Annales Mathematicae et Informaticae
***39** (2012) pp. 57–70*

Proceedings of the Conference on Stochastic Models and their Applications
Faculty of Informatics, University of Debrecen, Debrecen, Hungary, August 22–24, 2011

A generalized allocation scheme*

István Fazekas[a], **Bettina Porvázsnyik**[b]

[a]University of Debrecen, Faculty of Informatics
Debrecen, Hungary
e-mail: fazekasi@inf.unideb.hu

[b]University of Debrecen, Faculty of Science and Technology
Debrecen, Hungary
e-mail: porv.bettina@gmail.com

Dedicated to Mátyás Arató on his eightieth birthday

Abstract

The generalized allocation scheme was introduced by V.F. Kolchin [1]. Let $\xi_1, \xi_2, \dots, \xi_N$ be independent identically distributed non-negative integer valued non-degenerate random variables. Consider the random variables $\eta'_1, \dots, \eta'_N$ with joint distribution

$$\mathbb{P}\{\eta'_1 = k_1, \dots, \eta'_N = k_N\} = \mathbb{P}\Big\{\xi_1 = k_1, \dots, \xi_N = k_N \ \Big| \ \sum_{i=1}^{N} \xi_i = n\Big\}.$$

Let ξ_i have Poisson distribution, then $(\eta'_1, \dots, \eta'_N)$ has polynomial distribution. Therefore $\{\eta'_1 = k_1, \dots, \eta'_N = k_N\}$ means that the contents of the boxes are $k_1, \dots, k_N$ after allocating n balls into N boxes during the usual allocation procedure.

Our aim is to study random variables $\eta_1, \dots, \eta_N$ with joint distribution

$$\mathbb{P}\{\eta_1 = k_1, \dots, \eta_N = k_N\} = \mathbb{P}\Big\{\xi_1 = k_1, \dots, \xi_N = k_N \ \Big| \ \sum_{i=1}^{N} \xi_i \geq n\Big\}.$$

It can be considered as a general allocation scheme when we place at least n balls into N boxes. Let μ_{nN} denote the number of cases when $\{\eta_i = r\}$. That is μ_{nN} is the number of boxes containing r balls. We shall prove limit theorems for $\mathbb{P}\{\mu_{nN} = k\}$. Moreover, we shall consider the asymptotic behaviour of $\mathbb{P}\{\max_{1\leq i\leq N} \eta_i \leq r\}$ and $\mathbb{P}\{\min_{1\leq i\leq N} \eta_i \leq r\}$.

Keywords: generalized allocation scheme, conditional probability, law of large numbers, central limit theorem, Poisson distribution.

MSC: 60C05, 60F05

*Supported by the Hungarian Scientific Research Fund under Grant No. OTKA T079128/2009. Supported by the TÁMOP-4.2.2/B-10/1-2010-0024 project. The project is co-financed by the European Union and the European Social Fund.

1. Introduction

The usual allocation scheme is the following. Let n balls be placed successively and independently into N boxes. At any allocation the ball can fall into each box with probability $1/N$. This model was widely studied. See the early papers Weiss [13], Rényi [12], Békéssy [1] and the monograph Kolchin-Sevast'yanov-Chistyakov [8]. See also Chuprunov-Fazekas [2] for certain recent results.

A generalization of the usual allocation scheme was introduced by V.F. Kolchin (see the monographs of Kolchin [7] and Pavlov [10]). Let $\eta'_1, \eta'_2, \dots, \eta'_N$ be non-negative integer-valued random variables. In Kolchin's generalized allocation scheme the joint distribution of $\eta'_1, \eta'_2, \dots, \eta'_N$ can be represented as

$$\mathbb{P}\{\eta'_1 = k_1, \dots, \eta'_N = k_N\} = \mathbb{P}\Big\{\xi_1 = k_1, \dots, \xi_N = k_N \;\Big|\; \sum\nolimits_{i=1}^N \xi_i = n\Big\}, \qquad (1.1)$$

where $\xi_1, \xi_2, \dots, \xi_N$ are independent identically distributed non-negative integer valued non-degenerate random variables and $k_1, k_2, \dots, k_N$ are arbitrary non-negative integers, $k_1 + k_2 + \dots + k_N = n$. This scheme contains the usual allocation procedure, certain random forests, and several other models (see the monographs of Kolchin [7] and Pavlov [10]).

The usual allocation scheme is obtained as follows. Let ξ_i have Poisson distribution, i.e. $\mathbb{P}(\xi_i = k) = \frac{\lambda^k}{k!}e^{-\lambda}$, $k = 0, 1, \dots$. Then

$$\mathbb{P}\{\xi_1 = k_1, \dots, \xi_N = k_N \mid \xi_1 + \dots + \xi_N = n\} = \frac{n!}{k_1! \dots k_N!}\left(\frac{1}{N}\right)^n$$

if $k_1 + \dots + k_N = n$. That is $(\eta'_1, \dots, \eta'_N)$ has polynomial distribution. Now $\{\eta'_1 = k_1, \dots, \eta'_N = k_N\}$ means that the cell contents are $k_1, \dots, k_N$ after allocating n particles into N cells considering the usual allocation procedure.

The connection of the random forest and the generalized allocation scheme is the following. Let $\mathcal{T}_{n,N}$ denote the set of forests containing N labelled roots and n labelled non-root vertices. By Cayley's theorem, $\mathcal{T}_{n,N}$ has $N(n+N)^{n-1}$ elements. Consider uniform distribution on $\mathcal{T}_{n,N}$. Let η'_i denote the number of the non-root vertices of the ith tree. Then

$$\mathbb{P}\{\eta'_1 = k_1, \dots, \eta'_N = k_N\} = \frac{n!}{k_1! \dots k_N!}\frac{(k_1+1)^{k_1-1} \dots (k_N+1)^{k_N-1}}{N(N+n)^{n-1}}.$$

Now let ξ_i have Borel distribution (see [5], [9]) $\mathbb{P}(\xi_i = k) = \frac{\lambda^k(1+k)^{k-1}}{k!}e^{-(k+1)\lambda}$, $k = 0, 1, \dots$, $\lambda > 0$. Then

$$\begin{aligned}&\mathbb{P}\{\xi_1 = k_1, \dots, \xi_N = k_N \mid \xi_1 + \dots + \xi_N = n\}\\ &= \frac{n!}{k_1! \dots k_N!}\frac{(k_1+1)^{k_1-1} \dots (k_N+1)^{k_N-1}}{N(N+n)^{n-1}}\end{aligned}$$

if $k_1 + \dots + k_N = n$. See [7, 2, 10]. Therefore $\eta'_1, \dots, \eta'_N$ satisfy (1.1).

