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Moment-type estimates with asymptotically optimal structure for the accuracy of the normal approximation*

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Dedicated to Mátyás Arató on his eightieth birthday

Abstract

For the uniform distance Δ_n between the distribution function of the standard normal law and the distribution function of the standardized sum of independent random variables $X_1,\ldots,X_n$ with $\mathsf{E}X_j=0$, $\mathsf{E}|X_j|=\beta_{1,j}$, $\mathsf{E}X_j^2=\sigma_j^2$, $j=1,\ldots,n$, for all $n\geqslant 1$ the bounds

$$\Delta_n\leqslant\frac{2\ell_n}{3\sqrt{2\pi}}+\frac{1}{2\sqrt{2\pi}B_n^3}\sum_{j=1}^n\beta_{1,j}\,\sigma_j^2+R(\ell_n),$$

$$\Delta_n\leqslant\inf_{c\geqslant 2/(3\sqrt{2\pi})}\left\{c\ell_n+\frac{K(c)}{B_n^3}\sum_{j=1}^n\sigma_j^3+R_c(\ell_n)\right\},$$

are proved, where $B_n^2=\sum_{j=1}^n\sigma_j^2$, $\ell_n=B_n^{-3}\sum_{j=1}^n\mathsf{E}|X_j|^3$, $R(\ell_n)\leqslant 6\ell_n^{5/3}$, $R_c(\ell_n)\leqslant\min\{3\ell_n^{7/6},A(c)\ell_n^{4/3}\}$ in the general case and $R(\ell_n)\leqslant 3\ell_n^2$, $R_c(\ell_n)\leqslant\min\{2\ell_n^{3/2},A(c)\ell_n^2\}$, if $X_1,\ldots,X_n$ are identically distributed, $A(c)>0$ being a decreasing function of c such that $A(c)\to\infty$ as $c\to 2/(3\sqrt{2\pi})$. Moreover, the function $K(c)$ is optimal for each $c\geqslant 2/(3\sqrt{2\pi})$. In particular, $K\left((\sqrt{10}+3)/(6\sqrt{2\pi})\right)=0$, $K\left(2/(3\sqrt{2\pi})\right)=\sqrt{(2\sqrt{3}-3)/(6\pi)}=0.1569\ldots$ It is shown that in the first inequality the coefficients $2/(3\sqrt{2\pi})$ and $\left(2\sqrt{2\pi}\right)^{-1}$

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are optimal and the lower bound $2/(3\sqrt{2\pi})$ for c in the second inequality is unimprovable. These results sharpen the well-known estimates due to H. Prawitz (1975), V. Bentkus (1991, 1994) and G. P. Chistyakov (1996, 2001). Also, an analog of the first inequality is proved for the case where the summands possess only the moments of order $2+\delta$ with some $0<\delta<1$. As a by-product, the von Mises inequality for lattice distributions is sharpened and generalized.

Keywords: central limit theorem, convergence rate estimate, normal approximation, Berry–Esseen inequality, asymptotically exact constant, characteristic function

MSC: 60F05, 60E10

1. Introduction

For $\delta\in[0,1]$ let $\mathcal{F}_{2+\delta}$ be the class of distribution functions (d.f.'s) $F(x)$ satisfying the conditions

$$\int\limits_{-\infty}^{+\infty} x\,dF(x)=0,\qquad \int\limits_{-\infty}^{+\infty}|x|^{2+\delta}\,dF(x)<\infty.$$

For $h>0$ let $\mathcal{F}_{2+\delta}^h$ denote the class of all lattice d.f.'s from $\mathcal{F}_{2+\delta}$ with span h. For $F\in\mathcal{F}_{2+\delta}$ set

$$\beta_r=\beta_r(F)=\int\limits_{-\infty}^{+\infty}|x|^r\,dF(x),\quad 0<r\leqslant 2+\delta,\quad \sigma^2=\beta_2.$$

For $\delta=0$ by $\mathcal{F}_2$ we mean the class of all d.f.'s with zero mean and finite second moment. It is easy to see that $\mathcal{F}_{2+\delta_1}\subset\mathcal{F}_{2+\delta_2}$ for any $0\leqslant\delta_1<\delta_2\leqslant 1$, and $\sigma^{2+\delta}\leqslant\beta_{2+\delta}$ for all $F\in\mathcal{F}_{2+\delta}$ and $\delta\in[0,1]$ by the Lyapounov inequality.

Let $X_1,\ldots,X_n$ be independent random variables (r.v.'s) defined on some probability space $(\Omega,\mathcal{A},\mathsf{P})$ with the corresponding d.f.'s $F_1,\ldots,F_n\in\mathcal{F}_{2+\delta}$. Denote

$$\sigma_j^2=\mathsf{E}X_j^2,\quad \beta_{r,j}=\mathsf{E}|X_j|^r,\quad 0<r\leqslant 2+\delta,\quad j=1,2,\ldots,n,$$

$$B_n^2=\sum_{j=1}^n\sigma_j^2,\quad \ell_n=\frac{1}{B_n^{2+\delta}}\sum_{j=1}^n\beta_{2+\delta,j},$$

$$\overline{F}_n(x)=\mathsf{P}(X_1+\ldots+X_n<xB_n)=(F_1*\ldots*F_n)(xB_n),$$

$$\Delta_n=\Delta_n(F_1,\ldots,F_n)=\sup_x|\overline{F}_n(x)-\Phi(x)|,\quad n=1,2,\ldots,$$

$\Phi(x)$ being the standard normal d.f. Assume, that $B_n>0$. It is easy to verify that under the above assumptions for any $n\geqslant 1$ we have

$$\ell_n\geqslant\frac{1}{B_n^{2+\delta}}\sum_{j=1}^n\sigma_j^{2+\delta}\geqslant n^{-\delta/2}.$$

If the r.v.'s $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d.), then their common d.f. will be denoted by F $(= F_1 = \ldots = F_n)$. In this case we use the notation

$$\Delta_n(F) = \Delta_n(F_1, \ldots, F_n), \quad \sigma^2 = \mathsf{E}X_1^2 > 0, \quad \beta_{2+\delta} = \mathsf{E}|X_1|^{2+\delta}, \quad \beta_\delta = \mathsf{E}|X_1|^\delta.$$

Then

$$B_n = \sigma\sqrt{n}, \quad \ell_n = \frac{\beta_{2+\delta}}{\sigma^{2+\delta} n^{\delta/2}}.$$

In what follows, for a r.v. X the notation $X \in \mathcal{F}_{2+\delta}$ means that the d.f. $F(x) = \mathsf{P}(X < x)$, $x \in \mathbf{R}$, belongs to the class $\mathcal{F}_{2+\delta}$.

As is known, the rate of convergence in the central limit theorem of probability theory obeys the Berry–Esseen inequality

$$\Delta_n \leqslant C_{\text{BE}}(\delta) \cdot \ell_n, \quad n \geqslant 1, \ F_1, \ldots, F_n \in \mathcal{F}_{2+\delta}, \tag{1.1}$$

where $C_{\text{BE}}(\delta)$ depends only on δ [4, 8, 9]. Omitting the history of improvement of the constant $C_{\text{BE}}(1)$ the details of which can be found, for example, in the papers [19, 20], note that

$$0.4097\ldots = \frac{\sqrt{10}+3}{6\sqrt{2\pi}} \leqslant C_{\text{BE}}(1) \leqslant \begin{cases} 0.5600, & \text{in the general case,} \\ 0.4784, & \text{if } F_1 = \ldots = F_n, \end{cases}$$

see [10, 28, 20].[1] In 1966–1967 V. M. Zolotarev [37, 38, 39] suggested that $C_{\text{BE}}(1) = (\sqrt{10}+3)/(6\sqrt{2\pi})$. This hypothesis has been neither proved nor rejected yet.

For $0 < \delta < 1$ the best known upper estimates of the constants $C_{\text{BE}}(\delta)$ were obtained by W. Tysiak [30] for the general case (the second line in table 1) and by M. Grigorieva and I. Shevtsova [13] for the case of identically distributed summands (the third line in table 1). The first lower estimates were recently obtained by the author [29] (the fourth line in table 1).

In the case of identically distributed summands ($F_1 = \ldots = F_n = F$) and $\delta = 1$, inequality (1.1) takes the form

$$\Delta_n \leqslant C_{\text{BE}}(1) \cdot \frac{\beta_3}{\sigma^3\sqrt{n}}, \quad n \geqslant 1, \ F \in \mathcal{F}_3, \tag{1.2}$$

and along with the information concerning the two first moments also uses the value of the third absolute moment β_3.

[1] Recently, the presented upper bounds for $C_{\text{BE}}(1)$ were improved to $C_{\text{BE}}(1) \leqslant 0.5591$ in the general case by Ilya Tyurin (see "An improvement of the remainder in the Lyapounov theorem", Theory Probab. Appl., 2011, vol. 56, No. 4, p. 808-811 (in Russian)) and to $C_{\text{BE}}(1) \leqslant 0.4748$ in the i.i.d.-case by the author (see "On the absolute constants in the Berry–Esseen type inequalities for identically distributed summands", arXiv:1111.6554, 28 November 2011), the latest one — as a corollary to the estimate with an improved structure $\Delta_n \leqslant 0.33554(\beta_3/\sigma^3 + 0.415)/\sqrt{n}$, since $0.33554(\beta_3/\sigma^3 + 0.415) \leqslant 0.33554 \cdot 0.415\beta_3/\sigma^3 < 0.4748\beta_3/\sigma^3$ by virtue of the Lyapounov inequality. Independently, an estimate $C_{\text{BE}}(1) \leqslant 0.4774$ for the i.i.d.-case was obtained in the paper of I. Tyurin.

δ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$C_{\text{BE}}(\delta) \leqslant$	1.102	1.076	1.008	0.950	0.902	0.863	0.833	0.812	0.802
$C_{\text{BE}}(\delta) \leqslant$	0.6028	0.6094	0.6195	0.6342	0.6413	0.6276	0.6026	0.5723	0.5383
$C_{\text{BE}}(\delta) \geqslant$	0.4097	0.3603	0.3257	0.3000	0.2803	0.2651	0.2534	0.2446	0.2383

Table 1: Two-sided estimates of the constants $C_{\text{BE}}(\delta)$ from inequality (1.1) for some $\delta \in (0,1)$. The second line: the upper estimates in the general case [30]; the third line: improved estimates for the case of identically distributed summands [13]; the fourth line: the lower estimates [29].

