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Random walk on half-plane half-comb structure

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Dedicated to Mátyás Arató on his eightieth birthday

Abstract

We study limiting properties of a random walk on the plane, when we have a square lattice on the upper half-plane and a comb structure on the lower half-plane, i.e., horizontal lines below the x-axis are removed. We give strong approximations for the components with random time changed Wiener processes. As consequences, limiting distributions and some laws of the iterated logarithm are presented. Finally, a formula is given for the probability that the random walk returns to the origin in $2N$ steps.

Keywords: Anisotropic random walk; Strong approximation; Wiener process; Local time; Laws of the iterated logarithm;

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1. Introduction and main results

The properties of a simple symmetric random walk on the square lattice $\mathbb{Z}^2$ have been extensively investigated in the literature since Dvoretzky and Erdős (1951), and Erdős and Taylor (1960). For these and further results we refer to Révész (2005).

Subsequent investigations concern random walks on other structures of the plane. For example, a simple random walk on the 2-dimensional comb lattice that is obtained from $\mathbb{Z}^2$ by removing all horizontal lines off the x-axis was studied by Weiss and Havlin (1986), Bertacchi and Zucca (2003), Bertacchi (2006), Csáki et al. (2009, 2011).

These are particular cases of the so-called anisotropic random walk on the plane. The general case is given by the transition probabilities

$$\begin{aligned}&\mathbf{P}(\mathbf{C}(N+1)=(k+1,j)|\mathbf{C}(N)=(k,j))\\&=\mathbf{P}(\mathbf{C}(N+1)=(k-1,j)|\mathbf{C}(N)=(k,j))=\frac{1}{2}-p_j,\end{aligned}$$

$$\begin{aligned}&\mathbf{P}(\mathbf{C}(N+1)=(k,j+1)|\mathbf{C}(N)=(k,j))\\&=\mathbf{P}(\mathbf{C}(N+1)=(k,j-1)|\mathbf{C}(N)=(k,j))=p_j,\end{aligned}$$

for $(k,j)\in\mathbb{Z}^2$, $N=0,1,2,\ldots$ with $0<p_j\le 1/2$ and $\min_{j\in\mathbb{Z}}p_j<1/2$. See Seshadri et al. (1979), Silver et al. (1977), Heyde (1982) and Heyde et al. (1982). The simple symmetric random walk corresponds to the case $p_j=1/4$, $j=0,\pm1,\pm2,\ldots$, while $p_0=1/4$, $p_j=1/2$, $j=\pm1,\pm2,\ldots$ defines random walk on the comb.

In this paper we combine the simple symmetric random walk with random walk on a comb, when $p_j=1/4$, $j=0,1,2,\ldots$ and $p_j=1/2$, $j=-1,-2,\ldots$, i.e., we have a square lattice on the upper half-plane, and a comb structure on the lower half-plane. We call this model Half-Plane Half-Comb (HPHC) and denote the random walk on it by $\mathbf{C}(N)=(C_1(N),C_2(N))$, $N=0,1,2,\ldots$.

For the second component of the HPHC walk a theorem of Heyde et al. (1982) gives in this particular case, the following strong limit theorem.

Theorem A. *On an appropriate probability space one can construct a sequence $C_2^{(N)}(\cdot)$ and a process $Y(\cdot)$ such that*

$$\lim_{N\to\infty}\sup_{0\le t\le M}\left|\frac{C_2^{(N)}([Nt])}{\sqrt{N}}-Y(t)\right|=0\quad a.s.,$$

where $Y(\cdot)$ is an oscillating Brownian motion (Wiener process) and $M>0$ is arbitrary.

Our first result is a strong approximation of both components of the random walk $\mathbf{C}(\cdot)$ by certain time-changed Wiener processes (Brownian motions) with rates of convergence. Before stating it, we need some definitions. Assume that we have two independent standard Wiener processes $W_1(t), W_2(t),\ t \geq 0$, and consider

$$\alpha_2(t) := \int_0^t I\{W_2(s) \geq 0\}\, ds,$$

i.e., the time spent by W_2 on the non-negative side during the interval $[0, t]$. The process $\gamma_2(t) := \alpha_2(t) + t$ is strictly increasing, hence we can define its inverse: $\beta_2(t) := (\gamma_2(t))^{-1}$. Observe that the processes $\alpha_2(t)$, $\beta_2(t)$ and $\gamma_2(t)$ are defined in terms of $W_2(t)$ so they are independent from $W_1(t)$. Moreover, it can be seen that $0 \leq \alpha_2(t) \leq t$, and $t/2 \leq \beta_2(t) \leq t$.