We can say that in the generalized allocation scheme we place n balls into N boxes. In the framework of the generalized allocation scheme several asymptotic results can be obtained. Let μ'_r be the number of the random variables $\eta'_1, \eta'_2, \ldots, \eta'_N$ being equal to r $(r = 0, 1, \ldots, n)$.

Observe that

$$\mu'_r = \mu'_{rnN} = \mu'_{nN} = \sum_{i=1}^{N} \mathbb{I}_{\{\eta'_i = r\}} \tag{1.2}$$

can be considered as the number of boxes containing r balls. Here $\mathbb{I}_A$ is the indicator of the set A, i.e. $\mathbb{I}_A(x) = 1$ if $x \in A$ and $\mathbb{I}_A(x) = 0$ if $x \notin A$. (μ'_r, μ'_{rnN}, and μ'_{nN} are just different notations for the same quantity.)

Limit results for μ'_r can be obtained in the following way. Let ξ_0 be a random variable with the same distribution as ξ_1. Let $p_r = \mathbb{P}\{\xi_0 = r\}$ and $\mathbb{E}\xi_0 = a$. Introduce notation $S_N = \sum_{i=1}^{N} \xi_i$.

Denote by $\xi_0^{(r)}$ a random variable with distribution

$$\mathbb{P}\{\xi_0^{(r)} = k\} = \mathbb{P}\{\xi_0 = k \mid \xi_0 \neq r\}. \tag{1.3}$$

The expectation and the second moment of $\xi_0^{(r)}$ are the following $a_r = \mathbb{E}\xi_0^{(r)} = \dfrac{a - rp_r}{1 - p_r}$ and $\mathbb{E}\left(\xi_0^{(r)}\right)^2 = \dfrac{\mathbb{E}\xi_0^2 - r^2 p_r}{1 - p_r}$. Let $\xi_1^{(r)}, \ldots, \xi_N^{(r)}$ be independent copies of $\xi_0^{(r)}$. Let $S_N^{(r)} = \sum_{i=1}^{N} \xi_i^{(r)}$. Denote by C_N^k the binomial coefficient $C_N^k = \binom{N}{k}$.

V.F. Kolchin proved in [7] the following lemma.

Lemma 1.1. *Let μ'_{nN} and $\xi_0^{(r)}$ be defined by* (1.2) *and* (1.3)*, respectively. Then*

$$\mathbb{P}\{\mu'_{nN} = k\} = C_N^k p_r^k (1 - p_r)^{N-k} \frac{\mathbb{P}\{S_{N-k}^{(r)} = n - kr\}}{\mathbb{P}\{S_N = n\}}. \tag{1.4}$$

Using this representation, normal and Poisson limit theorems were obtained (see [7], and [10]).

In [4] a modification of the generalized allocation scheme was studied, that is in (1.1) the condition was changed for $\sum_{i=1}^{N} \xi_i \leq n$.

In this paper we introduce another scheme, i.e. we use in (1.1) condition of the form $\sum_{i=1}^{N} \xi_i \geq n$. It can be considered as a general allocation scheme when we place at least n balls into N boxes. Let μ_{nN} denote the number of cases when $\{\eta_i = r\}$. That is μ_{nN} is the number of boxes containing r balls.

We shall prove limit theorems for $\mathbb{P}\{\mu_{nN} = k\}$. Moreover, we shall consider the asymptotic behaviour of $\mathbb{P}\{\max_{1 \leq i \leq N} \eta_i \leq r\}$ and $\mathbb{P}\{\min_{1 \leq i \leq N} \eta_i \leq r\}$.

In Section 2 $\mathbb{P}\{\mu_{nN} = k\}$ is studied. In sections 3 and 4 $\mathbb{P}\{\max_{1 \leq i \leq N} \eta_i \leq r\}$ and $\mathbb{P}\{\min_{1 \leq i \leq N} \eta_i \leq r\}$ are considered, respectively.

2. Another generalized allocation scheme

Let $\xi_1, \xi_2, \ldots, \xi_N$ be independent identically distributed non-negative integer-valued non-degenerate random variables. Consider random variables $\eta_1, \eta_2, \ldots, \eta_N$ with joint distribution

$$\mathbb{P}\{\eta_1 = k_1, \dots, \eta_N = k_N\} = \mathbb{P}\Big\{\xi_1 = k_1, \dots, \xi_N = k_N \;\Big|\; \sum\nolimits_{i=1}^{N} \xi_i \geq n\Big\}. \quad (2.1)$$

In this case, we place at least n balls into N boxes.

Example 2.1. Let ξ_i have Poisson distribution, i.e. $\mathbb{P}(\xi_i = k) = \frac{\lambda^k}{k!} e^{-\lambda}$, $k = 0, 1, \dots$. Then

$$\mathbb{P}\{\xi_1 = k_1, \dots, \xi_N = k_N \mid \xi_1 + \dots + \xi_N \geq n\} = \frac{1}{k_1! \dots k_N!} \lambda^{k_0} \Big/ \sum_{k=n}^{\infty} \frac{(N\lambda)^k}{k!} \quad (2.2)$$

if $k_1 + \dots + k_N = k_0 \geq n$. Now, we place η (random number) balls into N boxes. Assume that $\eta \geq n$. Let η_i denote the number of balls in the ith box. Then

$$\begin{aligned} \mathbb{P}\{\eta_1 = k_1, \dots, \eta_N = k_N\} &= \sum_{i=n}^{\infty} \mathbb{P}(\eta_1 = k_1, \dots, \eta_N = k_N \mid \eta = i) \mathbb{P}(\eta = i) \\ &= \frac{k_0!}{k_1! \dots k_N!} \left(\frac{1}{N}\right)^{k_0} \mathbb{P}(\eta = k_0), \end{aligned} \quad (2.3)$$

if $k_1 + \dots + k_N = k_0 \geq n$. If we choose the a'priori distribution of η as Poisson distribution truncated from below, i.e

$$\mathbb{P}(\eta = i) = \frac{(N\lambda)^i}{i!} e^{-N\lambda} \Big/ \sum_{k=n}^{\infty} \frac{(N\lambda)^k}{k!} e^{-N\lambda}, \quad i = n, n+1, \dots,$$

then we obtain (2.2). That is our scheme (2.1) with ξ_i having Poisson distribution describes the usual allocation when the number of balls is given by a truncated Poisson distribution.