On the other hand, as $n \to \infty$, if the summands are i.i.d. with arbitrary *fixed* (independent of n) d.f. $F \in \mathcal{F}_3$, then, as it was established in 1945 by Esseen [9], uniformly in x

$$\overline{F}_n(x) = \Phi(x) + \frac{\mathsf{E}X_1^3}{6\sigma^3} \cdot \frac{(1-x^2)e^{-x^2/2}}{\sqrt{2\pi n}} + \frac{h}{\sigma} \cdot \frac{H_n(x)e^{-x^2/2}}{\sqrt{2\pi n}} + o\left(\frac{1}{\sqrt{n}}\right), \tag{1.3}$$

where $h = hH_n(x) \equiv 0$, if F is non-lattice, and

$$H_n(x) = \frac{1}{2} - \left\{\left(x\sqrt{n} - \frac{an}{\sigma}\right)\frac{\sigma}{h}\right\}, \quad |H_n(x)| \leqslant \frac{1}{2},$$

if F is concentrated on the lattice $\{a + kh,\ k = 0, \pm 1, \pm 2, \ldots\}$ with span h, $\{x\}$ being the fractional part of $x \in \mathbf{R}$, whence Esseen deduced [10] that

$$\limsup_{n\to\infty} \Delta_n(F)\sqrt{n} = \frac{|\mathsf{E}X_1^3| + 3h\sigma^2}{6\sqrt{2\pi}\sigma^3}, \ \ F \in \mathcal{F}_3^h. \tag{1.4}$$

So, unlike (1.2), in the asymptotic relations (1.3) and (1.4) the third absolute moment $\mathsf{E}|X_1|^3$ does not take part at all whereas only the first three *original* moments are used as well as the parameter h, carrying the information on *the structure* of the basic distribution. The numerical characteristics mentioned above satisfy the relation [10, 40]

$$\sup_{h>0}\sup_{X \in \mathcal{F}_3^h} \frac{|\mathsf{E}X^3| + 3h\mathsf{E}X^2}{\mathsf{E}|X|^3} = \sqrt{10} + 3, \tag{1.5}$$

with supremum attained at the two-point distribution $\mathsf{P}(X = -h(4-\sqrt{10})/2) = (\sqrt{10}-2)/2$, $\mathsf{P}(X = h(\sqrt{10}-2)/2) = (4-\sqrt{10})/2$, called *the Esseen distribution.* From (1.4) and (1.5) it follows that for any $\mathcal{F} \in \mathcal{F}_3$

$$\limsup_{n\to\infty} \Delta_n(F)\sqrt{n} \leqslant \frac{\sqrt{10}+3}{6\sqrt{2\pi}} \cdot \frac{\beta_3}{\sigma^3}. \tag{1.6}$$

With the supremum attained at the Esseen distribution. This remark makes it possible to establish the lower estimate $C_{\text{BE}}(1) \geqslant (\sqrt{10}+3)/(6\sqrt{2\pi})$ as it was done by Esseen [10]. It is worth noticing for the sake of completeness that the normalized value of the third absolute moment of the Esseen distribution delivering the extremum in (1.5) and equality in (1.6) have the form

$$\beta_3/\sigma^3 = \sqrt{20(\sqrt{10}-3)/3} = 1.0401\ldots$$

So, if in (1.5) the supremum is sought not over all $X \in \mathcal{F}_3^h$, but under additional requirement that the ratio $\mathsf{E}|X|^3/(\mathsf{E}X^2)^{3/2}$ should be large enough, then the extremal value becomes smaller and hence, the lower estimate of the constant $C_{\mathrm{BE}}(1)$ in (1.2) becomes more optimistic. This remark generates the hope (and explains) that the larger the value of the Lyapounov ratio β_3/σ^3, the smaller the upper estimate of the constant $C_{\mathrm{BE}}(1)$ in (1.1) is.

Apparently, S. Zahl was the first to notice this [35, 36]. In 1963 he presented the structural improvement of inequality (1.1)

$$\Delta_n \leqslant \frac{0.651}{B_n^3}\sum_{j=1}^n \beta'_{3,j},$$

where

$$\beta'_{3,j} = \begin{cases} \beta_{3,j}, & \beta_{3,j} \geqslant 3\sigma_j^3/\sqrt{2}, \\ \sigma_j^3/\left(0.7804 - 0.1457\beta_{3,j}/\sigma_j^3\right), & \beta_{3,j} < 3\sigma_j^3/\sqrt{2}, \end{cases}$$

which more efficiently uses the information concerning the first three moments of random summands.

The next step in this direction was made in 1975 by H. Prawitz, from whose paper [25] one can deduce the estimate

$$\Delta_n \leqslant \ell_n \cdot A_1(\ell_n) + \frac{1}{2\sqrt{2\pi}B_n^3}\sum_{j=1}^n \sigma_j^3 + \frac{1}{4\pi B_n^4}\sum_{j=1}^n \sigma_j^4, \tag{1.7}$$

where $A_1(\ell)$ is a positive function of $\ell > 0$ with a complicated structure such that $A_1(\ell)$ does not increase for ℓ small enough and

$$\lim_{\ell\to 0} A_1(\ell) = \frac{1.0253}{6\sqrt{2\pi}} + \frac{1}{2\sqrt{2\pi}} = \frac{2}{3\sqrt{2\pi}} + \frac{0.0253}{6\sqrt{2\pi}} = 0.2676\ldots$$

Prawitz also described an algorithm for the computation of $A_1(\ell)$ for concrete values of ℓ. Since

$$\frac{1}{B_n^3}\sum_{j=1}^n \sigma_j^3 \leqslant \frac{1}{B_n^3}\sum_{j=1}^n \beta_{3,\,j} = \ell_n, \quad \frac{1}{B_n^4}\sum_{j=1}^n \sigma_j^4 \leqslant \ell_n^{4/3} = o(\ell_n),\ \ \ell_n \to 0,$$

from (1.7) it follows that

$$\Delta_n \leqslant \ell_n \cdot A_2(\ell_n), \tag{1.8}$$

where $A_2(\ell)$ is a positive function of $\ell > 0$ such that $A_2(\ell)$ does not increase for ℓ small enough and

$$\lim_{\ell\to 0} A_2(\ell) = \frac{1.0253}{6\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}} = \frac{7}{6\sqrt{2\pi}} + \frac{0.0253}{6\sqrt{2\pi}} = 0.4671\ldots\ .$$

Inequality (1.8) with concrete values of A_2 plays an important role in the problem of determination of upper estimates of the absolute constant $C_{\text{BE}}(1)$ in the Berry–Esseen inequality (1.1), since the algorithms which are traditionally used for these purposes cannot obtain the values of this constant which are less than A_2.

In the same paper [25], for identically distributed summands and $n \geqslant 2$, Prawitz announced the inequality

$$\Delta_n \leqslant \frac{2}{3\sqrt{2\pi}} \cdot \frac{\beta_3}{\sigma^3\sqrt{n-1}} + \frac{1}{2\sqrt{2\pi(n-1)}} + A_3 \cdot \ell_{n-1}^2, \tag{1.9}$$

where A_3 is an absolute positive constant and stated that the coefficient

$$\frac{2}{3\sqrt{2\pi}} = 0.2659\ldots$$

at the Lyapounov fraction in (1.9) cannot be made smaller. Unfortunately, the proof of this statement as well as that of inequality (1.9) were not published by Prawitz.

A strict proof of Prawitz' inequality (1.9), however, with a little worse remainder, follows from the papers of V. Bentkus [2, 3], in which for the case of arbitrary $F_1, \ldots, F_n \in \mathcal{F}_3$ and $n \geqslant 1$ the estimate

$$\Delta_n \leqslant \frac{2\ell_n}{3\sqrt{2\pi}} + \frac{1}{2\sqrt{2\pi}B_n^3}\sum_{j=1}^n \sigma_j^3 + A_4 \cdot \ell_n^{4/3} \leqslant \frac{7\ell_n}{6\sqrt{2\pi}} + A_4 \cdot \ell_n^{4/3} \tag{1.10}$$

was obtained, where A_4 is an absolute constant. The worse order of the remainder in (1.10) as compared with (1.9) is due to that the estimate (1.10) holds for arbitrary (not necessarily identical) $F_1, \ldots, F_n \in \mathcal{F}_3$.

So, even if the value of the constant A_4 in (1.10) were known, it would not be possible to obtain an estimate of the absolute constant $C_{\text{BE}}(1)$ in the Berry–Esseen inequality (1.1) lower than $7/(6\sqrt{2\pi}) = 0.4654\ldots$. For further progress in this problem, one has to improve the main term of asymptotic estimate (1.10).

In 1953 A. N. Kolmogorov [17] (also see the monographs of I. A. Ibragimov and Yu. V. Linnik [16] and V. M. Zolotarev [40]) formulated the problem of calculation of the so-called asymptotically exact constant

$$C_{\text{AE}} = \limsup_{\ell \to 0} \sup_{n \geqslant 1,\, F_1, \ldots, F_n \,:\, \ell_n = \ell} \frac{\Delta_n(F_1, \ldots, F_n)}{\ell},$$

for which from the papers of Esseen [10] and Bentkus [2, 3] it follows that

$$0.4097\ldots = \frac{\sqrt{10}+3}{6\sqrt{2\pi}} \leqslant C_{\text{AE}} \leqslant \frac{7}{6\sqrt{2\pi}} = 0.4654\ldots .$$

V. M. Zolotarev [38, 39, 40] held the opinion that C_{AE} coincides with its lower bound and together with A. N. Kolmogorov considered the problem of calculation of C_{AE} to be intermediate or auxiliary for the problem of calculation of the exact value of the absolute constant $C_{\text{BE}}(1)$ in (1.1). The gap of approximately 0.06 between the upper and lower bounds of C_{AE} presented above is due to the fact that the information on the *original* moments of summands is not taken into account in [25, 2, 3]. Since the summands are centered, the only informative original moment is the third one. S. V. Nagaev and V. I. Chebotarev [21] also noticed this and for the i.i.d. two-point summands proved the estimate $C_{\text{BE}}(1) \leqslant 0.4215$.

