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Parameter estimation in linear regression driven by a Wiener sheet*

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Dedicated to Mátyás Arató on his eightieth birthday

Abstract

The problem of estimating the parameters of linear regression $Z(s,t) = m_1 g_1(s,t) + \cdots + m_p g_p(s,t) + W(s,t)$ based on observations of Z on a spatial domain G of special shape is considered, where the driving process W is a standard Wiener sheet and $g_1, \dots, g_p$ are known functions. We provide an expression for the maximum likelihood estimator of the unknown parameters based on the observation of the process Z on the set G. Simulation results are also presented, where the driving random sheets are simulated with the help of their Karhunen-Loève expansions.

Keywords: Wiener sheet, maximum likelihood estimation, Radon-Nikodym derivative.

MSC: 60G60; 62M10; 62M30.

1. Introduction

The Wiener sheet is one of the most important examples of Gaussian random fields. It has various applications in statistical modelling. Wiener sheet appears as limiting process of some random fields defined on the interface of the Ising model [12], it is used to model random polymers [9], to describe the dynamics of Heath–Jarrow–Morton type forward interest rate models [10] or to model random mortality surfaces [6]. Further, [7] considers the problem of estimation of the mean

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in a nonparametric regression on a two-dimensional regular grid of design points and constructs a Wiener sheet process on the unit square with a drift that is almost the mean function in the nonparametric regression.

In this paper we consider a linear regression driven by a Wiener sheet, that is random field

$$Z(s,t) := m_1 g_1(s,t) + \cdots + m_p g_p(s,t) + W(s,t) \tag{1.1}$$

observed on a domain G, where $g_1, \dots, g_p$ are known functions and W is a standard Wiener sheet, and we determine the maximum likelihood estimator (MLE) of the unknown parameters $m_1, \dots, m_p$.

In principle, the Radon-Nikodym derivative of Gaussian measures might be derived from the general Feldman-Hajek theorem [11], but in most of the cases explicit calculations cannot be carried out. For example, if $p = 1$ and $g_1 \equiv 1$ (shifted Wiener sheet), the MLE of the unknown parameter is given in [13] and the estimator is expressed as a function of a usually unknown random variable satisfying some characterizing equation. In several cases the exact form of this random variable can be derived by a method proposed in [14] based on linear stochastic partial differential equations.

Special case 1: Baran et al. [3] studied the case, when $p = 1$ and the random field Z is observed on a rectangular domain $G := [a_1, a_2] \times [b_1, b_2]$, $[a_1, a_2], [b_1, b_2] \subset (0, \infty)$. Assuming that g_1 is absolutely continuous with respect to the Lebesgue measure and $\partial_1\partial_2 g_1 \in L^2(G)$, they proved that the MLE of the shift parameter m_1 has the form $\widehat{m}_1 = A^{-1}\zeta$, where

$$A = \frac{g_1^2(a_1,b_1)}{a_1 b_1} + \int_{a_1}^{a_2} \frac{[\partial_1 g_1(u,b_1)]^2}{b_1} \mathrm{d}u + \int_{b_1}^{b_2} \frac{[\partial_2 g_1(a_1,v)]^2}{a_1} \mathrm{d}v \tag{1.2}$$
$$+ \iint_G [\partial_1\partial_2 g_1(u,v)]^2 \mathrm{d}u\mathrm{d}v,$$

$$\zeta = \frac{g_1(a_1,b_1)Z(a_1,b_1)}{a_1 b_1} + \int_{a_1}^{a_2} \frac{\partial_1 g_1(u,b_1)}{b_1} Z(\mathrm{d}u, b_1) + \int_{b_1}^{b_2} \frac{\partial_2 g_1(a_1,v)}{a_1} Z(a_1, \mathrm{d}v) \tag{1.3}$$
$$+ \iint_G \partial_1\partial_2 g_1(u,v) Z(\mathrm{d}u, \mathrm{d}v),$$

and it has normal distribution with mean m_1 and variance $1/A$. For $g_1 \equiv 1$ we have $\widehat{m}_1 = Z(a_1, b_1)$.

Special case 2: Arató N.M. [2] considered the case $p = 1$ and $g_1 \equiv 1$ and using Rozanov's method found the MLE of the shift parameter m_1 when the process is observed on a special domain

$$G \subset \widetilde{G} := \{(s,t) \in \mathbb{R}^2 \ : \ a \le s \le b,\ t \ge \gamma(s) \text{ or } s > b,\ t \ge \gamma(b)\}$$

containing an ε–strip of $\Gamma := \{(s, \gamma(s)) : s \in (a, b)\}$, i.e. for some $\varepsilon > 0$

$$\{(s,t) \in \mathbb{R}^2 : \; s \in [a, a+\varepsilon], t \in [\gamma(s), \gamma(a)] \text{ or } s \in [a+\varepsilon, b], t \in [\gamma(s), \gamma(s)+\varepsilon]\} \subset G,$$

where $\gamma : [a, b] \subset (0, \infty) \to \mathbb{R}$ is continuous and strictly decreasing with $\gamma(b) > 0$.