Theorem 1.1. *On an appropriate probability space for the HPHC random walk* $\{\mathbf{C}(N) = (C_1(N), C_2(N)); N = 0, 1, 2, \ldots\}$ *with* $p_j = 1/4,\ j = 0, 1, 2, \ldots,\ p_j = 1/2,\ j = -1, -2, \ldots$ *one can construct two independent standard Wiener processes* $\{W_1(t);\ t \geq 0\}$, $\{W_2(t);\ t \geq 0\}$ *such that, as* $N \to \infty$, *we have with any* $\varepsilon > 0$

$$|C_1(N) - W_1(N - \beta_2(N))| + |C_2(N) - W_2(\beta_2(N))| = O(N^{3/8+\varepsilon}) \quad a.s.$$

We note that the process $W_2(\beta_2(t))$ is identical with $Y(t)$ of Theorem A, i.e., an oscillating Brownian motion. It is a diffusion with speed measure (see Heyde et al., 1982)

$$m(dy) = \begin{cases} 4\,dy & \text{for} \quad y \geq 0, \\ 2\,dy & \text{for} \quad y < 0. \end{cases}$$

For more details on oscillating Brownian motion we refer to Keilson and Wellner (1978).

2. Preliminaries

First we want to redefine our walk $\mathbf{C}(\cdot)$ as follows: On a suitable probability space consider two independent simple symmetric (one-dimensional) random walks $S_1(\cdot)$, and $S_2(\cdot)$. We may assume that on the same probability space we have a sequence of independent geometric random variables $\{G_i,\ i = 1, 2, \ldots\}$, independent from $S_1(\cdot), S_2(\cdot)$, with distribution

$$\mathbf{P}(G_i = k) = \frac{1}{2^{k+1}},\ k = 0, 1, 2, \ldots$$

Now horizontal steps will be taken consecutively according to $S_1(\cdot)$, and vertical steps consecutively according to $S_2(\cdot)$ in the following way. Start from $(0, 0)$, take G_1 horizontal steps (possibly $G_1 = 0$) according to $S_1(\cdot)$, then take 1 vertical step. If this arrives to the upper half-plane ($S_2(1) = 1$), then take G_2 horizontal steps. If, however, the first vertical step is on the negative direction ($S_2(1) = -1$), then continue with another vertical step, and so on. In general, if the random walk is on the upper half-plane ($y \geq 0$) after a vertical step, then take a random number of horizontal steps according to the next (so far) unused G_j, independently from the previous steps. On the other hand, if the random walk is on the lower half-plane ($y < 0$) then continue with vertical steps according to $S_2(\cdot)$ until it reaches the x-axis, and so on.

Now we define the local times of a random walk and a Wiener process. Let $\{S(n);\, n = 0, 1, \ldots\}$ be a simple symmetric random walk on the line, i.e., $S(0) = 0$, $S(n) = X_1 + \ldots + X_n$, where $\{X_1, X_2, \ldots\}$ are i.i.d. random variables with $\mathbf{P}(X_i = 1) = \mathbf{P}(X_i = -1) = 1/2$. The local time is defined by

$$\xi(x, n) := \sum_{i=0}^{n} I\{S(i) = x\}, \quad x \in \mathbb{Z},\ n = 0, 1, \ldots,$$

where $I\{\cdot\}$ is the indicator function. The local time $\eta(x, t)$ of a Wiener process $W(\cdot)$ is defined via

$$\int_A \eta(x, t)\, dx = \lambda\{s :\, 0 \leq s \leq t,\, W(s) \in A\}$$

for any $x \in \mathbb{R}$, $t \geq 0$, where $A \subset \mathbb{R}$ is any Borel set and λ is the Lebesgue measure.

Now we state some results needed to prove our Theorem 1.1. First we quote a result of Révész (1981), that amounts to the first simultaneous strong approximation of a simple symmetric random walk and that of its local time process on the integer lattice $\mathbb{Z}$.

Lemma A. *On an appropriate probability space for a simple symmetric random walk $\{S(n);\, n = 0, 1, 2, \ldots\}$ with local time $\{\xi(x, n);\, x = 0, \pm 1, \pm 2, \ldots;\, n = 0, 1, 2, \ldots\}$ one can construct a standard Wiener process $\{W(t);\, t \geq 0\}$ with local time process $\{\eta(x, t);\, x \in \mathbb{R};\, t \geq 0\}$ such that, as $n \to \infty$, we have for any $\varepsilon > 0$*

$$S(n) - W(n) = O(n^{1/4+\varepsilon}) \quad a.s.$$

and

$$\sup_{x \in \mathbb{Z}} |\xi(x, n) - \eta(x, n)| = O(n^{1/4+\varepsilon}) \quad a.s.,$$

simultaneously.

The following strong invariance principle is given in Horváth (1998).