In 2001–2002 G. P. Chistyakov [7] obtained a new asymptotic expansion generalizing that due to Esseen (1.3) to the case of non-identically distributed summands. This new expansion allowed Chistyakov, as an intermediate step, to use the information concerning the original moments and other characteristics of the initial distributions and, as a result, to deduce the estimate

$$\Delta_n \leqslant \frac{\sqrt{10}+3}{6\sqrt{2\pi}} \cdot \ell_n + A_5 \cdot \ell_n^{40/39} |\ln \ell_n|^{7/6}, \tag{1.11}$$

where A_5 is an absolute constant. From (1.11) it follows that

$$C_{\text{AE}} = \frac{\sqrt{10}+3}{6\sqrt{2\pi}} = 0.4097\ldots,$$

thus Chistyakov proved the validity of Zolotarev's hypothesis concerning the exact value of the asymptotically exact constant C_{AE}.

Unfortunately, the particular value of the absolute constant A_5 in Chistyakov's inequality (1.11) was not given, so this fundamental result cannot be used for practical calculations, in particular, for the evaluation of the absolute constant $C_{\text{BE}}(1)$ in the Berry–Esseen inequality.

Nevertheless, the inequalities of Prawitz (1.9) and Bentkus (1.10) are interesting because in these inequalities the coefficient at the Lyapounov fraction is less than in Chistyakov's inequality (1.11):

$$0.2659\ldots = \frac{2}{3\sqrt{2\pi}} < \frac{\sqrt{10}+3}{6\sqrt{2\pi}} = 0.4097\ldots,$$

and hence, with large values of the ratio

$$\sum_{j=1}^{n} \beta_{3,\,j} \Big/ \sum_{j=1}^{n} \sigma_j^3$$

inequalities (1.9) and (1.10) are more precise than (1.11). This ratio may be arbitrarily large even in the case of identically distributed summands, for example, in the double array scheme where $\beta_3/\sigma^3 = \beta_3(n)/\sigma^3(n) \to \infty$, so that

$$\frac{1}{B_n^3} \sum_{j=1}^{n} \sigma_j^3 = \frac{1}{\sqrt{n}} = o(\ell_n) \quad \text{as } \ell_n = \frac{\beta_3(n)}{\sigma^3(n)\sqrt{n}} \to 0.$$

So, the unproved Prawitz' assertion that the coefficient $2/(3\sqrt{2\pi})$ at the Lyapounov fraction is unimprovable becomes exceptionally important. This assertion was proved only recently in [29] where the so-called *lower asymptotically exact constant*

$$\underline{C_{\text{AE}}} = \limsup_{\ell\to 0}\limsup_{n\to\infty}\sup_{F\colon \beta_3=\sigma^3\ell\sqrt{n}}\frac{\Delta_n(F)}{\ell}$$

was introduced (for the scheme of summation of identically distributed summands), which is an obvious lower bound for the coefficient under discussion, and it was demonstrated that $\underline{C_{\text{AE}}} = 2/(3\sqrt{2\pi})$.

The unimprovability of the first term in (1.9) naturally puts forward the question concerning the accuracy of the second term. No suggestions concerning the "exactness" of the coefficient at the second term in (1.9), (1.10) were stated by Prawitz or Bentkus. Actually, this question can be formulated in an even more general form: for any $c \geqslant \underline{C_{\text{AE}}}$ find the least possible value $K(c)$ providing the validity of the asymptotic estimate

$$\sup_{F\in\mathcal{F}_3\colon \beta_3=\rho\sigma^3}\Delta_n(F) \leqslant \frac{c\rho}{\sqrt{n}} + \frac{K(c)}{\sqrt{n}} + r_n(\rho)\cdot\frac{\rho}{\sqrt{n}}, \quad n,\ \rho\geqslant 1,$$

in which the remainder $r_n(\rho)\geqslant 0$ satisfies the conditions

$$\limsup_{\ell\to 0}\limsup_{n\to\infty} r_n(\ell\sqrt{n}) = 0, \quad \sup_{\rho\geqslant 1}\limsup_{n\to\infty} r_n(\rho) = 0. \tag{1.12}$$

Apparently, for the first time this question was formulated in [29], where lower estimates of $K(c)$ were presented for $\underline{C_{\text{AE}}} \leqslant c \leqslant C_{\text{AE}}$. In particular, for $c = \underline{C_{\text{AE}}}$ in [29] it was shown that

$$K\left(\frac{2}{3\sqrt{2\pi}}\right) \geqslant \sqrt{\frac{2\sqrt{3}-3}{6\pi}} = 0.1569\ldots,$$

which is strictly less than the value of the coefficient $\left(2\sqrt{2\pi}\right)^{-1} = 0.1994\ldots$ at the second term in inequalities (1.9) and (1.10). Thus, the question of the "exactness" of the second term in (1.9) and (1.10) remained unanswered.

In the present paper we will prove that: for all $n\geqslant 1$ and $F_1,\ldots,F_n\in\mathcal{F}_3$

$$\Delta_n \leqslant \inf_{c\geqslant \underline{C_{\text{AE}}}}\left\{c\ell_n + \frac{K(c)}{B_n^3}\sum_{j=1}^n \sigma_j^3 + \min\left\{2.7176\ell_n^{7/6}, A(c)\ell_n^{4/3}\right\}\right\},$$

and for identically distributed summands

$$\Delta_n \leqslant \inf_{c\geqslant \underline{C_{\text{AE}}}}\left\{\frac{c\beta_3}{\sigma^3\sqrt{n}} + \frac{K(c)}{\sqrt{n}} + \min\left\{1.7002\ell_n^{3/2}, A(c)\ell_n^2\right\}\right\},$$

with the function $K(c)$ optimal for each $c\geqslant \underline{C_{\text{AE}}}$ (the optimality of this function is proved in remark 4.16), $A(c)>0$ being a decreasing function of c such that

$A(c) \to \infty$ as $c \to 2/(3\sqrt{2\pi})$. The function $K(c)$ decreases monotonically alternating its sign in a single point $c = (\sqrt{10}+3)/(6\sqrt{2\pi})$. So, the second term in the estimates presented above is negative for $c > (\sqrt{10}+3)/(6\sqrt{2\pi})$. The presence of a negative summand in the main term is rather unusual in estimates of the accuracy of the normal approximation, but makes it possible to obtain asymptotically exact estimates as simple corollaries of the results presented above even for symmetric Bernoulli distributions (see corollary 4.19) which distinguishes these results from previously known. In particular, for $c = C_{\text{AE}}$ we have

$$\Delta_n \leqslant \frac{\sqrt{10}+3}{6\sqrt{2\pi}} \cdot \ell_n + 3.4314 \cdot \ell_n^{4/3}, \quad n \geqslant 1,\ F_1, \ldots, F_n \in \mathcal{F}_3,$$

$$\Delta_n \leqslant \frac{\sqrt{10}+3}{6\sqrt{2\pi}} \cdot \frac{\beta_3}{\sigma^3\sqrt{n}} + 2.5786 \cdot \ell_n^2, \quad n \geqslant 1,\ F_1 = \ldots = F_n \in \mathcal{F}_3,$$

which improves Chistyakov's inequality (1.11) with respect to the remainder, whereas for $c = \underline{C_{\text{AE}}}$ we have

$$\Delta_n \leqslant \frac{2\ell_n}{3\sqrt{2\pi}} + \sqrt{\frac{2\sqrt{3}-3}{6\pi} \sum_{j=1}^{n} \frac{\sigma_j^3}{B_n^3}} + 2.7176 \cdot \ell_n^{7/6}, \quad n \geqslant 1,\ F_1, \ldots, F_n \in \mathcal{F}_3,$$

$$\Delta_n \leqslant \frac{2}{3\sqrt{2\pi}} \cdot \frac{\beta_3}{\sigma^3\sqrt{n}} + \sqrt{\frac{2\sqrt{3}-3}{6\pi n}} + 1.7002 \cdot \ell_n^{3/2}, \quad n \geqslant 1,\ F_1 = \ldots = F_n \in \mathcal{F}_3,$$

which improves Prawitz' and Bentkus' inequalities (1.9), (1.10) with respect to the second term. Moreover, we will obtain the absolute improvements of Prawitz' and Bentkus' inequalities (1.9) and (1.10):

$$\Delta_n \leqslant \frac{2\ell_n}{3\sqrt{2\pi}} + \frac{1}{2\sqrt{2\pi}B_n^3} \sum_{j=1}^{n} \beta_{1,j}\, \sigma_j^2 + 5.4527 \cdot \ell_n^{5/3}, \quad n \geqslant 1,\ F_1, \ldots, F_n \in \mathcal{F}_3,$$

$$\Delta_n \leqslant \frac{2}{3\sqrt{2\pi}} \cdot \frac{\beta_3}{\sigma^3\sqrt{n}} + \frac{1}{2\sqrt{2\pi}} \cdot \frac{\beta_1}{\sigma\sqrt{n}} + 2.4606 \cdot \ell_n^2, \quad n \geqslant 1,\ F_1 = \ldots = F_n \in \mathcal{F}_3,$$

in which the remainders have no worse order of decrease than in (1.9) and (1.10) but with specified constants and an improved function $\sum_{j=1}^n \beta_{1,j}\, \sigma_j^2 \leqslant \sum_{j=1}^n \sigma_j^3$ of the two first moments in the second term with the same coefficient as in (1.9), (1.10). Below it will be shown that the value of the coefficient $\big(2\sqrt{2\pi}\big)^{-1}$ at this improved function of the two first moments yet cannot be made less (see remark 4.9). As well, similar estimates will be obtained for the case $0 < \delta < 1$, generalizing and sharpening the results of [11], where only the case of identically distributed summands was considered.

To prove the main results we use a combination of the method of characteristic functions (ch.f.'s) with the truncation method as well as some methods of convex analysis based on the works of W. Hoeffding [15] and V. M. Zolotarev [40].