Baran et al. [4] considered the same model and under much weaker conditions $\gamma \in C^2(a, b)$, $\gamma'(a) := \lim_{s\downarrow a} \gamma'(s) \in [-\infty, 0]$ and $\gamma'(b) := \lim_{s\uparrow b} \gamma'(s) \in [-\infty, 0]$ exist, and

$$\int_a^b \frac{|\gamma'(s)\gamma''(s)|}{(1+\gamma'(s)^2)^2} \,\mathrm{d}s < \infty,$$

they proved the result of [2, Theorem 2]. They showed that the MLE of the shift parameter m_1 has the form $\widehat{m_1} = A^{-1}\zeta$, where

$$A = \frac{1}{b\gamma(b)} + \int_a^b \frac{\mathrm{d}s}{s^2\gamma(s)}, \qquad \zeta = c_1 Z(a, \gamma(a)) + c_2 Z(b, \gamma(b)) + \int_\Gamma y_1 Z + \int_\Gamma y_2 \partial_{\mathrm{n}} Z, \tag{1.4}$$

c_1, c_2 are constants depending on γ and γ' at a and b, y_1 and y_2 are functions of $\gamma, \gamma', \gamma''$, and $\partial_{\mathrm{n}} Z$ denotes the normal derivative of Z [4, Definition 4.1].

If $\gamma'(a) = -\infty$ we have

$$\zeta = \frac{Z(b, \gamma(b))}{b\gamma(b)} + \int_a^b \frac{Z(s, \gamma(s))}{s^2\gamma(s)} \mathrm{d}s - \int_a^b \frac{1}{s\gamma(s)} Z(\mathrm{d}s, \gamma(s)).$$

In the present paper we consider the same type of domain G as in [5] and give a natural extension of their result for the general model (1.1). We also present some simulation results to illustrate the theoretical ones where the Wiener sheet is simulated with the help of its Karhunen-Loève expansion (see e.g. [8]).

2. Model and estimator

Consider the model (1.1) with some given functions $g_1, \ldots, g_p : \mathbb{R}_+^2 \to \mathbb{R}$ and with unknown regression parameters $m_1, \ldots, m_p \in \mathbb{R}$. Let $[a, c] \subset (0, \infty)$ and $b_1, b_2 \in (a, c)$, let $\gamma_{1,2} : [a, b_1] \to \mathbb{R}$ and $\gamma_0 : [b_2, c] \to \mathbb{R}$ be continuous, strictly decreasing functions and let $\gamma_1 : [b_1, c] \to \mathbb{R}$ and $\gamma_2 : [a, b_2] \to \mathbb{R}$ be continuous, strictly increasing functions with $\gamma_{1,2}(b_1) = \gamma_1(b_1) > 0$, $\gamma_2(b_2) = \gamma_0(b_2)$, $\gamma_{1,2}(a) = \gamma_2(a)$ and $\gamma_1(c) = \gamma_0(c)$. Consider the curve $\Gamma := \Gamma_{1,2} \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_0$, where

$$\Gamma_{1,2} := \Big\{ \big(s, \gamma_{1,2}(s)\big) : s \in [a, b_1] \Big\}, \qquad \Gamma_1 := \Big\{ \big(s, \gamma_1(s)\big) : s \in [b_1, c] \Big\},$$
$$\Gamma_2 := \Big\{ \big(s, \gamma_2(s)\big) : s \in [a, b_2] \Big\}, \qquad \Gamma_0 := \Big\{ \big(s, \gamma_0(s)\big) : s \in [b_2, c] \Big\},$$

and for a given $\varepsilon > 0$ let $\Gamma_{1,2}^{\varepsilon}$, Γ_1^{ε}, Γ_2^{ε} and Γ_0^{ε} denote the inner ε-strip of $\Gamma_{1,2}$, Γ_1, Γ_2 and Γ_0, respectively, that is e.g.

$$\Gamma_{1,2}^{\varepsilon} := \big\{(s,t) \in \mathbb{R}^2 : s \in [a, a+\varepsilon],\ t \in [\gamma_{1,2}(s), \gamma_{1,2}(a)] \text{ or } s \in [a+\varepsilon, b_1],\ t \in [\gamma_{1,2}(s), \gamma_{1,2}(s)+\varepsilon]\big\}.$$