Let

$$\mu_r = \mu_{rnN} = \mu_{nN} = \sum\nolimits_{i=1}^{N} \mathbb{I}_{\{\eta_i = r\}}$$

be the number of the boxes containing r balls. Then we have the following analogue of Kolchin's formula (1.4) for our model. Recall that $\xi_0^{(r)}$ is defined by (1.3).

Theorem 2.2. *For all $k = 0, 1, 2, \dots, N$ we have*

$$\mathbb{P}\{\mu_{nN} = k\} = C_N^k p_r^k (1 - p_r)^{N-k} \frac{\mathbb{P}\{S_{N-k}^{(r)} \geq n - kr\}}{\mathbb{P}\{S_N \geq n\}}. \quad (2.4)$$

Proof. (2.4) can be proved by a certain modification of the proof of Lemma 1.1.

Let $A_k^{(r)}$ be the event that exactly k of the random variables $\xi_1, \dots, \xi_N$ are equal to r. By (2.1), we have

$$\mathbb{P}\{\mu_{nN} = k\} = \mathbb{P}(A_k^{(r)} \mid S_N \geq n) = \frac{\mathbb{P}(A_k^{(r)}, S_N \geq n)}{\mathbb{P}(S_N \geq n)}.$$

Furthermore,

$$\begin{aligned}
&\mathbb{P}(A_k^{(r)}, S_N \geq n) = \mathbb{P}(S_N \geq n |\, A_k^{(r)})\mathbb{P}(A_k^{(r)}) \\
&= C_N^k p_r^k (1-p_r)^{N-k} \mathbb{P}(S_N \geq n |\, \xi_1 \neq r, \ldots, \xi_{N-k} \neq r, \xi_{N-k+1} = r, \ldots, \xi_N = r) \\
&= C_N^k p_r^k (1-p_r)^{N-k} \mathbb{P}(S_{N-k}^{(r)} \geq n - kr).
\end{aligned}$$

Here we have used that $\xi_1, \ldots, \xi_N$ are independent random variables and the event $A_k^{(r)}$ can occur C_N^k different ways, moreover

$$\begin{aligned}
&\mathbb{P}(S_N \geq n |\, A_k^{(r)}) = \frac{\mathbb{P}(S_N \geq n\,, A_k^{(r)})}{\mathbb{P}(A_k^{(r)})} \\
&= \frac{\mathbb{P}(\xi_1 + \cdots + \xi_N \geq n\,, \xi_1 \neq r, \ldots, \xi_{N-k} \neq r, \xi_{N-k+1} = r, \ldots, \xi_N = r)}{\mathbb{P}(\xi_1 \neq r, \ldots, \xi_{N-k} \neq r, \xi_{N-k+1} = r, \ldots, \xi_N = r)} \\
&= \frac{\mathbb{P}(\xi_1 + \cdots + \xi_{N-k} \geq n - kr\,, \xi_1 \neq r, \ldots, \xi_{N-k} \neq r, \xi_{N-k+1} = r, \ldots, \xi_N = r)}{\mathbb{P}(\xi_1 \neq r, \ldots, \xi_{N-k} \neq r, \xi_{N-k+1} = r, \ldots, \xi_N = r)} \\
&= \frac{\mathbb{P}(\xi_1 + \cdots + \xi_{N-k} \geq n - kr\,, \xi_1 \neq r, \ldots, \xi_{N-k} \neq r)}{\mathbb{P}(\xi_1 \neq r, \ldots, \xi_{N-k} \neq r)}.
\end{aligned}$$

□

The proofs of our limit theorems are based on representation (2.4). First we consider two theorems with normal limiting distribution. Let $\alpha_{nN} = \frac{n}{N}$.

Theorem 2.3. *Let $\mathbb{E}\xi_0 = a$ be finite, $\mathbb{E}\xi_0^{(r)} = a_r$, $s_r^2 = p_r(1-p_r)$.*

(1) Let $d < a$. Then, uniformly for $\alpha_{nN} < d$, we have

$$\mathbb{P}\{\mu_{nN} = k\} = \frac{1}{\sqrt{2\pi N} s_r} e^{-u^2/2}(1 + \mathrm{o}(1)), \tag{2.5}$$

as $n, N \to \infty$ and $u = \frac{k - Np_r}{s_r N^{1/2}}$ belongs to an arbitrary bounded fixed interval.

(2) Suppose that $a_r < a$. Let $a_r < d_1 < d < a$. If k belongs to a bounded interval, then we have

$$\lim_{n, N \to \infty,\, d_1 < \alpha_{nN} < d} \mathbb{P}\{\mu_{nN} = k\} = 0. \tag{2.6}$$

Proof. (1) By the Moivre-Laplace Theorem we have

$$C_N^k p_r^k (1-p_r)^{N-k} = \frac{1}{\sqrt{2\pi N} s_r} e^{-u^2/2}(1 + \mathrm{o}(1)), \tag{2.7}$$

as $N \to \infty$ uniformly if $u = \frac{k - Np_r}{s_r N^{1/2}}$ belongs to a bounded fixed interval, where $s_r^2 = p_r(1-p_r)$.

As $\alpha_{nN} < d < a$, applying Kolmogorov's law of large numbers, we obtain

$$\begin{aligned}
&\lim_{n, N \to \infty,\, \alpha_{nN} < d} \mathbb{P}\left\{\sum_{i=1}^{N-k} \xi_i^{(r)} \geq n - kr\right\} = 1, \\
&\lim_{n, N \to \infty,\, \alpha_{nN} < d} \mathbb{P}\left\{\sum_{i=1}^{N} \xi_i \geq n\right\} = 1.
\end{aligned} \tag{2.8}$$

Now (2.4), (2.7) and (2.8) imply (2.5).

(2) Let $d_1 < \alpha_{nN} < d$. By Kolmogorov's law of large numbers, we have

$$\lim_{n,N\to\infty,\, d_1<\alpha_{nN}<d} \mathbb{P}\left\{\sum_{i=1}^{N-k} \xi_i^{(r)} \geq n - kr\right\} = 0. \tag{2.9}$$

We obtain (2.6) from (2.4), if we apply (2.7) and (2.9). □

Remark 2.4. It is easy to see that $a < a_r$, $a > a_r$ and $a = a_r$ if and only if $a > r$, $a < r$ and $a = r$, respectively.

