the corresponding four points the two generators mentioned in the introduction and the points $A_{\pm}^{(1)}(2t^3-1,2t,2t^3+1,k,l,m)$ and $B_{\pm}^{(1)}(2t^3-1,2t,2t^3+1,k,l,m)$ (from Proposition 2.2) are independent. The proof for $S_{-}^{(2)}$ is analogous, by picking an adequate specialization. □

3. Subfamily of generic rank ≥ 5

Remark 3.1.

- In [5, Theorem 3.5.1.] a rational function is given
$$M(m) = \frac{1017m^4 - 8487m^3 + 19298m^2 - 14145m + 2825}{(3m^2-5)^2},$$
with the property that the rank of $E_{M(m)}$ over $\mathbb{Q}(m)$ is ≥ 4.
- We have two additional points coming from [5, Theorem 3.5.1.], R_3 with first coordinate
$$-\frac{69m^2 - 414m + 295}{3m^2-5}$$
and the point R_4 with first coordinate
$$\frac{357m^2 - 410m + 95}{3m^2-5}.$$
- This rational function $M(m)$ is equal $T_+^{(1)}\left(0, 9, 27, n, -\frac{1}{3}\frac{9nm^2-20m^2+69m-15n-35}{3m^2-5}, 1\right)$ in Theorem 2.1. The third point R_3 in [5] is equal $(0, T_+^{(1)}) + (-1, T_+^{(1)}) - C_+^{(1)}$, where $C_+^{(1)}$ is the third independent point in Theorem 2.1.
- The rational function $M(m)$ is also equal
$$T_+^{(1)}(0, 25, 125, n, -\frac{1}{25}\frac{75nm^2 - 102m^2 + 205m - 125n - 175}{3m^2-5}, 1).$$
The fourth point R_4 in [5] is equal $(-1, T_+^{(1)}) - C_+^{(1)}$, where $C_+^{(1)}$ is the third independent point in Theorem 2.1.
- In [5] an elliptic surface over a curve is found for which the Mordell-Weil group has rank ≥ 5. Here we give another example of an infinite family of elliptic curves of generic rank ≥ 5.

Theorem 3.2. *The elliptic curve*
$$Y^2 = X^3 - X + \left(\frac{3723875}{729}n^2 + \frac{155}{9}n - \frac{3723875}{729}\right)^2$$
over the function field $\mathbb{Q}(m,n)$ *where* $\left((3m^2-5)\left(\frac{48050}{81}n+1\right)\right)^2 =$
$$= \frac{2257735321}{729}m^4 + 584660m^3 - \frac{25995527290}{2187}m^2 + \frac{2923300}{3}m + \frac{56443383025}{6561},$$
has rank ≥ 5 *with five independent points with first coordinates*
$$0,\ -1,\ -\frac{69m^2 - 414m + 295}{3m^2-5},\ \frac{357m^2 - 410m + 95}{3m^2-5},\ \frac{24025}{81}n^2 - \frac{24025}{81}.$$

Proof. Here we will intersect $M(m)$ with $T_+^{(1)}(0,u^2,u^3,n,k,l)$ from Theorem 2.1 to obtain a subfamily of higher rank:

$$M(m)=T_+^{(1)}(0,u^2,u^3,n,k,l)=u^3l(n+k+\frac{1}{2u^2l})^2-\frac{1}{4}\frac{(2u^2l^2-2ul+1)(2u^2l^2+2ul+1)}{ul}.$$

This gives $(2u^2l(3m^2-5)(n+k+\frac{1}{2u^2l}))^2=$

$$=(9+36(ul)^4+4068(ul))m^4-33948(ul)m^3+(-30+77192ul-120(ul)^4)m^2$$
$$-56580(ul)m+25+100(ul)^4+11300(ul).$$

So, the point $m=1$ will be the solution of the above equation if $c=ul$ is the first coordinate on

$$\square=16c^4+2032c+4.$$

The corresponding elliptic curve is of rank five and from one of the generators of the free part we get $c=ul=-\frac{155}{9}$ (chosen such that the specialization $m=1$ gives the independence of points). So we take $k=0$, $l=1$ and we look at the intersection

$$M(m)=T_+^{(1)}\left(0,\left(-\frac{155}{9}\right)^2,\left(-\frac{155}{9}\right)^3,n,0,1\right)=-\frac{3723875}{729}n^2-\frac{155}{9}n+\frac{3723875}{729},$$

and we get that (m,n) lies on

$$\left((3m^2-5)\left(\frac{48050}{81}n+1\right)\right)^2=\frac{2257735321}{729}m^4+584660m^3$$
$$-\frac{25995527290}{2187}m^2+\frac{2923300}{3}m+\frac{56443383025}{6561}. \qquad (3.1)$$

So (m,n) on (3.1) gives five points from the claim of the theorem (where the third and fourth point are from [5] and the last point is from Theorem 2.1).