Suppose that there exists an $\varepsilon > 0$ such that

$$\Gamma_1^{\varepsilon} \cap \Gamma_2^{\varepsilon} = \emptyset \quad \text{and} \quad \Gamma_{1,2}^{\varepsilon} \cap \Gamma_0^{\varepsilon} = \emptyset, \tag{2.1}$$

and consider the set $G := G_1 \cup G_2 \cup G_3$, where

$$G_1 := \big\{(s,t) \in \mathbb{R}^2 : s \in [a, b_1 \wedge b_2],\ t \in [\gamma_{1,2}(s), \gamma_2(s)]\big\},$$
$$G_2 := \begin{cases} \big\{(s,t) \in \mathbb{R}^2 : s \in [b_1, b_2],\ t \in [\gamma_1(s), \gamma_2(s)]\big\}, & \text{if } b_1 \le b_2, \\ \big\{(s,t) \in \mathbb{R}^2 : s \in [b_2, b_1],\ t \in [\gamma_{1,2}(s), \gamma_0(s)]\big\}, & \text{if } b_1 > b_2, \end{cases}$$
$$G_3 := \big\{(s,t) \in \mathbb{R}^2 : s \in [b_1 \vee b_2, c],\ t \in [\gamma_1(s), \gamma_0(s)]\big\}.$$

An example of such a set of observations can be seen of Figure 1.

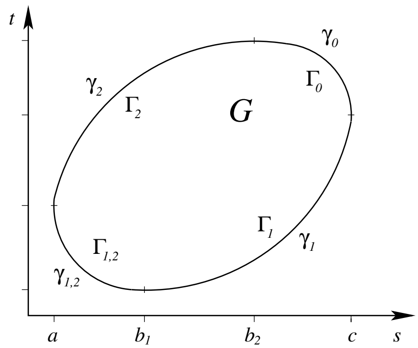

Figure 1: An example of a set of observations G

The following theorem is an extension of Theorem 2.1 of [5] and can be proved in a similar way. The proof is based on the discrete approximation method described in [3, 4, 5], which relies on the results of [1, Section 2.3.2].

Theorem 2.1. *If $g_1, \ldots, g_p$ are twice continuously differentiable inside G and the partial derivatives $\partial_1 g_i, \partial_2 g_i$ and $\partial_1\partial_2 g_i$, $i = 1, \ldots, p$, can be continuously extended to G then the probability measures P_Z and P_W, generated on $C(G)$ by the sheets Z and W, respectively, are equivalent and the Radon-Nikodym derivative of P_Z with respect to P_W equals*

$$\frac{\mathrm{d}\mathsf{P}_Z}{\mathrm{d}\mathsf{P}_W}(Z) = \exp\left\{-\frac{1}{2}\big(\mathbf{m}^\top A \mathbf{m} - 2\zeta^\top \mathbf{m}\big)\right\},$$

where $A := \big(A_{k,\ell}\big)_{k,\ell=1}^{p}$, $\mathbf{m} := (m_1, \dots, m_p)^\top$ *and* $\zeta := \big(\zeta_1, \dots, \zeta_p\big)^\top$ *with*

$$
\begin{aligned}
A_{k,\ell} :=\ & \frac{g_k\big(b_1, \gamma_{1,2}(b_1)\big)\, g_\ell\big(b_1, \gamma_{1,2}(b_1)\big)}{b_1 \gamma_{1,2}(b_1)} \qquad (2.2)\\
&+ \int_a^{b_1} \frac{\big[g_k\big(s, \gamma_{1,2}(s)\big) - s\partial_1 g_k\big(s, \gamma_{1,2}(s)\big)\big]\big[g_\ell\big(s, \gamma_{1,2}(s)\big) - s\partial_1 g_\ell\big(s, \gamma_{1,2}(s)\big)\big]}{s^2 \gamma_{1,2}(s)}\, \mathrm{d}s\\
&+ \int_{b_1}^{c} \frac{\partial_1 g_k\big(s, \gamma_1(s)\big)\, \partial_1 g_\ell\big(s, \gamma_1(s)\big)}{\gamma_1(s)}\, \mathrm{d}s + \int_{\gamma_2(a)}^{\gamma_2(b_2)} \frac{\partial_2 g_k\big(\gamma_2^{-1}(t), t\big)\, \partial_2 g_\ell\big(\gamma_2^{-1}(t), t\big)}{\gamma_2^{-1}(t)}\, \mathrm{d}t\\
&+ \int_{\gamma_{1,2}(b_1)}^{\gamma_{1,2}(a)} \frac{\partial_2 g_k\big(\gamma_{1,2}^{-1}(t), t\big)\, \partial_2 g_\ell\big(\gamma_{1,2}^{-1}(t), t\big)}{\gamma_{1,2}^{-1}(t)}\, \mathrm{d}t + \iint_G \partial_1\partial_2 g_k(s,t)\, \partial_1\partial_2 g_\ell(s,t) \mathrm{d}s\, \mathrm{d}t,
\end{aligned}
$$