It is worth noticing that in the preceding works dealing with the accuracy of the normal approximation, Prawitz' smoothing inequality was used, besides Prawitz himself, only by V. Bentkus [2, 3]. G. P. Chistyakov in [7] used Esseen's traditional smoothing inequality with the normal smoothing kernel, while in Prawitz' inequality, the smoothing function has a compact Fourier transform and does not have any probabilistic interpretation.

The paper is arranged as follows. In the second section we present new estimates for ch.f.'s implying, in particular, a generalization and improvement of the von Mises inequality for lattice distributions: for any $h>0$, $\delta\in(0,1]$ and $F\in\mathcal{F}^h_{2+\delta}$

$$\frac{h}{\sigma}\leqslant\frac{\beta_{2+\delta}}{\sigma^{2+\delta}}+\frac{\beta_\delta}{\sigma^\delta},$$

whereas in the original von Mises inequality $\delta=1$ and on the right-hand side there is $2\beta_3/\sigma^3$. In the third section a moment inequality is proved which improves (1.5) and plays the key role for the construction of the optimal function of moments in the resulting estimates. In the fourth section we formulate and prove new moment-type estimates of the accuracy of the normal approximation with optimal structure.

2. Estimates for characteristic functions

Denote

$$\varepsilon_n=B_n^{-(2+\delta)}\sum_{j=1}^n(\beta_{2+\delta,j}+\beta_{\delta,j}\,\sigma_j^2)=\ell_n+B_n^{-(2+\delta)}\sum_{j=1}^n\beta_{\delta,j}\,\sigma_j^2,$$

$$f_j(t)=\mathsf{E}e^{itX_j},\ j=1,2,\ldots,n,\quad \overline{f}_n(t)=\prod_{j=1}^n f_j\left(\frac{t}{B_n}\right),$$

$$r_n(t)=\left|\overline{f}_n(t)-e^{-t^2/2}\right|,\quad t\in\mathbf{R}.$$

As is well-known, if $X_1,\ldots,X_n$ are identically distributed, then

$$\overline{f}_n(t)=\left(f_1\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n,\quad t\in\mathbf{R}.$$

In this section new estimates for $|\overline{f}_n(t)|$ and $r_n(t)$ will be obtained.

Let $\theta_0(\delta)$ be the unique root of the equation

$$\delta\theta^2+2\theta\sin\theta+2(2+\delta)(\cos\theta-1)=0$$

within the interval $(0,2\pi)$. As this is so, $\pi<\theta_0(\delta)<2\pi$ for all $0<\delta\leqslant1$. Let

$$\varkappa_\delta\equiv\sup_{x>0}\frac{|\cos x-1+x^2/2|}{x^{2+\delta}}=\frac{\cos\theta_0(\delta)-1+\theta_0^2(\delta)/2}{\theta_0^{2+\delta}(\delta)}=\frac{\theta_0(\delta)-\sin\theta_0(\delta)}{(2+\delta)\theta_0^{1+\delta}(\delta)}.\quad(2.1)$$

Obviously,

$$\varkappa_\delta \leqslant \frac{1}{2\theta_0^\delta(\delta)} \leqslant \frac{1}{2\pi^\delta} \leqslant 1/2, \quad 0<\delta\leqslant 1. \tag{2.2}$$

For $\varepsilon>0$ let

$$\psi_\delta(t,\varepsilon)=\begin{cases} t^2/2-\varkappa_\delta\varepsilon|t|^{2+\delta}, & |t|<\theta_0(\delta)\varepsilon^{-1/\delta},\\ \dfrac{1-\cos\left(\varepsilon^{1/\delta}t\right)}{\varepsilon^{2/\delta}}, & \theta_0(\delta)\leqslant\varepsilon^{1/\delta}|t|\leqslant 2\pi,\\ 0, & |t|>2\pi\varepsilon^{-1/\delta}. \end{cases}$$

It is easy to see that the function $\psi_\delta(t,\varepsilon)$ decreases monotonically in ε for each fixed $t\in\mathbf{R}$ and all $0<\delta\leqslant 1$. Moreover, $\psi_\delta(t,\varepsilon)\geqslant 0$ for all $t\in\mathbf{R}$.

The following lemma plays the key role for the construction of estimates of the absolute value of a ch.f.

Lemma 2.1 (see [26]). *For any $x\in\mathbf{R}$ and $\theta_0(\delta)\leqslant\theta\leqslant 2\pi$*

$$\cos x\leqslant 1-a(\delta,\theta)x^2+b(\delta,\theta)|x|^{2+\delta},$$

where

$$a(\delta,\theta)=\frac{2+\delta}{\delta}\cdot\frac{1-\cos\theta}{\theta^2}-\frac{1}{\delta}\cdot\frac{\sin\theta}{\theta},$$
$$b(\delta,\theta)=\frac{2}{\delta}\cdot\frac{1-\cos\theta}{\theta^{2+\delta}}-\frac{1}{\delta}\cdot\frac{\sin\theta}{\theta^{1+\delta}}.$$

Theorem 2.2. *For any $F_1,\ldots,F_n\in\mathcal{F}_{2+\delta}$ and any $t\in\mathbf{R}$*

$$|\overline{f}_n(t)|\leqslant\left[1-\frac{2}{n}\,\psi_\delta(t,\varepsilon_n)\right]^{n/2}\leqslant\exp\{-\psi_\delta(t,\varepsilon_n)\}\leqslant\exp\left\{-t^2/2+\varkappa_\delta\varepsilon_n|t|^{2+\delta}\right\}.$$

Proof. Let X_j' be an independent copy of the r.v. X_j, $j=1,\ldots,n$. Then

$$\left|\overline{f}_n(t)\right|^2=\prod_{j=1}^n\left|f_j\left(\frac{t}{B_n}\right)\right|^2=\prod_{j=1}^n\mathsf{E}\cos\frac{t(X_j-X_j')}{B_n}.$$

Using lemma 2.1 and relations $\mathsf{E}(X_j-X_j')^2=2\sigma_j^2$, $\mathsf{E}|X_j-X_j'|^{2+\delta}\leqslant 2\big(\beta_{2+\delta,j}+\beta_{\delta,j}\,\sigma_j^2\big)$ (see, e. g., [34, p. 74, lemma 2.1.7]) we obtain

$$\begin{aligned}|\overline{f}_n(t)|^2&\leqslant\prod_{j=1}^n\left(1-a(\delta,\theta)\frac{t^2\mathsf{E}(X_j-X_j')^2}{B_n^2}+b(\delta,\theta)\frac{|t|^{2+\delta}\mathsf{E}|X_j-X_j'|^{2+\delta}}{B_n^{2+\delta}}\right)\\&\leqslant\prod_{j=1}^n\left(1-2a(\delta,\theta)t^2\frac{\sigma_j^2}{B_n^2}+2b(\delta,\theta)|t|^{2+\delta}\frac{\beta_{2+\delta,j}+\beta_{\delta,j}\,\sigma_j^2}{B_n^{2+\delta}}\right).\end{aligned}$$

The expression in brackets is an upper bound for the squared absolute value of the ch.f. $f_j(t)$ and, hence, is nonnegative. Since the geometric mean of nonnegative numbers is no greater than their arithmetic mean, for all $t \in \mathbf{R}$ and $\theta \in [\theta_0(\delta), 2\pi]$ we obtain

$$|\overline{f}_n(t)|^2 \leqslant \left[1 - \frac{2}{n}\sum_{j=1}^{n}\left(a(\delta,\theta)t^2\frac{\sigma_j^2}{B_n^2} - b(\delta,\theta)|t|^{2+\delta}\frac{\beta_{2+\delta,j} + \beta_{\delta,j}\,\sigma_j^2}{B_n^{2+\delta}}\right)\right]^n$$
$$= \left[1 - \frac{2}{n}\left(a(\delta,\theta)t^2 - b(\delta,\theta)\varepsilon_n|t|^{2+\delta}\right)\right]^n \equiv \left[1 - \frac{2}{n}\psi_\delta(t,\varepsilon_n,\theta)\right]^n,$$

where

$$\psi_\delta(t,\varepsilon,\theta) = a(\delta,\theta)t^2 - b(\delta,\theta)\varepsilon|t|^{2+\delta}, \quad t \in \mathbf{R},\ \varepsilon > 0,\ \theta_0(\delta) \leqslant \theta \leqslant 2\pi.$$

It can be made sure (see, e. g., [26]) that for any fixed $t \in \mathbf{R}$ the minimum of the right-hand side of the last estimate for $|\overline{f}_n(t)|^2$ is attained at

$$\theta = \min\left\{\max\left\{\theta_0(\delta), \varepsilon_n^{1/\delta}|t|\right\}, 2\pi\right\},$$

and

$$\psi_\delta(t,\varepsilon) = \max_{\theta_0(\delta)\leqslant\theta\leqslant 2\pi}\psi_\delta(t,\varepsilon,\theta) \geqslant \psi_\delta(t,\varepsilon,\theta_0(\delta)) = t^2/2 - \varkappa_\delta\varepsilon|t|^{2+\delta},$$

whence follows the statement of the lemma. □

For $n = 1$ from theorem 2.2 we obtain

Corollary 2.3. *For any r.v. $X \in \mathcal{F}_{2+\delta}$ for all $t \in \mathbf{R}$ there hold the estimates*

$$\left|\mathsf{E}e^{itX}\right|^2 \leqslant 1 - 2\psi_\delta\left(\sigma t, \beta_{2+\delta}/\sigma^{2+\delta} + \beta_\delta/\sigma^\delta\right) \leqslant 1 - \sigma^2t^2 + 2\varkappa_\delta\left(\beta_{2+\delta} + \beta_\delta\sigma^2\right)|t|^{2+\delta}.$$

Remark 2.4. For $\delta = 1$, in the paper of H. Prawitz [24] the first inequality of corollary 2.3 is proved as well as the second inequality of theorem 2.2. In the book of N. G. Ushakov [34] the second inequality of corollary 2.3 is proved for arbitrary $0 < \delta \leqslant 1$.