Let Φ denote the standard normal distribution function. Recall that $a = \mathbb{E}\xi_0$, $a_r = \mathbb{E}\xi_0^{(r)}$ and $s_r^2 = p_r(1-p_r)$.

Theorem 2.5. *Suppose that* $\mathbb{E}\xi_0^2 < \infty$. *Denote by* σ^2 *the variance of* ξ_0 *and by* σ_r^2 *the variance of* $\xi_0^{(r)}$. *Assume* $0 < \sigma^2, \sigma_r^2 < \infty$. *Let* $-\infty \leq C < \infty$. *Then, as* $n, N \to \infty$ *such that* $\sqrt{N}(\alpha_{nN} - a) \to C$, *we have*

$$\mathbb{P}\{\mu_{nN} = k\} = \frac{1}{\sqrt{2\pi N}s_r} e^{-u^2/2} \left(\frac{1 - \Phi\left(\frac{C + us_r \frac{a-r}{1-p_r}}{\sqrt{1-p_r}\sigma_r}\right)}{1 - \Phi\left(\frac{C}{\sigma}\right)} + \mathrm{o}(1) \right), \tag{2.10}$$

for $u = \frac{k - Np_r}{s_r N^{1/2}}$ *belonging to any bounded fixed interval.*

Proof. As $\sigma^2 = \mathbb{D}^2(\xi_0) < \infty$ and $\sigma_r^2 = \mathbb{D}^2(\xi_0^{(r)}) < \infty$, by the central limit theorem, we obtain

$$\mathbb{P}\left\{\sum_{i=1}^{N} \xi_i \geq n\right\} = \mathbb{P}\left\{\frac{\sum_{i=1}^{N} \xi_i - Na}{\sqrt{N}\sigma} \geq \frac{\sqrt{N}(\alpha_{nN} - a)}{\sigma}\right\}$$
$$= 1 - \Phi\left(\frac{\sqrt{N}(\alpha_{nN} - a)}{\sigma}\right) + \mathrm{o}(1), \tag{2.11}$$

and similarly we obtain

$$\mathbb{P}\left\{\sum_{i=1}^{N-k} \xi_i^{(r)} \geq n - kr\right\} = 1 - \Phi\left(\frac{\sqrt{N}(\alpha_{nN} - a) + us_r \dfrac{a-r}{1-p_r}}{\sqrt{1-p_r}\sigma_r}\right) + \mathrm{o}(1). \tag{2.12}$$

Using (2.11), (2.12), and (2.7), relation (2.4) implies the desired result. □

Using large deviation theorems we can describe the relation between μ_{nN} and μ'_{nN}.

Let $X_1, X_2, \ldots$ be independent identically distributed non-negative non-degenerate random variables with lattice distribution (assume that the span of the distribution of X_1 is 1). Suppose that Cramér's condition is satisfied, that is $\mathbb{E}e^{\lambda_0 X_1} < \infty$ for some $\lambda_0 > 0$. Let $Z_N = X_1 + \cdots + X_N$. Introduce notation

$$M(h) = \mathbb{E}e^{hX_1}, \quad a(h) = (\ln(M(h)))', \quad v^2(h) = a'(h).$$

As X_1 is non-degenerate, therefore $a'(h) > 0$, so $a(\cdot)$ is strictly increasing.

We have the following lemma from [11].

Lemma 2.6. *Let x be an integer number and let $h = a^{-1}(\frac{x}{N})$. Then, as $N \to \infty$, we have*

$$\mathbb{P}(Z_N = x) = \frac{1}{v(h)\sqrt{2\pi N}} M^N(h) e^{-hx} \left(1 + \mathrm{O}\left(\frac{1}{N}\right)\right),$$
$$\mathbb{P}(Z_N \geq x) = \frac{1}{v(h)\sqrt{2\pi N}} M^N(h) e^{-hx} (1 - e^{-h})^{-1} \left(1 + \mathrm{O}\left(\frac{1}{N}\right)\right)$$

uniformly for x, with $Na(\varepsilon) \leq x \leq Na(\lambda_0 - \varepsilon)$, where ε is an arbitrary small positive number. In particular

$$\frac{\mathbb{P}(Z_N \geq x)}{\mathbb{P}(Z_N = x)} = (1 - e^{-h})^{-1}(1 + \mathrm{o}(1)). \tag{2.13}$$

Introduce notation

$$L(\lambda) = \mathbb{E}e^{\lambda \xi_0}, \quad L_r(\lambda) = \mathbb{E}e^{\lambda \xi_0^{(r)}}$$

where we assume that there exist positive constants $\lambda_0 > 0$ and $\lambda_0^{(r)} > 0$ such that $\mathbb{E}e^{\lambda_0 \xi_0} < \infty$ and $\mathbb{E}e^{\lambda_0^{(r)} \xi_0^{(r)}} < \infty$ (Cramér's condition). Let

$$m(\lambda) = (\ln(L(\lambda)))', \quad \sigma^2(\lambda) = m'(\lambda), \quad 0 \leq \lambda \leq \lambda_0\,,$$

$$m_r(\lambda) = (\ln(L_r(\lambda)))', \quad \sigma_r^2(\lambda) = m_r'(\lambda), \quad 0 \leq \lambda \leq \lambda_0^{(r)}.$$

As ξ_0 is non-degenerate, therefore $m(.)$ is strictly increasing. Assume that $0 < \mathbb{P}(\xi_0 = 0) < 1$. Moreover, if we additionally assume that $r \neq 0$ and $\mathbb{P}(\xi_0 = 0) + \mathbb{P}(\xi_0 = r) < 1$, then $\xi_0^{(r)}$ is non-degenerate, therefore similar property is valid for the function $m_r(.)$.

Let $h = m^{-1}(\alpha_{nN})$, $h_r = m_r^{-1}(\alpha_{nN})$, and $\beta(\alpha_{nN}) = \dfrac{1 - e^{-h}}{1 - e^{-h_r}}$.