and

$$
\begin{aligned}
\zeta_k :=\ & \frac{g_k\big(b_1, \gamma_{1,2}(b_1)\big) Z\big(b_1, \gamma_{1,2}(b_1)\big)}{b_1 \gamma_{1,2}(b_1)} + \int_{b_1}^{c} \frac{\partial_1 g_k\big(s, \gamma_1(s)\big)}{\gamma_1(s)}\, Z\big(\mathrm{d}s, \gamma_1(s)\big) \qquad (2.3)\\
&+ \int_a^{b_1} \frac{\big[g_k\big(s, \gamma_{1,2}(s)\big) - s\partial_1 g_k\big(s, \gamma_{1,2}(s)\big)\big]}{s^2 \gamma_{1,2}(s)} \Big[Z\big(s, \gamma_{1,2}(s)\big)\, \mathrm{d}s - sZ\big(\mathrm{d}s, \gamma_{1,2}(s)\big)\Big]\\
&+ \int_{\gamma_2(a)}^{\gamma_2(b_2)} \frac{\partial_2 g_k\big(\gamma_2^{-1}(t), t\big)}{\gamma_2^{-1}(t)}\, Z\big(\gamma_2^{-1}(t), \mathrm{d}t\big) + \int_{\gamma_{1,2}(b_1)}^{\gamma_{1,2}(a)} \frac{\partial_2 g_k\big(\gamma_{1,2}^{-1}(t), t\big)}{\gamma_{1,2}^{-1}(t)}\, Z\big(\gamma_{1,2}^{-1}(t), \mathrm{d}t\big)\\
&+ \iint_G \partial_1\partial_2 g_k(s,t)\, Z(\mathrm{d}s, \mathrm{d}t).
\end{aligned}
$$

If $\det(A) \neq 0$ *then the maximum likelihood estimator of the parameter vector* $\mathbf{m}$ *based on the observations* $\{Z(s,t) : (s,t) \in G\}$ *has the form* $\widehat{\mathbf{m}} = A^{-1}\zeta$ *and has a* p*-dimensional normal distribution with mean* $\mathbf{m}$ *and covariance matrix* A^{-1}*.*

Remark 2.2. Observe that all six terms of matrix A are non-negative definite matrices, so A is non-negative definite, too. Hence, to ensure $\det(A) \neq 0$ it suffices to have at least one positive definite among the terms, which fulfils e.g. if $g_1, \dots, g_p$ are linearly independent.

Remark 2.3. We remark that the weighted L^2-Riemann integrals of partial derivatives of the Wiener sheet (and of other L^2-processes) along a curve are defined in the sense of [5, Definition 4.1]. This means that if Z is an L^2-process given along an ε-neighborhood of a curve $\Gamma := \big\{(s, \gamma(s)) : s \in [a,b]\big\}$, where $\gamma : [a,b] \to \mathbb{R}$ is strictly monotone and $y : [a,b] \to \mathbb{R}$ is a function, then

$$\int_a^b y(s)\, Z(\mathrm{d}s, \gamma(s)) := \operatorname*{l.i.m.}_{h\to 0} \frac{1}{h} \int_a^b y(s)\big[Z(s+h,\gamma(s)) - Z(s,\gamma(s))\big]\,\mathrm{d}s,$$

$$\int_{\gamma(a)}^{\gamma(b)} y(\gamma^{-1}(t))\, Z(\gamma^{-1}(t), \mathrm{d}t) := \operatorname*{l.i.m.}_{h\to 0} \frac{1}{h} \int_{\gamma(a)}^{\gamma(b)} y(\gamma^{-1}(t))\big[Z(\gamma^{-1}(t), t+h) - Z(\gamma^{-1}(t), t)\big]\,\mathrm{d}t,$$

if the right hand sides exist.

Example 2.4. Consider the model

$$Z(s,t) = m_1(s^2+t^2) + m_2(s+t) + m_3(s\cdot t) + W(s,t), \qquad (s,t)\in G,$$

where $W(s,t),\ (s,t) \in [u-r, u+r]\times[v-r, v+r]$ is a standard Wiener sheet and G is a circle with center at (u,v) and radius r. Thus

$$\begin{aligned}
\gamma_{1,2}(s) &= v - \sqrt{r^2-(s-u)^2}, \quad & s&\in[u-r,u],\\
\gamma_1(s) &= v - \sqrt{r^2-(s-u)^2}, \quad & s&\in[u,u+r],\\
\gamma_2(s) &= v + \sqrt{r^2-(s-u)^2}, \quad & s&\in[u-r,u],\\
\gamma_0(s) &= v + \sqrt{r^2-(s-u)^2}, \quad & s&\in[u,u+r],\\
\gamma_{1,2}^{-1}(t) &= u - \sqrt{r^2-(t-v)^2}, \quad & t&\in[v-r,v],\\
\gamma_2^{-1}(t) &= u - \sqrt{r^2-(t-v)^2}, \quad & t&\in[v,v+r].
\end{aligned}$$