Lemma B. *On the probability space of* Lemma A, *for any $\varepsilon > 0$, as $n \to \infty$, we have*

$$\left| \sum_{k=0}^{n} g(S(k)) - \int_0^n g(W(t))\, dt \right|$$

$$= \left| \sum_{j=-\infty}^{\infty} g(j)\xi(j,n) - \int_{-\infty}^{\infty} g(x)\eta(x,n)\,dx \right| = O(n^{a/2+3/4+\varepsilon}) \quad a.s.,$$

where $g(t) \geq 0$, $t \in \mathbb{R}$ is a function such that for $k \in \mathbb{Z}$ we have $g(t) = g(k)$, $k \leq t < k+1$ and

$$g(t) \leq C(|t|^a + 1)$$

for some $C > 0$ and $0 \leq a$.

For $n \geq 1$ let

$$A(n) := \sum_{i=0}^{n-1} I\{S(i) \geq 0\} = \sum_{j=0}^{\infty} \xi(j, n-1), \tag{2.1}$$

i.e., the time spent by the random walk $S(\cdot)$ on the non-negative side during the first $n-1$ steps. Let furthermore

$$\alpha(t) = \int_0^t I\{W(s) \geq 0\}\,ds = \int_0^{\infty} \eta(x,t)\,dx.$$

Applying Lemma B with $g(t) = I\{t \geq 0\}$, $a = 0$, and taking into account that $A(n+1) - A(n) \leq 1$, we have the following consequence.

Corollary A. *On the probability space of* Lemma A, *for any $\varepsilon > 0$, as $n \to \infty$, we have almost surely*

$$A(n) - \alpha(n) = O(n^{3/4+\varepsilon}).$$

Concerning the increments of the Wiener process we quote the following result from Csörgő and Révész (1981).

Lemma C. *Let $0 < a_T \leq T$ be a non-decreasing function of T. Then, as $T \to \infty$, we have almost surely*

$$\sup_{0 \leq t \leq T - a_T} \sup_{s \leq a_T} |W(t+s) - W(t)| = O(a_T^{1/2}(\log(T/a_T) + \log\log T)).$$

Put

$$f_v(z,y)\,dz\,dy := \mathbf{P}(W(v) \in dz,\ \alpha(v) \in dy),$$

the joint density function of $(W(v), \alpha(v))$. For $f_v(z,y)$ the following two formulas are known in the literature. The first one is due to Karatzas and Shreve (1984), (see also Borodin and Salminen, 1996), the second one is given in Nikitin and Orsingher (2000).

Lemma D. *For $0 \leq y \leq v$ we have*

$$f_v(z,y) = \begin{cases} \int_0^{\infty} \frac{s(s+z)}{\pi y^{3/2}(v-y)^{3/2}} \exp\left(-\frac{s^2}{2(v-y)} - \frac{(s+z)^2}{2y}\right) ds, & z \geq 0, \\ \\ \int_0^{\infty} \frac{s(s-z)}{\pi y^{3/2}(v-y)^{3/2}} \exp\left(-\frac{s^2}{2y} - \frac{(s-z)^2}{2(v-y)}\right) ds, & z < 0, \end{cases}$$

$$f_v(z,y)=\begin{cases}\int_{v-y}^{v}\frac{z\exp\left(-\frac{z^2}{2(v-s)}\right)}{2\pi s^{3/2}(v-s)^{3/2}}\,ds, & z\geq 0,\\ \\ \int_{y}^{v}\frac{|z|\exp\left(-\frac{z^2}{2(v-s)}\right)}{2\pi s^{3/2}(v-s)^{3/2}}\,ds, & z<0.\end{cases}$$

3. Proof of Theorem 1.1

Start with the construction of HPHC given in Section 2. Let H_N and V_N, the number of horizontal and vertical steps, respectively of the two-dimensional random walk $\mathbf{C}(\cdot)$ during the first N steps, i.e., $H_N+V_N=N$. Consider the two independent simple symmetric random walks $S_1(\cdot)$ and $S_2(\cdot)$ and the sequence of i.i.d. geometric random variables, which is indepedent from these two walks, as it was described in Section 2. Define $A_2(n)$ as in (2.1), in terms of $S_2(\cdot)$, i.e., $A_2(n)=\sum_{j=0}^{\infty}\xi_2(j,n-1)$, where $\xi_2(\cdot,\cdot)$ is the local time of $S_2(\cdot)$. Assume furthermore that on the same probability space we have strong approximations of (S_1,ξ_1) by (W_1,η_1) and that of (S_2,ξ_2) by (W_2,η_2) as described in Lemma A, where W_1 and W_2 are two independent Wiener processes on the line, and η_1 and η_2 are their respective local times.