Remark 2.5. From corollary 2.3 it follows that $|f(t)|<1$ for $|t| < 2\pi(\beta_{2+\delta}/\sigma^2 + \beta_\delta)^{-1/\delta}$ for any d.f. $F \in \mathcal{F}_{2+\delta}$. A special role of the point $t = 2\pi(\beta_{2+\delta}/\sigma^2 + \beta_\delta)^{-1/\delta}$ is due to the fact that this is the least possible period of the ch.f. of a r.v. with fixed three absolute moments β_δ, σ^2 and $\beta_{2+\delta}$. Indeed, for the symmetric distribution $\mathsf{P}(X = \pm a) = 1/(2a^2)$, $\mathsf{P}(X = 0) = 1 - 1/a^2$ with $a = 1/\sqrt{2^\delta - 1}$ we have $\beta_\delta = a^{\delta-2}$, $\sigma^2 = 1$, $\beta_{2+\delta} = a^\delta$. It is easy to see that the ch.f. $f(t) = \mathsf{E}\cos(tX) = 1 - (1 - \cos(at))/a^2$ equals 1 for $t = \pi/a$, and with a specified above

$$\frac{\pi}{a} = \frac{2\pi}{a(1 + a^{-2})^{1/\delta}} = \frac{2\pi}{(\beta_{2+\delta} + \beta_\delta)^{1/\delta}}.$$

The fact mentioned in remark 2.5 can be used for the improvement of the von Mises inequality

$$\frac{h}{\sigma} \leqslant 2\frac{\beta_3}{\sigma^3},$$

relating the span of a lattice distribution with its moments. Namely, from corollary 2.3 it follows that

$$t_0 = \inf\{t > 0\colon |f(t)| = 1\} \geqslant 2\pi(\beta_{2+\delta}/\sigma^2 + \beta_\delta)^{-1/\delta}.$$

As is known, $t_0 < \infty$ if and only if $F \in \mathcal{F}^h_{2+\delta}$ with $h = 2\pi/t_0$. So, the following theorem holds.

Theorem 2.6. *For any $h > 0$ and $X \in \mathcal{F}^h_{2+\delta}$*

$$h \leqslant (\beta_{2+\delta}/\sigma^2 + \beta_\delta)^{1/\delta}. \tag{2.3}$$

For all $0 < \delta \leqslant 1$, this inequality is unimprovable in the sense that for any $h > 0$ we have

$$\sup\left\{h(\beta_{2+\delta}/\sigma^2 + \beta_\delta)^{-1/\delta}\colon X \in \mathcal{F}^h_{2+\delta}\right\} = 1, \quad 0 < \delta \leqslant 1,$$

moreover, the supremum is attained at the family of distributions of the form

$$\mathsf{P}\left(X = \frac{h}{1+u}\right) = \frac{u}{1+u} = 1 - \mathsf{P}\left(X = -\frac{uh}{1+u}\right), \quad u \to \infty.$$

For $\delta = 1$ the supremum is also attained at the extremal distribution $\mathsf{P}(X = h/2) = \mathsf{P}(X = -h/2) = 1/2$.

Theorem 2.2 and inequality (2.3) also improve the results of paper [26], in which $\sigma^\delta \geqslant \beta_\delta$ is used instead of β_δ.

Lemma 2.7. *For any $F_1, \ldots, F_n \in \mathcal{F}_{2+\delta}$ and $t \in \mathbf{R}$*

$$\begin{aligned} r_n(t) &\equiv \left|\overline{f}_n(t) - e^{-t^2/2}\right| \\ &\leqslant \sum_{j=1}^n \left| f_j\left(\frac{t}{B_n}\right) - \exp\left\{-\frac{\sigma_j^2 t^2}{2B_n^2}\right\}\right| \exp\left\{-\frac{t^2}{2}\left(1 - \frac{\sigma_j^2}{B_n^2}\right) + \varkappa_\delta \varepsilon_n |t|^{2+\delta}\right\}. \end{aligned}$$

Proof. In [25] it was proved that for any $A_j > 0$, $B_j \in \mathbb{C}$, $C_j \geqslant \max\{A_j, |B_j|\}$

$$\begin{aligned} \left|\prod_{j=1}^n B_j - \prod_{j=1}^n A_j\right| &\leqslant \frac{1}{2}\prod_{i=1}^n C_i \sum_{j=1}^n \frac{|B_j - A_j|}{C_j} + \frac{1}{2}\prod_{i=1}^n A_i \sum_{j=1}^n \frac{|B_j - A_j|}{A_j} \\ &\leqslant \sum_{j=1}^n \frac{|B_j - A_j|}{A_j} \prod_{i=1}^n C_i. \end{aligned}$$

Using this inequality with

$$B_j = f_j\left(\frac{t}{B_n}\right), \quad A_j = \exp\left\{-\frac{\sigma_j^2 t^2}{2B_n^2}\right\},$$

$$C_j = \exp\left\{-\frac{\sigma_j^2 t^2}{2B_n^2} + \varkappa_\delta(\beta_{2+\delta,j} + \beta_{\delta,j}\,\sigma_j^2)\frac{|t|^{2+\delta}}{B_n^{2+\delta}}\right\}$$

(the estimate $|B_j| \leqslant C_j$ follows from theorem 2.2), for $r_n(t)$ we obtain

$$r_n(t) = \left|\prod_{j=1}^n f_j\left(\frac{t}{B_n}\right) - \prod_{j=1}^n \exp\left\{-\frac{\sigma_j^2 t^2}{2B_n^2}\right\}\right| \leqslant$$
$$\leqslant \sum_{j=1}^n \left|f_j\left(\frac{t}{B_n}\right) - \exp\left\{-\frac{\sigma_j^2 t^2}{2B_n^2}\right\}\right| \exp\left\{-\frac{t^2}{2} + \varkappa_\delta \varepsilon_n |t|^{2+\delta} + \frac{\sigma_j^2 t^2}{2B_n^2}\right\}. \quad \square$$

The way we estimate $|f_j(t/B_n) - e^{-\sigma_j^2 t^2/(2B_n^2)}|$ in lemma 2.7 depends on whether $\delta = 1$ or not.

Lemma 2.8. *For any r.v. $X \in \mathcal{F}_{2+\delta}$ with the ch.f. $f(t)$ for all $t \in \mathbf{R}$ we have the estimates:*
if $\delta = 1$, then

$$\left|f(t) - e^{-\sigma^2 t^2/2}\right| \leqslant \frac{\beta_3 |t|^3}{6}, \tag{2.4}$$

$$\left|f(t) - e^{-\sigma^2 t^2/2}\right| \leqslant \frac{|t|^3}{6}\Big(\left|\mathsf{E}X^3 \mathbf{1}(|X| \leqslant U)\right| + \mathsf{E}|X|^3 \mathbf{1}(|X| > U)\Big) + $$
$$+ \frac{t^4}{24}\mathsf{E}|X|^4 \mathbf{1}(|X| \leqslant U) + \frac{\sigma^4 t^4}{8} \tag{2.5}$$

for all $U \geqslant 0$;
if $0 < \delta \leqslant 1$, then

$$\left|f(t) - e^{-\sigma^2 t^2/2}\right| \leqslant \gamma_\delta \beta_{2+\delta}\, |t|^{2+\delta} + \sigma^4 t^4/8, \tag{2.6}$$

where

$$\gamma_\delta = \sup_{x>0} \left|e^{ix} - 1 - ix - (ix)^2/2\right| / x^{2+\delta}$$
$$= \sup_{x>0} \sqrt{\left(\frac{\cos x - 1 + x^2/2}{x^{2+\delta}}\right)^2 + \left(\frac{\sin x - x}{x^{2+\delta}}\right)^2}.$$

The values of γ_δ for some $0 < \delta \leqslant 1$ are presented in the second column of table 3. In particular, $\gamma_1 = 1/6$. The estimates given in lemma 2.8 were apparently first obtained for the case $0 < \delta < 1$ by W. Tysiak [30]. Nevertheless, for completeness we give their simple proof as well.

Proof. The first estimate follows from the works of I. Tyurin [31, 32], in which the inequality

$$\left|f(t) - e^{-\sigma^2 t^2/2}\right| \leqslant e^{-t^2/2} \int_0^{|t|} \frac{\beta_3 s^2}{2}\, e^{s^2/2}\, ds \leqslant \int_0^{|t|} \frac{\beta_3 s^2}{2}\, ds = \frac{\beta_3 |t|^3}{6}, \quad t \in \mathbf{R},$$

was proved.

Further, using the inequality $|e^{-x}-1+x|\leqslant x^2/2$, $x\geqslant 0$, for all $t\in\mathbf{R}$ we obtain

$$\begin{aligned}|f(t)-e^{-\sigma^2t^2/2}| &\leqslant \left|\mathsf{E}\left(e^{itX}-1-itX+\frac{t^2X^2}{2}\right)\right|+\left|e^{-\sigma^2t^2/2}-1+\frac{\sigma^2t^2}{2}\right|\\ &\leqslant R(t)+\frac{\sigma^4t^4}{8},\end{aligned}$$

where

$$R(t)=\left|\mathsf{E}\left(e^{itX}-1-itX-\frac{(itX)^2}{2}\right)\right|\leqslant R_1(t,U)+R_2(t,U),$$

$$R_1(t,U)=\left|\mathsf{E}\left(e^{itX}-1-itX-\frac{(itX)^2}{2}\right)\mathbf{1}(|X|\leqslant U)\right|,$$

$$R_2(t,U)=\mathsf{E}\left|e^{itX}-1-itX-\frac{(itX)^2}{2}\right|\mathbf{1}(|X|>U)$$

for any $U\geqslant 0$.