In this case the distinct elements of the symmetric matrix A defined by (2.2) are the following

$$\begin{aligned}
A_{1,1} &= \frac{(u^2+(v-r)^2)^2}{u(v-r)} + \int_{u-r}^{u} \frac{(\gamma_{1,2}^2(s)-s^2)^2}{s^2\gamma_{1,2}(s)}\mathrm{d}s + 4\int_u^{u+r}\frac{s^2}{\gamma_1(s)}\mathrm{d}s\\
&\quad + 4\int_{v-r}^{v}\frac{t^2}{\gamma_{1,2}^{-1}(t)}\mathrm{d}t + 4\int_v^{v+r}\frac{t^2}{\gamma_2^{-1}(t)}\mathrm{d}t,\\
A_{1,2} &= \frac{(u^2+(v-r)^2)(u+v-r)}{u(v-r)} + \int_{u-r}^{u}\frac{\gamma_{1,2}^2(s)-s^2}{s^2}\mathrm{d}s\\
&\quad + 2\int_u^{u+r}\frac{s}{\gamma_1(s)}\mathrm{d}s + 2\int_{v-r}^{v}\frac{t}{\gamma_{1,2}^{-1}(t)}\mathrm{d}t + 2\int_v^{v+r}\frac{t}{\gamma_2^{-1}(t)}\mathrm{d}t,\\
A_{2,2} &= \frac{(u+v-r)^2}{u(v-r)} + \int_{u-r}^{u}\frac{\gamma_{1,2}(s)}{s^2}\mathrm{d}s + \int_u^{u+r}\frac{1}{\gamma_1(s)}\mathrm{d}s
\end{aligned}$$

$$+\int_{v-r}^{v}\frac{1}{\gamma_{1,2}^{-1}(t)}\mathrm{d}t+\int_{v}^{v+r}\frac{1}{\gamma_2^{-1}(t)}\mathrm{d}t, \tag{2.4}$$

$$\begin{aligned}
A_{1,3}&=u^2+(v-r)^2+2r(u+2v)+r^2,\\
A_{2,3}&=u+v+2r,\\
A_{3,3}&=u(v-r)+r(v+2u)-\frac{3\pi r^2}{4},
\end{aligned}$$

while the components of $\zeta=\big(\zeta_1,\zeta_2,\zeta_3\big)^\top$ defined by (2.3) are

$$\begin{aligned}
\zeta_1=&\frac{\big(u^2+(v-r)^2\big)Z\big(b_1,\gamma_{1,2}(b_1)\big)}{u(v-r)}+\int_{u}^{u+r}\frac{2s}{\gamma_1(s)}\,Z\big(\mathrm{d}s,\gamma_1(s)\big)\\
&+\int_{v-r}^{v}\frac{2t}{\gamma_2^{-1}(t)}\,Z\big(\gamma_2^{-1}(t),\mathrm{d}t\big)\\
&+\int_{u-r}^{u}\frac{(\gamma_{1,2}^2(s)-s^2)}{s^2\gamma_{1,2}(s)}\Big[Z\big(s,\gamma_{1,2}(s)\big)\,\mathrm{d}s-sZ\big(\mathrm{d}s,\gamma_{1,2}(s)\big)\Big]\\
&+\int_{v}^{v+r}\frac{2t}{\gamma_{1,2}^{-1}(t)}\,Z\big(\gamma_{1,2}^{-1}(t),\mathrm{d}t\big),\\
\zeta_2=&\frac{\big(u+v-r\big)Z\big(b_1,\gamma_{1,2}(b_1)\big)}{u(v-r)}+\int_{u}^{u+r}\frac{1}{\gamma_1(s)}\,Z\big(\mathrm{d}s,\gamma_1(s)\big)\\
&+\int_{v-r}^{v}\frac{1}{\gamma_2^{-1}(t)}\,Z\big(\gamma_2^{-1}(t),\mathrm{d}t\big)\\
&+\int_{u-r}^{u}\frac{1}{s^2}\Big[Z\big(s,\gamma_{1,2}(s)\big)\,\mathrm{d}s-sZ\big(\mathrm{d}s,\gamma_{1,2}(s)\big)\Big]+\int_{v}^{v+r}\frac{1}{\gamma_{1,2}^{-1}(t)}\,Z\big(\gamma_{1,2}^{-1}(t),\mathrm{d}t\big),\\
\zeta_3=&Z\big(b_1,\gamma_{1,2}(b_1)\big)+\int_{u}^{u+r}Z\big(\mathrm{d}s,\gamma_1(s)\big)+\int_{v-r}^{v}Z\big(\gamma_2^{-1}(t),\mathrm{d}t\big)\\
&+\int_{v}^{v+r}Z\big(\gamma_{1,2}^{-1}(t),\mathrm{d}t\big)+\iint_G Z(\mathrm{d}s,\mathrm{d}t).
\end{aligned} \tag{2.5}$$

3. Simulation results

To illustrate the theoretical results of [2, 3, 4, 5] and of Theorem 2.1 we performed computer simulations using Matlab 2010a. In order to simulate a Wiener sheet $W(s,t),\ 0 \le s \le S,\ 0 \le t \le T$, we considered its Karhunen-Loève expansion, that is

$$W(s,t) \approx \sum_{j,k=1}^{n} \omega_{j,k} \frac{8\sqrt{ST}}{(\pi^2)(2k-1)(2j-1)} \sin\left(\frac{\pi(2j-1)t}{2T}\right) \sin\left(\frac{\pi(2k-1)s}{2S}\right), \quad (3.1)$$

where $\{\omega_{j,k} : 1 \le j,k \le n\}$ are independent standard normal random variables [8]. Figure 2 shows an approximation of the Wiener sheet with $n = 150$. Obviously, there are other methods of simulating a Wiener sheet e.g. with the help of discretization and using the independence of increments (see e.g. [15]). However, in order to calculate our estimators we need a method which provides us whole realizations of the sheet.