Then, with $V_N=n$,

$$\sum_{j=1}^{A_2(n)}G_j\leq H_N\leq\sum_{j=1}^{A_2(n)+1}G_j$$

and since one term in the above sum is $O(\log N)$ a.s., and $EG_j=1$, with finite variance, we have

$$H_N=A_2(n)+O(A_2(n)^{1/2+\varepsilon})=A_2(n)+O(N^{1/2+\varepsilon})\quad a.s.,$$

as $N\to\infty$. Hence, using Corollary A, we have almost surely, as $N\to\infty$,

$$\alpha_2(n)+n=A_2(n)+O(N^{3/4+\varepsilon})+V_N=H_N+V_N+O(N^{3/4+\varepsilon})=N+O(N^{3/4+\varepsilon}).$$

Consequently,

$$V_N=n=\beta_2(\alpha_2(n)+n)=\beta_2(N+O(N^{3/4+\varepsilon}))=\beta_2(N)+O(N^{3/4+\varepsilon})$$

and

$$H_N=N-\beta_2(N)+O(N^{3/4+\varepsilon}).$$

Using Lemma C, this gives almost surely, as $N\to\infty$,

$$C_1(N)=S_1(H_N)=W_1(H_N)+O(H_N^{1/4+\varepsilon})=W_1(N-\beta_2(N))+O(N^{3/8+\varepsilon})$$

and

$$C_2(N)=S_2(V_N)=W_2(\beta_2(N))+O(N^{3/8+\varepsilon}),$$

proving Theorem 1.1. $\square$

Remark 3.1. In the above argument we used the fact, that for $u, v > 0$, $\beta(u+v) - \beta(u) \le v$. To see this recall that $\beta(t)$ is the inverse of $\gamma(t) = \alpha(t) + t$. Hence

$$v = \gamma(\beta(u+v)) - \gamma(\beta(u)) = \alpha(\beta(u+v)) + \beta(u+v) - \alpha(\beta(u)) - \beta(u) \ge \beta(u+v) - \beta(u),$$

as $\alpha(t)$ is nondecreasing.

4. Limiting densities and consequences

First we give an integral expression for the joint density of the vector $(W_1(t-\beta(t)), W_2(\beta(t)))$, using Lemma D. Here, and throughout this section, $\beta(t)$ stands for $\beta_2(t)$, hence it is independent from W_1. The joint density of $W_1(t-\beta(t))$, $W_2(\beta(t))$, $\beta(t)$ is given by

$$\begin{aligned}&\mathbf{P}(W_1(t-\beta(t)) \in du,\, W_2(\beta(t)) \in dz,\, \beta(t) \in dv)\\ &= \frac{1}{\sqrt{2\pi(t-v)}} \exp\left(-\frac{u^2}{2(t-v)}\right) f_v(z, t-v)\, du\, dz\, dv.\end{aligned}$$

From this we get

Lemma 4.1.

$$\begin{aligned}g_t(u,z)\,du\,dz &:= \mathbf{P}(W_1(t-\beta(t)) \in du,\, W_2(\beta(t)) \in dz)\\ &= \left(\int_{t/2}^{t} \frac{1}{\sqrt{2\pi(t-v)}} \exp\left(-\frac{u^2}{2(t-v)}\right) f_v(z,t-v)\, dv\right) du\, dz.\end{aligned}$$

The marginal density of $W_1(t-\beta(t))$ is given by

Lemma 4.2.

$$g_t^{(1)}(u)\,du := \mathbf{P}(W_1(t-\beta(t)) \in du) = \frac{1}{\pi\sqrt{2\pi t}} \exp\left(-\frac{u^2}{2t}\right) K_0\left(\frac{u^2}{2t}\right) du,$$

where $K_0(\cdot)$ is the modified Bessel function of the second kind.

Proof.

$$\mathbf{P}(W_1(t-\beta(t)) \in du) = \int_{t/2}^{t} \mathbf{P}(W_1(t-v) \in du,\, \beta(t) \in dv)$$

$$= \left(\int_{t/2}^{t} \frac{1}{\sqrt{2\pi(t-v)}} \exp\left(-\frac{u^2}{2(t-v)}\right) \frac{1}{\pi\sqrt{(t-v)(2v-t)}}\, dv\right) du$$

$$= \frac{1}{\pi\sqrt{2\pi}} \exp\left(-\frac{u^2}{t}\right) \int_0^\infty \frac{1}{\sqrt{y^2 t + y u^2}} e^{-y}\, dy\, du = \frac{1}{\pi\sqrt{2\pi t}} \exp\left(-\frac{u^2}{2t}\right) K_0\left(\frac{u^2}{2t}\right) du,$$

where the substitution

$$y = u^2 \left(\frac{1}{2(t-v)} - \frac{1}{t}\right)$$

was made and the formula

$$\int_0^\infty \frac{e^{-px}\, dx}{\sqrt{x(x+a)}} = e^{ap/2} K_0\left(\frac{ap}{2}\right)$$

was used (see Gradsteyn and Ryzhik, 1994, 3.364.3). □

For the marginal density of $W_2(\beta(t))$ as follows, we refer to Heyde et al. (1982).