Theorem 2.7. *Assume $r > 0$, $\mathbb{P}(\xi_0 = 0) > 0$, and $\mathbb{P}(\xi_0 = 0) + \mathbb{P}(\xi_0 = r) < 1$. Let $\max\{a, a_r\} < d_1 < d_2 < \min\{m(\lambda_0), m_r(\lambda_0^{(r)})\}$. Then, as $n, N \to \infty$, we have*

$$\mathbb{P}\{\mu_{nN} = k\} = \mathbb{P}\{\mu'_{nN} = k\} \beta(\alpha_{nN})(1 + \mathrm{o}(1)) \tag{2.14}$$

uniformly for $d_1 < \alpha_{nN} < d_2$.

Proof. We obtain Theorem 2.7 from (2.4) and from Lemma 1.1 , if we apply (2.13) both for ξ_i and for $\xi_i^{(r)}$. □

We shall use the so called power series distribution. Consider the random variable ξ_0 with the following distribution. Let $b_0,\ b_1, b_2, \dots$ be a sequence of non-negative numbers and let R denote the radius of convergence of the series

$$B(\theta) = \sum\nolimits_{k=0}^{\infty} \frac{b_k \theta^k}{k!}.$$

Assume that $R > 0$. Let $\xi_0 = \xi_0(\theta)$ have the following distribution

$$p_k = p_k(\theta) = \mathbb{P}\{\xi_0(\theta) = k\} = \frac{b_k \theta^k}{k! B(\theta)}, \quad k = 0, 1, 2, \dots. \tag{2.15}$$

Differentiating $B(\theta)$ for $0 \le \theta < R$, we obtain

$$\mathbb{E}\xi_0(\theta) = \frac{\theta B'(\theta)}{B(\theta)}, \quad \mathbb{D}^2\xi_0(\theta) = \frac{\theta^2 B''(\theta)}{B(\theta)} + \mathbb{E}\xi_0(\theta) - (\mathbb{E}\xi_0(\theta))^2$$

(see e.g. [7]).

We will assume that the distribution of the random variable $\xi_0(\theta)$ satisfies

$$b_0 > 0, \ b_1 > 0. \tag{2.16}$$

We emphasize that the distribution of $\xi_0 = \xi_0(\theta)$ is not fixed, it depends on θ.

We have the following Poisson limit theorem.

Theorem 2.8. *Suppose that the random variable $\xi_0 = \xi_0(\theta)$ has distribution* (2.15), *condition* (2.16) *is satisfied. Let $\theta \le K < R$. Let $r > 1$ and $\frac{n}{N^{1-\frac{1}{r}}} \to 0$. Let $N \to \infty$ such that $Np_r(\theta) \to \lambda$ for some $0 < \lambda < \infty$. Then for all $k \in \mathbb{N}$ we have*

$$\mathbb{P}\{\mu_{nN} = k\} = \frac{\lambda^k e^{-\lambda}}{k!}(1 + \mathrm{o}(1)). \tag{2.17}$$

Proof. Let $k \in \mathbb{N}$. By the Poisson limit theorem, one has

$$C_N^k p_r^k (1 - p_r)^{N-k} = \frac{\lambda^k e^{-\lambda}}{k!}(1 + \mathrm{o}(1)). \tag{2.18}$$

Relation $Np_r(\theta) \to \lambda$ implies that $\theta = \mathrm{o}(1)$, $B(\theta) = b_0 + \mathrm{o}(1)$, $B^{'}(\theta) = b_1 + \mathrm{o}(1)$ and $B^{''}(\theta) = b_2 + \mathrm{o}(1)$. Therefore $\theta = \left(\frac{r!(b_0\lambda + \mathrm{o}(1))}{Nb_r}\right)^{1/r}$.

We know that $\mathbb{E}\xi_0 = \frac{\theta B^{'}(\theta)}{B(\theta)}$. Therefore

$$\mathbb{E}\xi_0 = \frac{b_1}{b_0}\left(\frac{r!(b_0\lambda + \mathrm{o}(1))}{Nb_r}\right)^{1/r}(1 + \mathrm{o}(1)) = C\left(\frac{1}{N}\right)^{1/r}(1 + \mathrm{o}(1)). \tag{2.19}$$

Here and in what follows C denotes an appropriate constant (its value can be different in different formulae). Similarly

$$\mathbb{D}^2\xi_0 = C\left(\frac{1}{N}\right)^{1/r}(1 + \mathrm{o}(1)). \tag{2.20}$$

Now applying condition $\frac{n}{N^{1-\frac{1}{r}}} \to 0$, Chebishev's inequality and relations (2.19), (2.20), we obtain

$$\mathbb{P}\{S_N \geq n\} = (1 + \mathrm{o}(1)). \tag{2.21}$$

As $\mathbb{E}\xi_0^{(r)} = \dfrac{\mathbb{E}\xi_0 - rp_r}{1 - p_r}$, so (2.19) and condition $\frac{n}{N^{1-\frac{1}{r}}} \to 0$ imply that

$$\mathbb{E}\xi_0^{(r)} = C \left(\frac{1}{N}\right)^{1/r} (1 + \mathrm{o}(1)). \tag{2.22}$$

We have

$$\mathbb{D}^2\xi_0^{(r)} = \frac{\mathbb{E}\xi_0^2}{1 - p_r} - \frac{a^2}{(1 - p_r)^2} + \frac{2arp_r}{(1 - p_r)^2} - \frac{r^2 p_r}{(1 - p_r)^2}. \tag{2.23}$$

We obtain

$$\mathbb{D}^2\xi_0^{(r)} = C \left(\frac{1}{N}\right)^{1/r} (1 + \mathrm{o}(1)). \tag{2.24}$$

Now applying condition $\frac{n}{N^{1-\frac{1}{r}}} \to 0$, Chebishev's inequality and relations (2.22), (2.24), we obtain

$$\mathbb{P}\{S_{N-k}^{(r)} \geq n - kr\} = (1 + \mathrm{o}(1)). \tag{2.25}$$

Inserting (2.21), (2.25), and (2.18) into (2.4), we obtain (2.17). □

3. Limit theorems for $\max_{1 \leq i \leq N} \eta_i$

Let $\eta_{(N)} = \max_{1 \leq i \leq N} \eta_i$. $\eta_{(N)}$ is the maximal number of balls contained by any of the boxes.