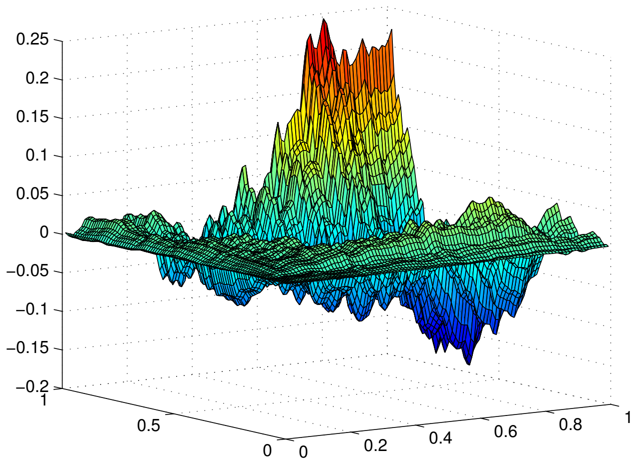

Figure 2: Simulation of Wiener sheet, $n = 150$

In each of the following examples 1000 independent samples of the driving Wiener sheet were simulated with n varying between 25 and 100 and the means of the estimates of the parameters and the empirical variances or covariance matrices of ζ defined by (1.3), (1.4) and (2.3), respectively, were calculated.

Example 3.1. Consider the model

$$Z(s,t) = W(s,t) + m(s^2 + t^2),$$

where $W(s,t),\ (s,t) \in G = [1,3]^2$ is a standard Wiener sheet (see Special case 1). Components of A and of the approximation of ζ can be given in the following closed form:

$$A = \frac{4b_2^3 - b_1^3}{3a_1} + \frac{4a_2^3 - a_1^3}{3b_1} + 2a_1b_1,$$

$$\begin{aligned}
&\zeta - mA \approx \\
&\sum_{j,k=1}^{n} \omega_{j,k} \frac{8\sqrt{ST}}{(\pi^2)(2k-1)(2j-1)} \left\{ \frac{a_1^2+b_1^2}{a_1b_1} \sin\left(\frac{\pi(2k-1)a_1}{2S}\right) \sin\left(\frac{\pi(2j-1)b_1}{2T}\right) \right. \\
&+ \frac{2}{b_1} \sin\left(\frac{\pi(2j-1)b_1}{2T}\right) \left[\frac{2S}{\pi(2k-1)} \left(\cos\left(\frac{\pi(2k-1)a_2}{2S}\right) - \cos\left(\frac{\pi(2k-1)a_1}{2S}\right) \right) \right. \\
&\left. + a_2 \sin\left(\frac{\pi(2k-1)a_2}{2S}\right) - a_1 \sin\left(\frac{\pi(2k-1)a_1}{2S}\right) \right] + \frac{2}{a_1} \sin\left(\frac{\pi(2k-1)a_1}{2S}\right) \\
&\times \left[\frac{2T}{\pi(2j-1)} \left(\cos\left(\frac{\pi(2j-1)b_2}{2T}\right) - \cos\left(\frac{\pi(2j-1)b_1}{2T}\right) \right) \right. \\
&\left.\left. + b_2 \sin\left(\frac{\pi(2j-1)b_2}{2T}\right) - b_1 \sin\left(\frac{\pi(2j-1)b_1}{2T}\right) \right] \right\},
\end{aligned}$$

where

$$\zeta = \frac{(a_1^2+b_1^2)Z(a_1,b_1)}{a_1b_1} + \int_{a_1}^{a_2} \frac{2u}{b_1} Z(\mathrm{d}u, b_1) + \int_{b_1}^{b_2} \frac{2v}{a_1} Z(a_1, \mathrm{d}v).$$

The theoretical parameter value is $m = 5$, while $A = 33.3333$. On Figure 3 the means of the estimates of the parameter and the estimated variances of ζ are plotted versus the level n of the approximation (3.1). In case of $n = 100$ we have $\widehat{m} = 5.0007$ and $\widehat{A} = 33.7233$.