Lemma E.

$$g_t^{(2)}(z)\, dz = \mathbf{P}(W_2(\beta(t)) \in dz) = \begin{cases} 2\sqrt{\frac{2}{\pi t}}(\sqrt{2}-1)e^{-z^2/t}\, dz, & z \geq 0 \\ \sqrt{\frac{2}{\pi t}}(\sqrt{2}-1)e^{-z^2/2t}\, dz, & z < 0. \end{cases}$$

As a consequence of these Lemmas, we now obtain the joint and marginal limiting distributions of the HPHC random walk.

Corollary 4.3.

$$\lim_{N\to\infty} \mathbf{P}\left(\frac{C_1(N)}{\sqrt{N}} \leq x,\ \frac{C_2(N)}{\sqrt{N}} \leq y\right) = \int_{-\infty}^{x} \int_{-\infty}^{y} g_1(u,z)\, du\, dz,$$

$$\lim_{N\to\infty} \mathbf{P}\left(\frac{C_1(N)}{\sqrt{N}} \leq x\right) = \int_{-\infty}^{x} g_1^{(1)}(u)\, du,$$

$$\lim_{N\to\infty} \mathbf{P}\left(\frac{C_2(N)}{\sqrt{N}} \leq y\right) = \int_{-\infty}^{y} g_1^{(2)}(z)\, dz.$$

Corollary 4.4. *The following laws of the iterated logarithm hold.*

(i) $$\limsup_{t\to\infty} \frac{W_1(t-\beta(t))}{\sqrt{t\log\log t}} = \limsup_{N\to\infty} \frac{C_1(N)}{\sqrt{N\log\log N}} = 1 \quad a.s.,$$

(ii) $$\liminf_{t\to\infty} \frac{W_1(t-\beta(t))}{\sqrt{t\log\log t}} = \liminf_{N\to\infty} \frac{C_1(N)}{\sqrt{N\log\log N}} = -1 \quad a.s.,$$

(iii) $$\limsup_{t\to\infty} \frac{W_2(\beta(t))}{\sqrt{t\log\log t}} = \limsup_{N\to\infty} \frac{C_2(N)}{\sqrt{N\log\log N}} = 1 \quad a.s.,$$

(iv) $\liminf_{t\to\infty} \frac{W_2(\beta(t))}{\sqrt{t\log\log t}} = \liminf_{N\to\infty} \frac{C_2(N)}{\sqrt{N\log\log N}} = -\sqrt{2} \quad a.s.$

Proof. We give short proofs in the case of W_1 and W_2. The results for C_1 and C_2 then follow from Theorem 1.1. In the proof we repeatedly use the inequality

$$\frac{t}{2} \le \beta(t) \le t.$$

Proof of (i) and (ii). By the law of the iterated logarithm for W_1 we have for all large enough t

$$\begin{aligned} W_1(t-\beta(t)) &\le (1+\varepsilon)(2(t-\beta(t))\log\log(t-\beta(t)))^{1/2} \\ &\le (1+\varepsilon)(t\log\log t)^{1/2}, \end{aligned}$$

which gives an upper bound in (i).

To give a lower bound in (i), for any sufficiently small $\delta > 0$ define the events

$$A_n = \{W_1(u_n) \ge (1-\delta)(2u_n\log\log u_n)^{1/2}\}, \quad B_n = \{\alpha(u_n(1+\delta)) > u_n\},$$

$n = 1, 2, \ldots$. Then, with some sequence $\{u_n\}$ ($u_n = a^n$ with sufficiently large a will do), we have

$$\mathbf{P}(A_n\ i.o.) = 1, \qquad \mathbf{P}(B_n) > c > 0.$$

It follows from Klass (1976) that

$$\mathbf{P}(A_nB_n\ i.o.) \ge c > 0.$$

By the 0-1 law this probability is equal to 1. Let t_n be defined by

$$u_n = t_n - \beta(t_n) = \alpha(\beta(t_n)).$$

Since

$$B_n = \{\alpha(u_n(1+\delta)) > \alpha(\beta(t_n))\},$$

B_n implies

$$u_n \ge \frac{\beta(t_n)}{1+\delta} \ge \frac{t_n}{2(1+\delta)}.$$

Hence A_nB_n implies

$$W_1(t-\beta(t_n)) \ge (1-\delta)\left(\frac{t_n\log\log t_n}{1+\delta}\right)^{1/2}.$$

Since $\delta > 0$ is arbitrary, this gives a lower bound in (i).

The proof of (ii) follows by symmetry.

Proof of (iii). We have infinitely often with probability 1

$$W_2(\beta(t)) \geq (1-\varepsilon)(2\beta(t)\log\log t)^{1/2} \geq (1-\varepsilon)(t\log\log t)^{1/2},$$

giving a lower bound in (iii).