Let $\xi_0^{(\leq r)}$ be a random variable with distribution

$$\mathbb{P}\{\xi_0^{(\leq r)} = k\} = \mathbb{P}\{\xi_0 = k \mid \xi_0 \leq r\}.$$

Let $\xi_i^{(\leq r)}$, $i = 1, \ldots, N$, be independent copies of $\xi_0^{(\leq r)}$. Let $S_N^{(\leq r)} = \sum_{i=1}^{N} \xi_i^{(\leq r)}$ and $\mathbb{E}\xi_0^{(\leq r)} = a_{\leq r}$. We can see that $a_{\leq r} \leq a$. Moreover, $a_{\leq r} = a$ if and only if $\mathbb{P}(\xi_0 \leq r) = 1$, that is ξ_0 and $\xi_0^{(\leq r)}$ have the same distribution.

The following representation of $\eta_{(N)}$ is useful to obtain limit results.

Theorem 3.1. *We have*

$$\mathbb{P}\{\eta_{(N)} \leq r\} = (1 - P_r)^N \frac{\mathbb{P}\{S_N^{(\leq r)} \geq n\}}{\mathbb{P}\{S_N \geq n\}}, \tag{3.1}$$

for all $r \in \mathbb{N}$ where $P_r = \mathbb{P}\{\xi_0 > r\}$.

Proof.

$$\begin{aligned}\mathbb{P}\{\eta_{(N)} \le r\} &= \mathbb{P}\{\eta_1 \le r, \dots, \eta_N \le r\} \\ &= \mathbb{P}\Big\{\xi_1 \le r, \dots, \xi_N \le r \;\Big|\; \sum\nolimits_{i=1}^{N} \xi_i \ge n\Big\} \\ &= \frac{\mathbb{P}\Big\{\xi_1 \le r, \dots, \xi_N \le r\,, S_N \ge n\Big\}}{\mathbb{P}\Big\{S_N \ge n\Big\}} \\ &= (\mathbb{P}\{\xi_1 \le r\})^N \, \frac{\mathbb{P}\Big\{S_N \ge n \;\Big|\; \xi_1 \le r, \dots, \xi_N \le r\Big\}}{\mathbb{P}\Big\{S_N \ge n\Big\}} \\ &= (1 - P_r)^N \frac{\mathbb{P}\{S_N^{(\le r)} \ge n\}}{\mathbb{P}\{S_N \ge n\}}. \end{aligned}$$

□

Theorem 3.2. *(1) Let $d < a_{\le r}$. Then for all fixed $r \in \mathbb{N}$, as $n, N \to \infty$, we have*

$$\mathbb{P}\{\eta_{(N)} \le r\} = (1 - P_r)^N (1 + \mathrm{o}(1)) \tag{3.2}$$

uniformly for $\alpha_{nN} < d$.

(2) Suppose that $a_{\le r} < a$ and $a_{\le r} < d_1 < d < a$. Then for all fixed $r \in \mathbb{N}$, as $n, N \to \infty$, we have

$$\mathbb{P}\{\eta_{(N)} \le r\} = (1 - P_r)^N \mathrm{o}(1) \tag{3.3}$$

uniformly for $d > \alpha_{nN} > d_1$.

Proof. (1) Apply Kolmogorov's law of large numbers for S_N and $S_N^{(\le r)}$ in (3.1). Then (3.2) follows.

(2) If $d_1 < \alpha_{nN} < d$ and we apply Kolmogorov's law of large numbers, then we obtain

$$\lim_{n,N\to\infty,\, d_1 < \alpha_{nN} < d} \mathbb{P}\left\{ \frac{S_N^{(\le r)}}{N} \ge \frac{n}{N} \right\} = 0. \tag{3.4}$$

□

Theorem 3.3. *Suppose that $\mathbb{E}\xi_0^2 < \infty$ and let $\sigma^2_{\le r}$ be the variance of $\xi_0^{(\le r)}$. Let $-\infty \le C < \infty$. Then, for all $r \in \mathbb{N}$, as $n, N \to \infty$ such that $\sqrt{N}(\alpha_{nN} - a) \to C$, we have*

$$\mathbb{P}\{\eta_{(N)} \le r\} = (1 - P_r)^N \left(\frac{1 - \Phi\left(\frac{C}{\sigma_{\le r}}\right)}{1 - \Phi\left(\frac{C}{\sigma}\right)} + \mathrm{o}(1) \right), \quad \textit{for} \quad a_{\le r} = a, \tag{3.5}$$

$$\mathbb{P}\{\eta_{(N)} \le r\} = (1 - P_r)^N \cdot \mathrm{o}(1), \quad \textit{for} \quad a_{\le r} < a. \tag{3.6}$$

Proof. By the central limit theorem, we have

$$\mathbb{P}\left\{ \sum_{i=1}^{N} \xi_i^{(\le r)} \ge n \right\} = \mathbb{P}\left\{ \frac{\sum_{i=1}^{N} \xi_i^{(\le r)} - N a_{\le r}}{\sqrt{N}\sigma_{\le r}} \ge \frac{\sqrt{N}(\alpha_{nN} - a_{\le r})}{\sigma_{\le r}} \right\}$$

$$= 1 - \Phi\left(\frac{\sqrt{N}(\alpha_{nN} - a_{\le r})}{\sigma_{\le r}}\right) + \mathrm{o}(1). \tag{3.7}$$

In relation (3.1) apply (2.11) and (3.7) to obtain (3.5) and (3.6). □

Let $\eta'_{(N)} = \max_{1\le i\le N}\eta'_i$ be the maximum in the usual generalized allocation scheme (1.1). Using large deviation results, we can describe the relation of $\eta'_{(N)}$ and $\eta_{(N)}$.

Introduce notation

$$L_{\le r}(\lambda) = \mathbb{E}e^{\lambda\xi_0^{(\le r)}}$$

where we assume that there exist a positive constant $\lambda_0^{(\le r)} > 0$, such that

$$\mathbb{E}e^{\lambda_0^{(\le r)}\xi_0^{(\le r)}} < \infty \quad \text{(Cramér's condition)}.$$

Let

$$m_{\le r}(\lambda) = (\ln(L_{\le r}(\lambda)))', \quad \sigma^2_{\le r}(\lambda) = m'_{\le r}(\lambda), \quad 0 \le \lambda \le \lambda_0^{(\le r)}.$$

Let $r \ge 1$. If $\mathbb{P}(\xi_0 = 0) > 0$ and $\mathbb{P}(\xi_0 \le r) > \mathbb{P}(\xi_0 = 0)$, then $\xi_0^{(\le r)}$ is non-degenerate, therefore $m_{\le r}(.)$ is a strictly increasing function.