Example 3.2. Consider the model

$$Z(s,t) = W(s,t) + m,$$

where $W(s,t), (s,t) \in G$, is a standard Wiener sheet and G is a set satisfying conditions of Special case 2 and Γ is a part of a circle with center at the origin, that is $\gamma(s) = \sqrt{r^2 - s^2}$ with some $r > 0$ and with $[a,b] \subset (0,r)$ [4, Example 1.2] . Then

$$A = \frac{1}{r^2}\left(\frac{\sqrt{r^2-a^2}}{a} + \frac{b}{\sqrt{r^2-b^2}}\right), \qquad c_1 = \frac{\sqrt{r^2-a^2}}{r^2a}, \qquad c_2 = \frac{b}{r^2\sqrt{r^2-b^2}},$$

$$y_1(s, \sqrt{r^2-s^2}) \equiv 0, \qquad y_2(s, \sqrt{r^2-s^2}) \equiv -\frac{1}{r^2}$$

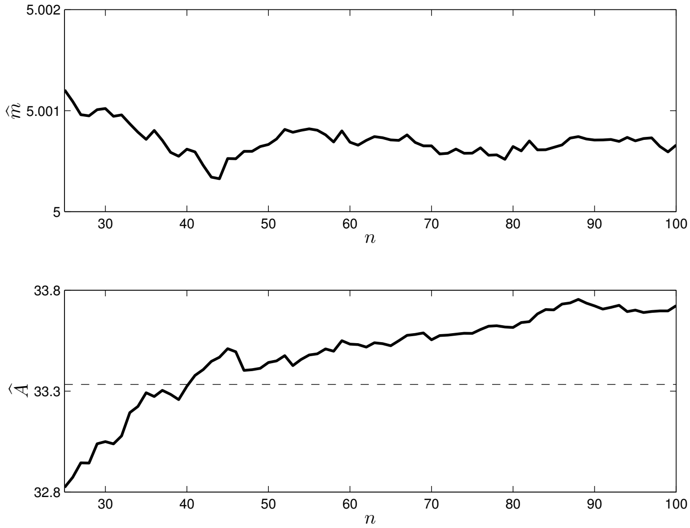

Figure 3: Means of the estimates of m and estimated variances of ζ in Example 3.1 for $25 \leq n \leq 100$

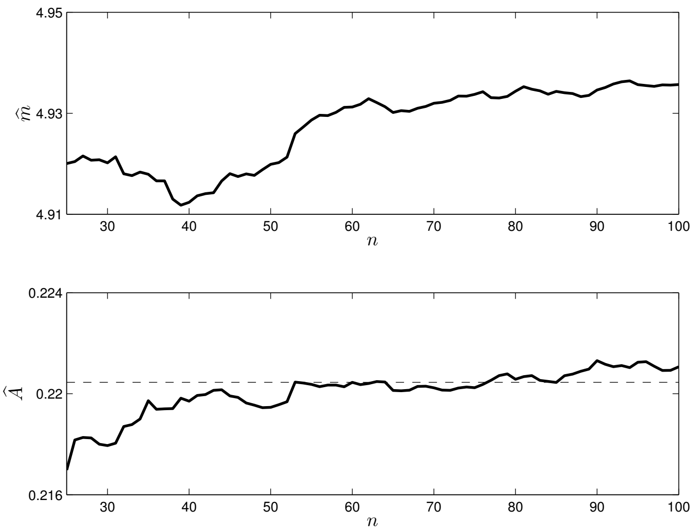

Figure 4: Means of the estimates of m and estimated variances of ζ in Example 3.2 for $25 \leq n \leq 100$

and

$$\zeta - mA \approx \sum_{j,k=1}^{n} \omega_{j,k} \frac{8\sqrt{ST}}{\pi^2 r^2 (2k-1)(2j-1)}$$

$$\times \left\{ \frac{\sqrt{r^2-a^2}}{a} \sin\left(\frac{\pi(2k-1)a}{2S}\right) \sin\left(\frac{\pi(2j-1)\sqrt{r^2-a^2}}{2T}\right) \right.$$
$$+ \frac{b}{\sqrt{r^2-b^2}} \sin\left(\frac{\pi(2k-1)b}{2S}\right) \sin\left(\frac{\pi(2j-1)\sqrt{r^2-b^2}}{2T}\right)$$
$$- \int_a^b \left\{ \frac{\pi(2k-1)s}{2S\sqrt{r^2-s^2}} \cos\left(\frac{\pi(2k-1)s}{2S}\right) \sin\left(\frac{\pi(2j-1)\sqrt{r^2-s^2}}{2T}\right) \right.$$
$$\left. \left. + \frac{\pi(2j-1)}{2T} \sin\left(\frac{\pi(2k-1)s}{2S}\right) \cos\left(\frac{\pi(2j-1)\sqrt{r^2-s^2}}{2T}\right) \right\} \mathrm{d}s \right\},$$

where ζ is defined by (1.4).

Let parameter value be $m = 5$ and choose $a = 1$, $b = 3$ and $r = 5$ yielding $A = 0.2205$. On Figure 4 the means of the estimates of the parameter and the estimated variances of ζ are plotted versus the level n of the approximation (3.1). In case of $n = 100$ we have $\widehat{m} = 4.9357$ and $\widehat{A} = 0.2213$.