To give an upper bound, we use the formula for the distribution of the supremum of $W_2(\beta(t))$ given in Corollary 2 of Keilson and Wellner (1978), which in our case is equivalent to

$$\mathbf{P}(\sup_{0\leq s\leq t} W_2(\beta(s)) > y)$$

$$= \frac{2\sqrt{2}}{1+\sqrt{2}} \sum_{k=0}^{\infty} \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^k \left(1-\Phi\left(\frac{(2k+1)y\sqrt{2}}{\sqrt{t}}\right)\right).$$

From this it is easy to give the estimation

$$\mathbf{P}(\sup_{0\leq s\leq t} W_2(\beta(s)) > y) \leq c\exp\left(-\frac{y^2}{t}\right)$$

with some constant c, from which the upper estimation in (iii) follows by the usual procedure.

Proof of (iv). The lower estimation is easy. Namely we have

$$W_2(\beta(t)) \geq -(1+\varepsilon)(2\beta(t)\log\log\beta(t))^{1/2} \geq -(1+\varepsilon)(2t\log\log t)^{1/2}.$$

It remains to prove an upper estimation in (iv). By the law of the iterated logarithm for W_2

$$W_2(v) \leq -((2-\varepsilon)v\log\log v)^{1/2} \tag{4.1}$$

almost surely for infinitely many v tending to infinity. Let $\zeta(v)$ be the last zero of W_2 before v, i.e.,

$$\zeta(v) = \max\{u \leq v : W_2(u) = 0\}.$$

By Theorem 1 of Csáki and Grill (1988), for large v satisfying (4.1) we have $\zeta(v) \leq \varepsilon v$, and hence also $\alpha(v) \leq \zeta(v) \leq \varepsilon v$. Now put $v = \beta(t)$, i.e., $\alpha(v)+v = t \leq (1+\varepsilon)v$, from which $v = \beta(t) \geq t/(1+\varepsilon)$. Hence

$$W_2(v) = W_2(\beta(t)) \leq -\left(\frac{(2-\varepsilon)t\log\log t}{1+\varepsilon}\right)^{1/2}.$$

Since $\varepsilon > 0$ is arbitrary, this gives an upper bound in (iv).

This completes the proof of Corollary 4.4. □

Some related distributions can also be determined. For example, we can obtain the following result for the supremum of the first component.

Lemma 4.5.

$$\begin{aligned}&\mathbf{P}(\sup_{0\le s\le t}|W_1(s-\beta(s))|\le u)\\&=\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\exp\left(-\frac{(2j+1)^2\pi^2 t}{32u^2}\right)I_0\left(\frac{(2j+1)^2\pi^2 t}{32u^2}\right),\end{aligned}$$

where I_0 is the modified Bessel function of the first kind given by

$$I_0(z)=\sum_{k=0}^{\infty}\frac{z^{2k}}{4^k(k!)^2}.$$

Proof.

$$\mathbf{P}(\sup_{0\le s\le t}|W_1(s-\beta(s))|\le u)=\int_{t/2}^{t}\mathbf{P}(\sup_{z\le t-v}|W_1(z)|\le u)\mathbf{P}(\beta(t)\in dv)$$

$$=\int_{t/2}^{t}\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\exp\left(-\frac{(2j+1)^2\pi^2(t-v)}{8u^2}\right)\frac{1}{\pi\sqrt{(t-v)(2v-t)}}\,dv,$$

and using 3.384.2 and 9.235.1 of Gradsteyn and Ryzhik (1994), and some calculations, we obtain Lemma 4.5. □

Corollary 4.6.

$$\begin{aligned}&\lim_{N\to\infty}\mathbf{P}\left(\frac{\sup_{0\le k\le N}|C_1(k)|}{\sqrt{N}}\le u\right)\\&=\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\exp\left(-\frac{(2j+1)^2\pi^2}{32u^2}\right)I_0\left(\frac{(2j+1)^2\pi^2}{32u^2}\right),\end{aligned}$$

5. Return probabilities

We give the probability that the random walk returns to the origin in 2N steps.