For the specialization $(m,n)\mapsto(1,-\frac{4753}{4805})$ we get the elliptic curve

$$E_{M_2(1)}=E_{T_+^{(1)}\left(0,(-\frac{155}{9})^2,(-\frac{155}{9})^3,-\frac{4753}{4805},0,1\right)}=E_{127}:Y^2=X^3-X+127^2,$$

with corresponding five independent points with first coordinates 0, -1, -25, -21, $-\frac{6136}{961}$. So the five points from the claim of the theorem are independent. □

Remark 3.3. Points (m,n) in the above theorem can be obtained with the transformation

$$m=\frac{11602011740X-139896435555764171800+47449Y}{47449Y+7099196538X-80704505760225548460},$$

where (X,Y) is a point on the curve

$$Y^2=X^3-4119006235730787327\,00X+32137586998783982379698901460\,00.$$

The value of n can be obtained from (3.1). This curve is of positive rank by [7], so the subfamily of elliptic curves from Theorem 3.2 is infinite.

4. Specializations of high rank

The highest rank found for the elliptic curve $E_t : Y^2 = X^3 - X + t^2$ over $\mathbb{Q}$ is ≥ 11 and is obtained for $t = 1118245045$. In this case we get the elliptic curve $E_{1118245045} : Y^2 = X^3 - X + 1118245045^2$ and eleven independent points

$$(1, 1118245045), (-1, 1118245045), (-149499, 1116750055), (-187723, 1115283209)$$

$$(208403, 1122284857), (-357751, 1097581405), (-369623, 1095433091),$$

$$(-398399, 1089604235), (402083, 1146942473), (506597, 1174940551),$$

$$(919987, 1424474279).$$

This was found using the sieve method explained in [4, 8, 10]. Here we observed $t = \frac{t_1}{t_2}$ $(1 \leq t_2 \leq 10000,\ 1 \leq t_1 \leq 100000)$, and elliptic curves E_t with $S(523, E_t) > 23$ for which $S(1979, E_t) > 43.5$. The lower bound was found using the command `Seek1` in Apecs [1]. In addition we observed integers $1 \leq t \leq 1130000000$, and elliptic curves E_t with $S(523, E_t) > 23$ for which $S(1979, E_t) > 41.5$ for the remaining ones. Here is the list of values t which we obtained with rank ≥ 8:

rank	t
≥ 8	$\frac{1567}{3025}$, $\frac{7247}{7688}$, $\frac{23618}{9025}$, $\frac{14809}{4800}$, $\frac{32971}{9072}$, $\frac{22069}{5329}$, $\frac{23581}{3481}$, $\frac{18353}{2197}$, $\frac{4882}{529}$, $\frac{88745}{8496}$, $\frac{74227}{6859}$, $\frac{47059}{3698}$, $\frac{6913}{242}$, $\frac{17489}{343}$, $\frac{53708}{529}$, $\frac{11689}{50}$, $\frac{29689}{2}$, 78560, 2011060, 14083286, 14083286, 21717559, 35498230, 38998023, 45321449, 58235977, 67190943, 67292109, 83402041, 86010677, 96384349, 101940616, 122421035, 159056061, 171981307, 200300248, 217135540, 230684707, 266349308, 307253369, 329132909, 331903387, 342825543, 349640440, 391942721, 423787655, 436687265, 484259053, 484594343, 566328793, 586597025, 594744835, 594782908, 594869501, 598442638, 620933242, 631151494, 747946597, 781809427, 787815289, 836422595, 851738165, 919540903, 1015597721, 1029670387, 1111072411
≥ 9	$\frac{20155}{7442}$, $\frac{90719}{9248}$, $\frac{36749}{1225}$, $\frac{51691}{1089}$, $\frac{83351}{1521}$, $\frac{70313}{845}$, 423515, 829999, 1741033, 2650019, 7030799, 11180651, 53958107, 70808669, 76758473, 97399947, 101469425, 154523221, 197903551, 281137843, 300361741, 304354681, 352968853, 355308367, 599768545, 863227439, 911227325, 1040969455
≥ 10	765617, 17708315, 64232534, 77799653, 236076508, 269371865, 337557943, 450112831, 808983247
≥ 11	1118245045

The greatest rank obtained in [5] was rank 6 for $t = 337$, while the greatest rank obtained in [2] was rank 10 for $t = 765617$.

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