By the definition of γ_δ, $\left|e^{ix}-1-ix-(ix)^2/2\right|\leqslant\gamma_\delta|x|^{2+\delta}$, $x\in\mathbf{R}$, whence for $R_2(t,U)$ we obtain

$$R_2(t,U)\leqslant\gamma_\delta|t|^{2+\delta}\mathsf{E}|X|^{2+\delta}\mathbf{1}(|X|>U).$$

Adding and subtracting $(itX)^3/6\cdot\mathbf{1}(|X|\leqslant U)$ under the sign of expectation in $R_1(t,U)$, taking account of the inequality $\left|e^{ix}-1-ix-(ix)^2/2-(ix)^3/6\right|\leqslant x^4/24$, $x\in\mathbf{R}$, for $R_1(t,U)$ we obtain

$$\begin{aligned}R_1(t,U)\leqslant&\left|\mathsf{E}\left(e^{itX}-1-itX-\frac{(itX)^2}{2}-\frac{(itX)^3}{6}\right)\mathbf{1}(|X|\leqslant U)\right|\\ &+\frac{|t|^3}{6}\left|\mathsf{E}X^3\mathbf{1}(|X|\leqslant U)\right|\leqslant\frac{t^4}{24}\mathsf{E}X^4\mathbf{1}(|X|\leqslant U)+\frac{|t|^3}{6}\left|\mathsf{E}X^3\mathbf{1}(|X|\leqslant U)\right|.\end{aligned}$$

So, for any $0<\delta\leqslant 1$ and $U\geqslant 0$ for all $t\in\mathbf{R}$ we have

$$\begin{aligned}|f(t)-e^{-\sigma^2t^2/2}|\leqslant&\frac{\sigma^4t^4}{8}+\gamma_\delta|t|^{2+\delta}\mathsf{E}|X|^{2+\delta}\mathbf{1}(|X|>U)\\ &+\frac{|t|^3}{6}\left|\mathsf{E}X^3\mathbf{1}(|X|\leqslant U)\right|+\frac{t^4}{24}\mathsf{E}X^4\mathbf{1}(|X|\leqslant U).\end{aligned}$$

Setting $U=0$ in this inequality, we obtain the second estimate of the lemma, setting $\delta=1$ we obtain the third one. The lemma is completely proved. □

Remark 2.9. Note that using new optimal estimates for ζ-metrics obtained in [33], we can as well prove an analog of the first estimate of lemma 2.8 for the case of an arbitrary $0<\delta<1$ in the form

$$\left|f(t)-e^{-t^2/2}\right|\leqslant\frac{\beta_{2+\delta}|t|^{2+\delta}}{(1+\delta)(2+\delta)}\sup_{x>0}\frac{|e^{ix}-1|}{x^\delta},$$

however, it turns out that for all $0<\delta<1$

$$\frac{1}{(1+\delta)(2+\delta)}\sup_{x>0}\frac{|e^{ix}-1|}{x^\delta}>\sup_{x>0}\frac{\left|e^{ix}-1-ix-(ix)^2/2\right|}{x^{2+\delta}}=\gamma_\delta,$$

that is, the coefficient at $\beta_{2+\delta}|t|^{2+\delta}$ in this estimate will be greater than that in the third estimate of lemma 2.8. This circumstance is critical for the estimation of the remainder in the central limit theorem since it is this coefficient that determines the value of the constant at the main term. This is the reason why the third estimate of lemma 2.8 is more preferable, and will be used for our purposes.

3. The moment inequality

Theorem 3.1. *For any r.v. $X\in\mathcal{F}_3$, for all $\lambda\geqslant 1$ the inequality*

$$|\mathsf{E}X^3|+3\mathsf{E}|X|\cdot\mathsf{E}X^2\leqslant\lambda\mathsf{E}|X|^3+M(p(\lambda),\lambda)(\mathsf{E}X^2)^{3/2}$$

holds, where

$$p(\lambda)=\frac12-\sqrt{\frac{\lambda+1}{\lambda+3}}\sin\left(\frac{\pi}{6}-\frac13\arctan\sqrt{\lambda^2+2\frac{\lambda-1}{\lambda+3}}\right),$$

$$M(p,\lambda)=\frac{1-\lambda+2(\lambda+2)p-2(\lambda+3)p^2}{\sqrt{p(1-p)}},\quad 0<p\leqslant\frac12,\ \lambda\geqslant 1,$$

with equality attained for each $\lambda\geqslant 1$ at the family of two-point distributions $\big\{\mathsf{P}\big(X=\sigma\sqrt{q/p}\big)=p=1-\mathsf{P}\big(X=-\sigma\sqrt{p/q}\big)\colon\sigma>0\big\}$, where $p=p(\lambda)$, $q=1-p(\lambda)$.

The optimal values of the parameter $\lambda=\lambda(\beta_3)$, delivering the minimum to the right-hand side of the inequality in theorem 3.1 and the corresponding values $p=p(\lambda(\beta_3))$ are presented for some $\beta_3=\mathsf{E}|X|^3/(\mathsf{E}X^2)^{3/2}$ in the fourth and seventh columns, respectively, of table 2 below.

Remark 3.2. It can be made sure that the function $p(\lambda)$ increases monotonically for $\lambda\geqslant 1$, varying within the limits

$$0.3169\ldots=\frac12\left(1-\sqrt{1-\sqrt3/2}\right)=p(1)\leqslant p(\lambda)\leqslant\lim_{\lambda\to\infty}p(\lambda)=\frac12.$$

Moreover, as it will be seen from the proof, the function $M(p(\lambda),\lambda)$ can be represented as

$$M(p(\lambda),\lambda)=\sup_{0<p\leqslant 1/2}(\alpha_3(p)-\lambda\beta_3(p)+3\beta_1(p)),$$

where $\alpha_3(p)$, $\beta_3(p)$, $\beta_1(p)$ are, respectively, the third original, third absolute and first absolute moments of the Bernoulli distribution assigning the probabilities p and $q=1-p$ to the points $\sqrt{q/p}$ and $-\sqrt{p/q}$:

$$M(p(\lambda),\lambda)=\sup\left\{\frac{q-p-\lambda(p^2+q^2)+6pq}{\sqrt{pq}}:0<p\leqslant\frac12,\ q=1-p\right\}.\tag{3.1}$$

From this representation, first, it follows that the function $M(p,\lambda)$ decreases monotonically in $\lambda \geqslant 1$ for each $0 < p \leqslant 1/2$. The same property is inherent in $M(p(\lambda),\lambda)$, since for any $\lambda_1 \geqslant \lambda_2 \geqslant 1$ we have

$$M(p(\lambda_1),\lambda_1) \leqslant M(p(\lambda_1),\lambda_2) \leqslant \sup_{0<p<1/2} M(p,\lambda_2) = M(p(\lambda_2),\lambda_2).$$

Second, evidently,

$$M(p(\lambda),\lambda) \geqslant \frac{q-p-\lambda(p^2+q^2)+6pq}{\sqrt{pq}}\bigg|_{p=q=1/2} = 3-\lambda, \quad \lambda \geqslant 1,$$

with equality attained at $\lambda \to \infty$, so that

$$\inf_{\lambda\geqslant 1}\big(\lambda + M(p(\lambda),\lambda)\big) = \lim_{\lambda\to\infty}\big(\lambda + M(p(\lambda),\lambda)\big) = 3.$$

Thus, the function $M(p(\lambda),\lambda)$ decreases monotonically for all $\lambda \geqslant 1$, varying within the limits

$$2.3599\ldots = 2\sqrt{3\sqrt{3}(2-\sqrt{3})} = M(p(1),1) \geqslant M(p(\lambda),\lambda) > \lim_{\lambda\to\infty} M(p(\lambda),\lambda) = -\infty,$$

whence it follows that $M(p(\lambda),\lambda)$ alters its sign at the unique point $\lambda = \sqrt{10}$ corresponding to the value $p(\sqrt{10}) = 2 - \sqrt{10}/2 = 0.4188\ldots$, so that

$$M(p(\lambda),\lambda) < 0 \quad \Longleftrightarrow \quad \lambda > \sqrt{10}.$$

Since $p^2+q^2-\sqrt{pq} = -2pq-\sqrt{pq}+1 > 0$ for all $p \in (0,1/2)$, $q = 1-p$, from (3.1) it also follows that the function

$$\lambda + M(p(\lambda),\lambda) = \sup\left\{\frac{q-p-\lambda(p^2+q^2-\sqrt{pq})+6pq}{\sqrt{pq}} : 0 < p \leqslant \frac{1}{2},\ q = 1-p\right\}$$

decreases monotonically, varying within the limits

$$3 < \lambda + M(p(\lambda),\lambda) \leqslant 1 + 2\sqrt{3\sqrt{3}(2-\sqrt{3})} = 3.3599\ldots, \quad \lambda \geqslant 1. \tag{3.2}$$

Using theorem 3.1 it is possible to improve a result due to C.-G. Esseen [10], according to which for a sequence of independent r.v.'s $X_1, X_2\ldots$ with the d.f. $F \in \mathcal{F}_3^h$ for some $h > 0$ such that $\mathsf{E}X_1^2 = 1$, $\mathsf{E}X_1^3 = \alpha_3$, $\mathsf{E}|X_1|^3 = \beta_3$, the relation

$$\psi(F) \equiv \limsup_{n\to\infty} \Delta_n\sqrt{n} = \frac{|\alpha_3|+3h}{6\sqrt{2\pi}} \leqslant \frac{\sqrt{10}+3}{6\sqrt{2\pi}}\,\beta_3 \equiv \psi_1(\beta_3)$$

holds (see (1.4) and (1.6)).