Theorem 5.1. *For $N\ge 1$*

$$\begin{aligned}&\mathbf{P}(\mathbf{C}(2N)=(0,0))\\&=\frac{1}{2^{4N}}\left(\binom{2N}{N}+\sum_{n=1}^{N}\sum_{k=1}^{n}\sum_{j=1}^{k}\binom{2N-2n}{N-n}a_j a_{n+1-j}\left(b(n,2k)+b(n,2k-1)\right)\right),\end{aligned}$$

where for $i=1,2,\dots,\quad n=1,2,\dots,N,\quad \ell=1,2,\dots,$

$$a_i=\frac{1}{2i-1}\binom{2i-1}{i},\qquad b(n,\ell)=\binom{2N-2n+\ell}{\ell}2^{2n-\ell}.$$

Proof. For $n \geq 1$ let

$$P(2n,r) = \mathbf{P}(S_2(2n) = 0, A_2(2n) = r), \quad Q(2n,r) = 2^{2n}P(2n,r).$$

Obviously $P(2n,r) = 0$ if $r > 2n$ or $r \leq 0$. Furthermore it is easy to see, that

$$P(2n,1) = \frac{1}{2n-1}\binom{2n-1}{n}\frac{1}{2^{2n}} = \frac{1}{2(2n-1)}\binom{2n}{n}\frac{1}{2^{2n}},$$
$$P(2n,2n) = \frac{1}{n+1}\binom{2n}{n}\frac{1}{2^{2n}}.$$

For $n = 1, 2\ldots, \quad r = 2, 3, \ldots 2n$, we have the following recursion for $P(2n,r)$.

$$\begin{aligned}
P(2n,r) &= \sum_{i=1}^{n} \mathbf{P}(S(1) < 0, \ldots, S(2i-1) < 0, S(2i) = 0)P(2n-2i, r-1) \\
&\quad + \sum_{i=1}^{n} \mathbf{P}(S(1) > 0, \ldots, S(2i-1) > 0, S(2i) = 0)P(2n-2i, r-2i) \\
&= \sum_{i=1}^{n} \frac{1}{2i-1}\binom{2i-1}{i}\frac{1}{2^{2i}}P(2n-2i, r-1) \\
&\quad + \sum_{i=1}^{n} \frac{1}{2i-1}\binom{2i-1}{i}\frac{1}{2^{2i}}P(2n-2i, r-2i),
\end{aligned}$$

where we define $P(0,0) = 1$. □

Now we need the following lemma.

Lemma 5.2. *For $n = 1, 2, \ldots, \quad k = 1, 2, \ldots, n,$ we have*

$$Q(2n, 2k-1) = Q(2n, 2k) \tag{5.1}$$

and

$$\begin{aligned}
Q(2n,2k) &= \sum_{j=1}^{k} a_j a_{n+1-j} \\
&= \sum_{j=1}^{k} \frac{1}{2j-1}\binom{2j-1}{j}\frac{1}{2n+1-2j}\binom{2n+1-2j}{n+1-j}.
\end{aligned} \tag{5.2}$$

Remark 5.3. It is obvious that

$$Q(2n+2, 1) = Q(2n, 2n).$$

Furthermore, we can conveniently reformulate the second statement as

$$Q(2n, 2k) = Q(2n, 2k-2) + a_k a_{n+1-k}.$$

In particular

$$Q(2n, 2n) = Q(2n+2, 2) = a_{n+1}.$$

Proof. We prove Lemma 5.2 with simultaneous induction. Clearly, for $n = 1$ and $k = 1$ both of our statements are correct. We suppose that (5.1) and (5.2) hold for all $m < n$ and $j \leq 2k - 2$. First we prove (5.1). By our recursion formula and the induction hypothesis we have

$$Q(2n, 2k-1) = \sum_{j=1}^{n-k+1} a_j Q(2n-2j, 2k-2) + \sum_{j=1}^{k-1} a_j\, Q(2n-2j, 2k-2j-1)$$
$$= \sum_{j=1}^{n-k+1} a_j Q(2n-2j, 2k-2) + \sum_{j=1}^{k-1} a_j\, Q(2n-2j, 2k-2j).$$

Moreover,

$$Q(2n, 2k) = \sum_{j=1}^{n-k} a_j Q(2n-2j, 2k-1) + \sum_{j=1}^{k-1} a_j\, Q(2n-2j, 2k-2j)$$
$$= \sum_{j=1}^{n-k} a_j Q(2n-2j, 2k) + \sum_{j=1}^{k-1} a_j\, Q(2n-2j, 2k-2j).$$

Then

$$Q(2n, 2k) - Q(2n, 2k-1)$$
$$= \sum_{j=1}^{n-k} a_j \left(Q(2n-2j, 2k) - Q(2n-2j, 2k-2)\right) - a_{n-k+1}\, Q(2k-2, 2k-2)$$
$$= \sum_{j=1}^{n-k} a_j a_k a_{n+1-k-j} - a_{n-k+1}\, a_k = a_k \sum_{j=1}^{n-k} a_j a_{n+1-k-j} - a_{n-k+1}\, a_k$$
$$= a_k Q(2n-2k, 2n-2k) - a_{n-k+1}\, a_k = a_k\, a_{n-k+1} - a_k\, a_{n-k+1} = 0,$$