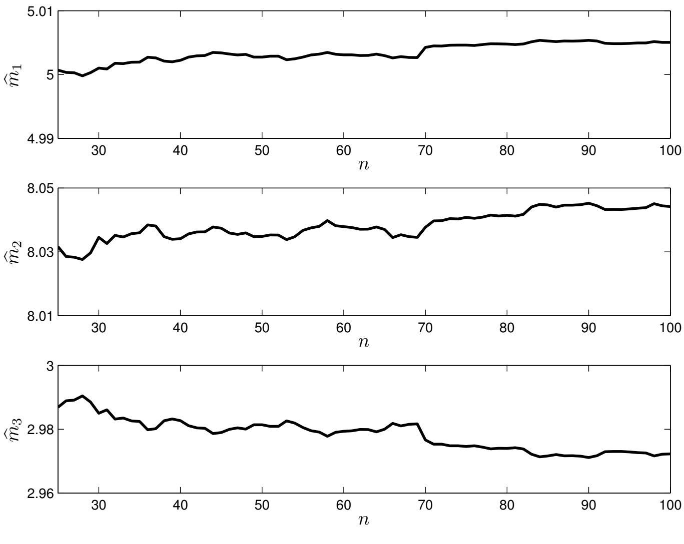

Figure 5: Means of the estimates of the components of $\mathbf{m}$ in Example 3.3 for $25 \le n \le 100$

Example 3.3. Consider the same model

$$Z(s,t) = m_1(s^2+t^2) + m_2(s+t) + m_3(s \cdot t) + W(s,t), \qquad (s,t) \in G,$$

as in Example 2.4, where $W(s,t)$, $(s,t) \in [0,8]^2$, is a standard Wiener sheet and G is a circle with center at $(6,6)$ and radius $r = 2$. In this case the entries of the matrix A defined by (2.4) and the approximation of the components of $\zeta = \big(\zeta_1, \zeta_2, \zeta_3\big)^\top$ defined by (2.5) can be calculated using numerical integration, where Matlab function `quad` is applied (recursive adaptive Simpson quadrature).

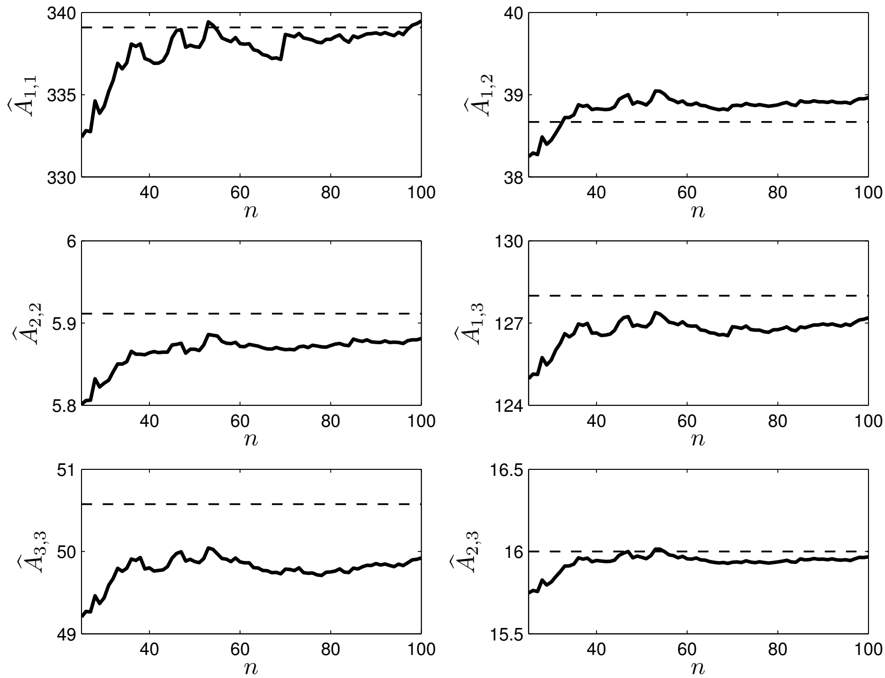

Figure 6: Estimated covariances of ζ in Example 3.3 for $25 \leq n \leq 100$

The theoretical parameter values are $m_1 = 5,\ m_2 = 8$ and $m_3 = 3$, while the theoretical covariance matrix of ζ equals

$$A = \begin{pmatrix} 339.0895 & 38.6688 & 128.0000 \\ 38.6688 & 5.9115 & 16.0000 \\ 128.0000 & 16.0000 & 50.5752 \end{pmatrix}.$$

On Figure 5 the means of the estimates of the three parameters, while on Figure 6 the estimated covariances of ζ are plotted versus the level n of the approximation (3.1). In case of $n = 100$ we have $(5.0050, 8.0442, 2.9723)$ for the mean and

$$\widehat{A} = \begin{pmatrix} 339.4824 & 38.9639 & 127.1914 \\ 38.9639 & 5.8811 & 15.9680 \\ 127.1914 & 15.9680 & 49.9207 \end{pmatrix}$$

for the covariance matrix.

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