On the other hand, according to (2.3) for h we have the estimate $h \leqslant \beta_3 + \beta_1$, whence it follows that in the case considered by Esseen

$$\psi(F) \leqslant \frac{|\alpha_3| + 3(\beta_3 + \beta_1)}{6\sqrt{2\pi}} \leqslant \inf_{\lambda \geqslant 1} \frac{(\lambda+3)\beta_3 + M(p(\lambda), \lambda)}{6\sqrt{2\pi}} =$$
$$= \inf_{c \geqslant 2/(3\sqrt{2\pi})} (c\beta_3 + K(c)) \equiv \psi_2(\beta_3), \quad (3.3)$$

where

$$K(c) = \frac{M(p(\lambda), \lambda)}{6\sqrt{2\pi}}\bigg|_{\lambda = 6\sqrt{2\pi}c - 3}.$$

Moreover, from theorem 3.1 it follows that c cannot be less than $2/(3\sqrt{2\pi}) = 0.2659\ldots$, and $K(c)$ in (3.3) can be made less for no $c \geqslant 2/(3\sqrt{2\pi})$. From (3.3) with $c = (\sqrt{10}+3)/(6\sqrt{2\pi}) = 0.4097\ldots$ (that corresponds to $\lambda = \sqrt{10}$, $K(c) = 0$) Esseen's bound follows, whereas (3.3) with $c = 2/(3\sqrt{2\pi})$ (that corresponds to $\lambda = 1$) implies the estimate

$$\psi(F) \leqslant \frac{2}{3\sqrt{2\pi}} \cdot \beta_3 + \sqrt{\frac{2\sqrt{3}-3}{6\pi}} < 0.2660\beta_3 + 0.1570, \quad (3.4)$$

which is more accurate than Esseen's bound $\psi(F) \leqslant \psi_1(\beta_3)$ for

$$\beta_3 \geqslant \frac{2\sqrt{3\sqrt{3}(2-\sqrt{3})}}{\sqrt{10}-1} = 1.0914\ldots,$$

although the value $c = 2/(3\sqrt{2\pi})$ (that is, $\lambda = 1$) is optimal in (3.3) only for $\beta_3 \geqslant 1.2185\ldots$

Comparing the functions $\psi_1(\beta_3)$ and $\psi_2(\beta_3)$, we conclude that their values coincide only at the unique point β_3 for which $c = (\sqrt{10}+3)/(6\sqrt{2\pi})$, $K(c) = 0$ (that corresponds to $\lambda = \sqrt{10}$, $p(\sqrt{10}) = 2 - \sqrt{10}/2$), that is, at the point

$$\beta_3 = \frac{p^2 + (1-p)^2}{\sqrt{p(1-p)}}\bigg|_{p = 2 - \sqrt{10}/2} = \sqrt{20(\sqrt{10}-3)/3} = 1.0401\ldots,$$

and for all the rest of the values of $\beta_3 \geqslant 1$ the strict inequality $\psi_1(\beta_3) > \psi_2(\beta_3)$ holds. In particular, for $\beta_3 = 1$ (that is, for the symmetric Bernoulli distribution) $\psi_1(1) = (\sqrt{10}+3)/(6\sqrt{2\pi}) = 0.4097\ldots$, while

$$\psi_2(1) = \lim_{c\to\infty}(c + K(c)) = \lim_{\lambda\to\infty} \frac{\lambda + 3 + M(p(\lambda), \lambda)}{6\sqrt{2\pi}}$$
$$= \frac{1}{\sqrt{2\pi}} = 0.3989\ldots < \psi_1(1) - 0.0107.$$

The values of the functions $\psi_1(\beta_3)$ and $\psi_2(\beta_3)$ for some $\beta_3 \geqslant 1$ are presented in the second and third columns of table 2. The corresponding values of $c = c(\beta_3)$ and $K = K(c(\beta_3))$ delivering the minimum in (3.3) are presented in the fifth and sixth columns of table 2.

β_3	ψ_1	ψ_2	λ	c	K	p
1	0.4097	0.3989	+ inf	+ inf	− inf	1/2
1.01	0.4138	0.4111	7.2034	0.6784	-0.2741	0.4592
1.02	0.4179	0.4170	4.8305	0.5206	-0.1141	0.4424
1.03	0.4220	0.4218	3.7862	0.4512	-0.0430	0.4296
1.04	0.4261	0.4261	3.1682	0.4101	-0.0005	0.4189
1.05	0.4302	0.4300	2.7497	0.3823	0.0286	0.4095
1.06	0.4343	0.4337	2.4432	0.3619	0.0501	0.4011
1.07	0.4384	0.4373	2.2070	0.3462	0.0668	0.3934
1.08	0.4425	0.4407	2.0182	0.3336	0.0803	0.3863
1.09	0.4466	0.4440	1.8633	0.3233	0.0915	0.3796
1.10	0.4507	0.4471	1.7335	0.3147	0.1009	0.3733
1.12	0.4589	0.4533	1.5275	0.3010	0.1161	0.3618
1.14	0.4670	0.4592	1.3707	0.2906	0.1279	0.3513
1.16	0.4752	0.4649	1.2470	0.2823	0.1374	0.3416
1.18	0.4834	0.4705	1.1470	0.2757	0.1451	0.3326
1.20	0.4916	0.4760	1.0645	0.2702	0.1517	0.3243
1.21	0.4957	0.4787	1.0284	0.2678	0.1546	0.3203
1.22	0.4998	0.4813	1.0000	0.2659	0.1569	0.3169

Table 2: The values of the functions $\psi_1(\beta_3)$ and $\psi_2(\beta_3)$ for some β_3; optimal values of $c = (\lambda+3)/(6\sqrt{2\pi})$ delivering the minimum to $\psi_2(\beta_3)$ (see (3.3)); the corresponding values of $K(c)$ in (3.3); the parameter $p(\lambda)$ of the extremal distribution.

Proof of theorem 3.1. Since for $\sigma^2 \equiv \mathsf{E}X^2 = 0$ the statement of the theorem is obvious, in what follows we assume that $\sigma > 0$. Consider the functional

$$J_{\lambda,\sigma}(X) = \left(|\mathsf{E}X^3| + 3\mathsf{E}|X|\sigma^2 - \lambda\mathsf{E}|X|^3\right)/\sigma^3, \quad X \in \mathcal{F}_3.$$

Then the statement of the theorem is equivalent to

$$\sup_{\sigma>0}\ \sup_{X\in\mathcal{F}_3\colon\, \mathsf{E}X=0,\ \mathsf{E}X^2=\sigma^2} J_{\lambda,\sigma}(X) = M(p(\lambda),\lambda).$$

On the other hand, for any $\sigma > 0$

$$\sup_{X\in\mathcal{F}_3\colon\, \mathsf{E}X=0,\ \mathsf{E}X^2=\sigma^2} J_{\lambda,\sigma}(X) = \sup_{X\in\mathcal{F}_3\colon\, \mathsf{E}X=0,\ \mathsf{E}X^2=\sigma^2} J_{\lambda,\sigma}(-X)$$
$$= \sup_{X\in\mathcal{F}_3\colon\, \mathsf{E}X=0,\ \mathsf{E}X^2=\sigma^2} \widetilde{J}_{\lambda,\sigma}(X),$$

where

$$\widetilde{J}_{\lambda,\sigma}(X) = \left(\mathsf{E}X^3 + 3\mathsf{E}|X|\sigma^2 - \lambda\mathsf{E}|X|^3\right)/\sigma^3.$$

With the account of the results of W. Hoeffding [15] and V. M. Zolotarev [40] it is easy to see that for each $\sigma > 0$ the extremum of the moment-type functional

$\widetilde{J}_{\lambda,\sigma}(X)$ linear with respect to $F \in \mathcal{F}_3$ under two moment-type restrictions

$$\mathsf{E}X = 0, \quad \mathsf{E}X^2 = \sigma^2,$$

is attained on distributions concentrated in at most three points. Without loss of generality assume that the r.v. X takes the values $x < y \leqslant 0 < z$ with the probabilities

$$\mathsf{P}(X = x) = \frac{\sigma^2 + yz}{(z - x)(y - x)}, \quad \mathsf{P}(X = y) = -\frac{\sigma^2 + xz}{(z - y)(y - x)},$$

$$\mathsf{P}(X = z) = \frac{\sigma^2 + xy}{(z - x)(z - y)}, \quad -yz \leqslant \sigma^2 \leqslant -xz.$$

Then

$$\mathsf{E}|X| = \frac{2z(\sigma^2 + xy)}{(x - z)(y - z)}, \quad 3\mathsf{E}|X|\sigma^2 = \frac{6z\sigma^4 + 6xyz\sigma^2}{(x - z)(y - z)},$$

$$\mathsf{E}|X|^3 = \frac{(z^3 + a)\sigma^2 - xyz(xy - xz - yz - z^2)}{(z - x)(z - y)},$$

$$\mathsf{E}X^3 = (x + y + z)\sigma^2 + xyz = \frac{(z^3 - a)\sigma^2 + xyz(xy - xz - yz + z^2)}{(z - x)(z - y)},$$

$$a = a(x, y, z) = z(x^2 + y^2 + xy) - xy(x + y) > 0, \quad x < y \leqslant 0 < z,$$

$$\widetilde{J}_{\lambda,\sigma}(X) = \big(6z\sigma + (6xyz - (\lambda - 1)z^3 - a(\lambda + 1))\sigma^{-1} + \\ + xyz((\lambda + 1)(xy - xz - yz) - (\lambda - 1)z^2)\sigma^{-3}\big)/((z - x)(z - y))$$

and

$$\sup_{\sigma>0} \sup_{X\in\mathcal{F}_3:\ \mathsf{E}X=0,\ \mathsf{E}X^2=\sigma^2} \widetilde{J}_{\lambda,\sigma}(X) = \sup_{X\in\mathcal{F}_3:\ \mathsf{E}X=0} \sup_{\sigma>0} \frac{g(\sigma)}{(z - x)(z - y)},$$

where

$$g(\sigma) = g(\sigma, x, y, z, \lambda) = 6z\sigma + (6xyz - (\lambda - 1)z^3 - a(\lambda + 1))\sigma^{-1} +$$

$$+ xyz((\lambda + 1)(xy - xz - yz) - (\lambda - 1)z^2)\sigma^{-3}.$$

Show that the function $g(\sigma)$ is quasi-convex for $\sigma > 0$, namely, either $g(\sigma)$ increases monotonically for $\sigma > 0$ or there exists a point $\sigma_1 > 0$ such that $g(\sigma)$ decreases monotonically for $0 < \sigma < \sigma_1$ and increases monotonically for $\sigma > \sigma_1$. For this purpose differentiate $g(\sigma)$ and find the stationary points. We have

$$g'(\sigma) = 6z + (a(\lambda + 1) + (\lambda - 1)z^3 - 6xyz)\sigma^{-2} \\ - 3xyz((\lambda + 1)(xy - xz - yz) - (\lambda - 1)z^2)\sigma^{-4} \geqslant 0$$

if and only if

$$6\sigma^4+(a(\lambda+1)/z+(\lambda-1)z^2-6xy)\sigma^2+3xy((\lambda-1)z^2-(\lambda+1)(xy-xz-yz))\geqslant 0.$$

So, the equation $g'(\sigma)=0$ is equivalent to the quadratic equation with respect to σ^2. The latter either has no real roots and then $g'(\sigma)>0$ and $g(\sigma)$ increases, or has one real root which is the point of reflection of $g(\sigma)$ and then $g(\sigma)$ increases, or has two different real roots $\sigma_1<\sigma_2$ so that σ_1 is the point of maximum and σ_2 is the point of minimum. The desired property of the function g will be proved if we show that the smaller root σ_1 of the equation $g'(\sigma)=0$ is non-positive.