which proves (5.1). To prove (5.2), consider

$$Q(2n, 2k) - Q(2n, 2k-2) = \sum_{j=1}^{n-k} a_j Q(2n-2j, 2k) + \sum_{j=1}^{k-1} a_j Q(2n-2j, 2k-2j)$$
$$- \left(\sum_{j=1}^{n+1-k} a_j Q(2n-2j, 2k-2) + \sum_{j=1}^{k-2} a_j Q(2n-2j, 2k-2-2j)\right)$$
$$= \sum_{j=1}^{n-k} a_j \left(Q(2n-2j, 2k) - Q(2n-2j, 2k-2)\right) - a_{n+1-k} Q(2k-2, 2k-2)$$
$$+ \sum_{j=1}^{k-2} a_j \left(Q(2n-2j, 2k-2j) - Q(2n-2j, 2k-2-2j)\right) + a_{k-1} Q(2n-2k+2, 2)$$

$$
\begin{aligned}
&= \sum_{j=1}^{n-k} a_j a_k a_{n-k+1-j} - a_{n+1-k} a_k + \sum_{j=1}^{k-2} a_j a_{k-j} a_{n+1-k} + a_{k-1}\, a_{n-k+1} \\
&= a_k \sum_{j=1}^{n-k} a_j a_{n-k+1-j} - a_{n+1-k} a_k + a_{n+1-k} \sum_{j=1}^{k-2} a_j a_{k-j} + a_{k-1} a_{n-k+1} \\
&= a_k Q(2n-2k, 2n-2k) - a_{n+1-k} a_k + a_{n+1-k} Q(2k-2, 2k-4) + a_{k-1} a_{n-k+1} \\
&= a_k a_{n-k+1} - a_{n+1-k} a_k + a_{n+1-k} \left(Q(2k-2, 2k-2) - a_1 a_{k-1} \right) + a_{k-1} a_{n-k+1} \\
&= a_{n+1-k} a_k - a_{n+1-k} a_{k-1} + a_{k-1} a_{n+1-k} = a_k a_{n+1-k},
\end{aligned}
$$

proving (5.2). □

Returning to the proof of Theorem 5.1, let V_N and H_N be the number of vertical and horizontal steps, resp. as in the proof of Theorem 1.1. We have

$$
\begin{aligned}
&\mathbf{P}(\mathbf{C}(2N) = (0,0)) = \mathbf{P}(H_{2N} = 2N, S_1(2N) = 0) \\
&+ \sum_{n=1}^{N} \sum_{r=1}^{2n} \mathbf{P}(H_{2N} = 2N - 2n | S_2(2n) = 0, A_2(2n) = r) \\
&\times P(2n, r) \mathbf{P}(S_1(2N - 2n) = 0).
\end{aligned}
$$

For $n \geq 1$ we show that

$$
\mathbf{P}(H_{2N} = 2N - 2n | S_2(2n) = 0, A_2(2n) = r) = \binom{2N - 2n + r}{r} \frac{1}{2^{2N-2n+r}}.
$$

Under the condition $S_2(2n) = 0, A_2(2n) = r$, we have

$$
H_{2N} = \sum_{i=1}^{r} G_i + G,
$$

where G_i are i.i.d. geometric variables with

$$
\mathbf{P}(G_i = k) = \frac{1}{2^{k+1}}, \quad k = 0, 1, \ldots
$$

and G denotes the number of horizontal steps after the $2n$-th vertical step up to the total number of $2N$ steps. So

$$
\begin{aligned}
&\mathbf{P}(H_{2N} = 2N - 2n | S_2(2n) = 0, A_2(2n) = r) = \sum_{k=0}^{2N-2n} \mathbf{P}\left(\sum_{i=1}^{r} G_i = k \right) \frac{1}{2^{2N-2n-k}} \\
&= \sum_{k=0}^{2N-2n} \binom{k + r - 1}{k} \frac{1}{2^{k+r}} \frac{1}{2^{2N-2n-k}} = \frac{1}{2^{2N-2n+r}} \sum_{k=0}^{2N-2n} \binom{k + r - 1}{k} \\
&= \binom{2N - 2n + r}{r} \frac{1}{2^{2N-2n+r}}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\mathbf{P}(\mathbf{C}(2N) = (0,0)) \\
&= \frac{1}{2^{4N}}\binom{2N}{N} + \sum_{n=1}^{N}\sum_{r=1}^{2n} P(2n,r)\binom{2N-2n}{N-n}\frac{1}{2^{2N-2n}}\binom{2N-2n+r}{r}\frac{1}{2^{2N-2n+r}} \\
&= \frac{1}{2^{4N}}\binom{2N}{N} + \sum_{n=1}^{N}\sum_{r=1}^{2n} Q(2n,r)\binom{2N-2n}{N-n}\frac{1}{2^{2N}}\binom{2N-2n+r}{r}\frac{1}{2^{2N-2n+r}}
\end{aligned}
$$

and using Lemma 5.2 completes the proof of our Theorem 5.1.

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