Let $h = m^{-1}(\alpha_{nN})$, $h_{\le r} = m^{-1}_{\le r}(\alpha_{nN})$, and $\beta_{\le r}(\alpha_{nN}) = \frac{1-e^{-h}}{1-e^{-h_{\le r}}}$.

Theorem 3.4. *Assume that* $r \ge 1$, $\mathbb{P}(\xi_0 = 0) > 0$, *and* $\mathbb{P}(\xi_0 \le r) > \mathbb{P}(\xi_0 = 0)$. *Let* $\max\{a, a_{\le r}\} < d_1 < d_2 < \min\{m(\lambda_0), m_{\le r}(\lambda_0^{(\le r)})\}$. *Then, for all* $r \in \mathbb{N}$ *as* $n, N \to \infty$, *we have*

$$\mathbb{P}\{\eta_{(N)} \le r\} = \mathbb{P}\{\eta'_{(N)} \le r\}\beta_{\le r}(\alpha_{nN})(1 + \mathrm{o}(1)) \tag{3.8}$$

uniformly for $d_1 < \alpha_{nN} < d_2$.

Proof. For the usual generalized allocation scheme, V.F. Kolchin in [7] obtained that

$$\mathbb{P}\{\eta'_{(N)} \le r\} = (1 - P_r)^N \frac{\mathbb{P}\{S_N^{(\le r)} = n\}}{\mathbb{P}\{S_N = n\}} \tag{3.9}$$

for all $r \in \mathbb{N}$ where $P_r = \mathbb{P}\{\xi_0 > r\}$.

Using (3.9) and (3.1) and applying (2.13) both for ξ_i and for $\xi_i^{(\le r)}$, the proof of Theorem 3.4 is complete. □

Theorem 3.5. *Suppose that the random variable* $\xi = \xi(\theta)$ *has distribution* (2.15), *condition* (2.16) *is satisfied and* $\theta \le K < R$. *Let* $r \in \mathbb{N}$. *Let* $\theta = \theta(N)$ *be such that* $Np_{r+1}(\theta) \to \lambda$ *where* $0 < \lambda < \infty$. *Then, as* $n, N \to \infty$ *such that* $\frac{n}{N^{r/(r+1)}} \to 0$, *we have*

$$\mathbb{P}\{\eta_{(N)} = r\} = e^{-\lambda} + \mathrm{o}(1), \tag{3.10}$$

$$\mathbb{P}\{\eta_{(N)} = r + 1\} = 1 - e^{-\lambda} + \mathrm{o}(1). \tag{3.11}$$

Proof. Relation $\frac{n}{N^{r/(r+1)}} \to 0$ implies that

$$B(\theta) = b_0 + \mathrm{o}(1) \quad \text{and} \quad \theta = \left(\frac{(r+1)!(b_0\lambda + \mathrm{o}(1))}{N b_{r+1}} \right)^{1/(r+1)}.$$

Using $r+1$ instead of r in the proof of Theorem 2.8, we obtain

$$\mathbb{P}\left\{S_N \geq n\right\} = (1 + \mathrm{o}(1)). \tag{3.12}$$

Let $r_1 \in \mathbb{N}$. Then

$$\begin{aligned} \mathbb{E}\xi_0^{(\leq r_1)} &= \frac{\sum_{k=1}^{r_1} k \frac{b_k}{k! b_0} \left(\left(\frac{(r+1)!(b_0\lambda + \mathrm{o}(1))}{N b_{r+1}} \right)^{1/(r+1)} \right)^k}{\sum_{k=0}^{r_1} \frac{b_k}{k! b_0} \left(\left(\frac{(r+1)!(b_0\lambda + \mathrm{o}(1))}{N b_{r+1}} \right)^{1/(r+1)} \right)^k} (1 + \mathrm{o}(1)) \\ &= C \left(\frac{1}{N} \right)^{1/(r+1)} (1 + \mathrm{o}(1)). \end{aligned} \tag{3.13}$$

Moreover,

$$\mathbb{D}^2 \xi_0^{(\leq r_1)} \leq C \left(\frac{1}{N} \right)^{1/(r+1)} (1 + \mathrm{o}(1)). \tag{3.14}$$

Using Chebishev's inequality, (3.13) and (3.14), we obtain

$$\mathbb{P}\{S_N^{(\leq r_1)} \geq n\} = (1 + \mathrm{o}(1)). \tag{3.15}$$

Using relations $\theta \to 0$ and $N p_{r+1}(\theta) \to \lambda$, we obtain

$$(1 - P_{r-1})^N = \mathrm{o}(1), \quad (1 - P_r)^N = e^{-\lambda} + \mathrm{o}(1), \quad (1 - P_{r+1})^N = 1 + \mathrm{o}(1). \tag{3.16}$$

Inserting (3.12), (3.15), and (3.16) into (3.1), we obtain

$$\mathbb{P}\{\eta_{(N)} \leq r - 1\} = \mathrm{o}(1), \quad \mathbb{P}\{\eta_{(N)} \leq r\} = e^{-\lambda} + \mathrm{o}(1), \quad \mathbb{P}\{\eta_{(N)} \leq r + 1\} = 1 + \mathrm{o}(1).$$

These relations imply (3.10) and (3.11). □

4. Limit theorems for $\min_{1 \leq i \leq N} \eta_i$

In this section we shall prove limit theorems for the minimal content of the boxes. Let $\eta_{(N-)} = \min_{1 \leq i \leq N} \eta_i$. Let $\xi_0^{(\geq r)}$ be a random variable with distribution $\mathbb{P}\{\xi_0^{(\geq r)} = k\} = \mathbb{P}\{\xi_0 = k \mid \xi_0 \geq r\}$. Let $\xi_i^{(\geq r)}$, $i = 1, \ldots, N$, be independent copies of $\xi_0^{(\geq r)}$. Let $S_N^{(\geq r)} = \sum_{i=1}^N \xi_i^{(\geq r)}$ and $\mathbb{E}\xi_0^{(\geq r)} = a_{\geq r}$. One can see that $\mathbb{E}\xi_0^{(\geq r)} \geq \mathbb{E}\xi_0$ and equality can happen if and only if $\xi_0^{(\geq r)} = \xi_0$.

We start with an appropriate representation of $\eta_{(N-)}$

Theorem 4.1. *We have*

$$\mathbb{P}\{\eta_{(N-)} \geq r\} = (1-Q_r)^N \frac{\mathbb{P}\{S_N^{(\geq r)} \geq n\}}{\mathbb{P}\{S_N \geq n\}}, \tag{4.1}$$

for all $r \in \mathbb{N}$ *where* $Q_r = \mathbb{P}\{\xi_0 < r\}$.

Proof.

$$\begin{aligned}
\mathbb{P}\{\eta_{(N-)} \geq r\} &= \mathbb{P}\{\eta_1 \geq r, \dots, \eta_N \geq r\} \\
&= \mathbb{P}\Big\{\xi_1 \geq r, \dots, \xi_N \geq r \;\Big|\; \sum\nolimits_{i=1}^N \xi_i \geq n\Big\} \\
&= \frac{\mathbb{P}\Big\{\xi_1 \geq r, \dots, \xi_N \geq r\,, S_N \geq n\Big\}}{\mathbb{P}\Big\{S_N \geq n\Big\}} \\
&= (\mathbb{P}\{\xi_1 \geq r\})^N \frac{\mathbb{P}\Big\{S_N \geq n \;\Big|\; \xi_1 \geq r, \dots, \xi_N \geq r\Big\}}{\mathbb{P}\Big\{S_N \geq n\Big\}} \\
&= (1-Q_r)^N \frac{\mathbb{P}\{S_N^{(\geq r)} \geq n\}}{\mathbb{P}\{S_N \geq n\}}.
\end{aligned}$$

□

Theorem 4.2. *Let* $d < a$. *Then for all* $r \in \mathbb{N}$, *as* $n, N \to \infty$, *we have*

$$\mathbb{P}\{\eta_{(N-)} \geq r\} = (1-Q_r)^N (1 + \mathrm{o}(1)) \tag{4.2}$$

uniformly for $\alpha_{nN} < d$.

Proof. We apply Kolmogorov's law of large numbers for S_N and $S_N^{(\geq r)}$ in (4.1). Then we obtain (4.2). □

Theorem 4.3. *Suppose that* $\mathbb{E}\xi_0^2 < \infty$ *and let* $\sigma_{\geq r}^2$ *be the variance of* $\xi_0^{(\geq r)}$. *Let* $-\infty \leq C < \infty$. *Then, for all* $r \in \mathbb{N}$, *as* $n, N \to \infty$ *such that* $\sqrt{N}(\alpha_{nN} - a) \to C$, *we have*

$$\mathbb{P}\{\eta_{(N-)} \geq r\} = (1-Q_r)^N \left(\frac{1 - \Phi\left(\frac{C}{\sigma_{\geq r}}\right)}{1 - \Phi\left(\frac{C}{\sigma}\right)} + \mathrm{o}(1) \right), \quad \textit{for} \quad a_{\geq r} = a, \tag{4.3}$$

$$\mathbb{P}\{\eta_{(N-)} \geq r\} = (1-Q_r)^N \left(\frac{1}{1 - \Phi\left(\frac{C}{\sigma}\right)} + \mathrm{o}(1) \right), \quad \textit{for} \quad a_{\geq r} > a. \tag{4.4}$$

Proof. By the central limit theorem, we have

$$\begin{aligned}
\mathbb{P}\left\{ \sum_{i=1}^N \xi_i^{(\geq r)} \geq n \right\} &= \mathbb{P}\left\{ \frac{\sum_{i=1}^N \xi_i^{(\geq r)} - N a_{\geq r}}{\sqrt{N}\sigma_{\geq r}} \geq \frac{\sqrt{N}(\alpha_{nN} - a_{\geq r})}{\sigma_{\geq r}} \right\} \\
&= 1 - \Phi\left(\frac{\sqrt{N}(\alpha_{nN} - a_{\geq r})}{\sigma_{\geq r}} \right) + \mathrm{o}(1).
\end{aligned} \tag{4.5}$$

In relation (4.1) apply (2.11) and (4.5). Then we obtain (4.3) and (4.4). □

References

[1] Békéssy, A. On classical occupancy problems. I. *Magy. Tud. Akad. Mat. Kutató Int. Közl.* **8** (1963), No.1-2, 59–71.

[2] Chuprunov, A.N. and Fazekas, I. Inequalities and strong laws of large numbers for random allocations. *Acta Math. Hungar.* **109** (2005), No.1-2, 163–182.

[3] Chuprunov, A.N. and Fazekas, I. An inequality for moments and its applications to the generalized allocation scheme. *Publ. Math. Debrecen,* **76** (2010), no. 3, 271–286.

[4] Chuprunov, A.N. and Fazekas, I. One some analogue of the generalized allocation scheme. (Russian) *Diskretnaya Matematika*, **21**(2012), No.1.

[5] Johnson, N. L., Kemp, A. W. and Kotz, S. *Univariate Discrete Distributions.* Third edition. Wiley Series in Probability and Statistics. Wiley-Interscience, Hoboken, NJ, 2005.

[6] Kolchin, A. V. and Kolchin, V. F. On transition of distributions of sums of independent identically distributed random variables from one lattice to another in the generalised allocation scheme. (Russian) *Diskret. Mat.* **18** (2006), no. 4, 113–127; translation in *Discrete Math. Appl.* **16** (2006), no. 6, 527–540.

[7] Kolchin, V.F. *Random Graphs.* Cambridge University Press, Cambridge, 1999.

[8] Kolchin, V.F. , Sevast'yanov, B.A. and Chistyakov, V.P. *Random allocations.* V.H. Winston & Sons, Washington D. C. 1978.

[9] Lerner, B., Lone, A. and Rao, M. On generalized Poisson distributions. *Probab. Math. Statist.* **17** (1997), no. 2, Acta Univ. Wratislav. No. 2029, 377–385.

[10] Pavlov, Yu. L. *Random Forests.* VSP, Utrecht, 2000.

[11] Rozovskii, L. V. Probabilities of large deviations for some classes of distributions satisfying the Cramér condition. (Russian) *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **298** (2003), *Veroyatn. i Stat.* 6, 161–185, 318–319; translation in *J. Math. Sci. (N. Y.)* **128** (2005), no. 1, 2585–2600.

[12] Rényi, A. Three new proofs and generalization of a theorem of Irving Weiss. *Magy. Tud. Akad. Mat. Kutató Int. Közl.* **7** (1962), No.1-2, 203–214.

[13] Weiss, I. Limiting distributions in some occupancy problems. *Ann. Math. Statist.* **29** (1958), No.3, 878–884.