The smaller root s_1 of the quadratic equation $s^2+bs+c=0$ with two different roots has the form $s_1=-b-\sqrt{b^2-4c}$. It is obvious that $s_1\leqslant 0$ if and only if either $b>0$, or $b\leqslant 0$ and $c\leqslant 0$, that is, if the condition $b\leqslant 0$ implies $c\leqslant 0$. Apply this reasoning to $s=\sigma^2$,

$$b=(a(\lambda+1)/z+(\lambda-1)z^2-6xy)/6,\quad c=\frac{xy}{2z}((\lambda-1)z^3-(\lambda+1)z(xy-xz-yz)).$$

Indeed, the condition $b\leqslant 0$ implies $(\lambda-1)z^3\leqslant 6xyz-a(\lambda+1)$ and

$$c\cdot\frac{2z}{(\lambda+1)xy}\leqslant\frac{6xyz}{\lambda+1}-a-z(xy-xz-yz)\leqslant 3xyz-a-z(xy-xz-yz)=$$

$$=xz(y-x)-y^2z+(x+y)(xy+z^2)\leqslant 0$$

for all $\lambda\geqslant 1$ and $x<y\leqslant 0<z$. So, the maximum value of the function $g(\sigma)$ on the interval $-yz\leqslant\sigma^2\leqslant -xz$ is attained either at $\sigma^2=-yz$ and then $\mathsf{P}(X=x)=0$, or at $\sigma^2=-xz$ and then $\mathsf{P}(X=y)=0$, that is, the extremum of the functional $\widetilde{J}_{\lambda,\sigma}(X)$ is attained at two-point distributions of the r.v. X.

Now let $\mathsf{P}(X=\sigma\sqrt{q/p})=p$, $\mathsf{P}(X=-\sigma\sqrt{p/q})=q=1-p$, $0<p<1$. Then

$$\mathsf{E}X^3=\frac{q-p}{\sqrt{pq}}\,\sigma^3,\quad \mathsf{E}|X|^3=\frac{p^2+q^2}{\sqrt{pq}}\,\sigma^3=\frac{1-2pq}{\sqrt{pq}}\,\sigma^3,\quad \mathsf{E}|X|=2\sqrt{pq}\sigma.$$

Since $\mathsf{E}X^3<0$ for $p<1/2$, the range of the values of p under consideration can be restricted to the semi-interval $(0,1/2]$. Further, the functional

$$\widetilde{J}_{\lambda,\sigma}(X)=\frac{\mathsf{E}X^3-\lambda\mathsf{E}|X|^3+3\mathsf{E}|X|\sigma^2}{\sigma^3}=\frac{q-p-\lambda(1-2pq)+6pq}{\sqrt{pq}}=$$

$$=\frac{1-\lambda+2(\lambda+2)p-2(\lambda+3)p^2}{\sqrt{p(1-p)}}\equiv M(p,\lambda)$$

does not depend on σ and hence,

$$\sup_{\sigma>0}\ \sup_{X\in\mathcal{F}_3\colon\,\mathsf{E}X=0,\mathsf{E}X^2=\sigma^2}\widetilde{J}_{\lambda,\sigma}(X)=\sup_{0<p\leqslant 1/2}M(p,\lambda).$$

It remains to show that for each λ, $M(p,\lambda)$ attains its maximum value at the point $p=p(\lambda)$ specified in the formulation of Theorem 3.1.

Consider the zeroes of the derivative $M'_p(p,\lambda)$. We have

$$M'_p(p,\lambda)\cdot 2(p(1-p))^{3/2} = 4(\lambda+3)p^3-6(\lambda+3)p^2+6p+\lambda-1 \equiv h(p), \quad 0<p\leqslant 1/2.$$

Since

$$h''(p) = (12(\lambda+3)(p^2-p)+6)'_p = 12(\lambda+3)(2p-1)\leqslant 0$$

for $p\leqslant 1/2$, the function $h(p)$ is concave on the interval $(0,\,1/2]$. Moreover, $h(0+)=\lambda-1\geqslant 0$, $h(1/2)=-1<0$, that is, the function

$$h(p) = M'_p(p,\lambda)\cdot 2(p(1-p))^{3/2}$$

changes its sign at the unique point on the interval $(0,\,1/2]$, which delivers the maximum to the function $M(p,\lambda)$ for each $\lambda\geqslant 1$. It is easy to see that

$$p(\lambda) = \frac{1}{2}-\sqrt{\frac{\lambda+1}{\lambda+3}}\sin\left(\frac{\pi}{6}-\frac{1}{3}\arctan\sqrt{\lambda^2+2\frac{\lambda-1}{\lambda+3}}\right)\in(0,1/2],$$

for all $\lambda\geqslant 1$ and $h(p(\lambda))\equiv 0$, that is, $p(\lambda)$ is the point of the maximum of the function $M(p,\lambda)$. □

4. Estimates of the accuracy of the normal approximation to the distributions of sums of independent random variables

In addition to the notation introduced in section 1, let

$$\nu_n = 1+\sum_{j=1}^n \beta_{\delta,j}\,\sigma_j^2\Big/\sum_{j=1}^n\beta_{2+\delta,j}.$$

It is easy to see that the quantities ν_n, $\ell_n=\sum_{j=1}^n\beta_{2+\delta,j}$ are linked with the quantity ε_n introduced in section 2 by the relation $\varepsilon_n=\nu_n\ell_n$. Furthermore, by the Lyapounov inequality we have $1\leqslant\nu_n\leqslant 2$, and in the case of identically distributed summands we have

$$\nu_n = 1+\frac{\beta_\delta\sigma^2}{\beta_{2+\delta}}\leqslant 1+\frac{1}{n^{\delta/2}\ell_n}\leqslant 2. \tag{4.1}$$

We will also use the following inequality proved by H. Prawitz in [25]:

$$\sum_{j=1}^n\beta_{2+\delta,j}^r\leqslant\left(\sum_{j=1}^n\beta_{2+\delta,j}\right)^r = (B_n^{2+\delta}\ell_n)^r,\quad r\geqslant 1. \tag{4.2}$$

Before we proceed to the construction of new estimates of the accuracy of the normal approximation, note that

$$\kappa\equiv\sup_{F\in\mathcal{F}_2}\sup_x|F(x)-\Phi(x)| = \sup_{b>0}\left(\frac{1}{1+b^2}-\Phi(-b)\right) = 0.54093\ldots \tag{4.3}$$

This relation is a consequence of lemma 12.3 from the monograph [5], establishing an upper bound for the uniform distance between F and Φ, and the paper [18] where the extremal two-point distribution was constructed. Relation (4.3) provides a universal estimate for all distributions with finite second moment. We will use this estimate for the purpose of bounding the range of the values of ℓ_n under consideration.

Recall that in section 2 by $f_j(t)$ we denoted the characteristic functions of the r.v.'s X_j, $j = 1, \ldots, n$, $\overline{f}_n(t) = \prod_{j=1}^n f_j(t/B_n)$, $r_n(t) = |\overline{f}_n(t) - e^{-t^2/2}|$.

The key role in the construction of estimates for Δ_n is played by Prawitz' smoothing inequality presented in the following lemma.

Lemma 4.1 (see [23]). *For all $n \geqslant 1$ and arbitrary d.f.'s $F_1, \ldots, F_n$ with zero expectations for any $0 < t_0 \leqslant 1$ and $T > 0$ there holds the inequality*

$$\Delta_n \leqslant 2\int_0^{t_0} |K(t)| r_n(Tt)\, dt + 2\int_{t_0}^{1} |K(t)| \cdot |\overline{f}_n(Tt)| dt +$$

$$+2\int_0^{t_0} \left| K(t) - \frac{i}{2\pi t}\right| e^{-T^2t^2/2} dt + \frac{1}{\pi}\int_{t_0}^{\infty} e^{-T^2t^2/2}\frac{dt}{t},$$

where

$$K(t) = \frac{1}{2}(1-|t|) + \frac{i}{2}\left[(1-|t|)\cot \pi t + \frac{\mathrm{sign} t}{\pi}\right], \quad -1 \leqslant t \leqslant 1,$$

furthermore, the function $K(t)$ satisfies the inequalities

$$|K(t)| \leqslant \frac{1.0253}{2\pi|t|}, \quad \left|K(t) - \frac{i}{2\pi t}\right| \leqslant \frac{1}{2}\left(1 - |t| + \frac{\pi^2 t^2}{18}\right), \quad -1 \leqslant t \leqslant 1.$$

The following lemma is important for the calculation of constants in the estimates of the normal approximation, to be constructed below. By $\mathcal{D}$ denote the class of real continuous nonnegative functions $J(z)$ defined for $z \geqslant 0$, which have a unique maximum and do not have a minimum for $z > 0$.

Lemma 4.2 (see [25, 11]). *Let $a < b$ and $k > 0$ be arbitrary constants, $g(s)$ and $G(s)$ be positive monotonically increasing differentiable functions on $a \leqslant s \leqslant b$. If the function*

$$\varphi(s) = \frac{G(s) - G(a)}{g^k(s)}, \quad a \leqslant s \leqslant b,$$

increases monotonically, then the function

$$J(z) = z^k \int_a^b e^{-zg(s)}\, dG(s), \quad z \geqslant 0,$$

belongs to the class $\mathcal{D}$.

If $G(a) = g(a) = 0$, *then the condition that* $\varphi(s)$ *increases can be relaxed the requirement that the function*

$$\psi(s) = \frac{G'(s)}{(g^k(s))'}, \quad a \leqslant s \leqslant b,$$

increases.

Lemma 4.1 for all $F_1, \dots, F_n \in \mathcal{F}_{2+\delta}$, $n \geqslant 1$, $0 < \delta \leqslant 1$, $0 < t_0 \leqslant t_1 \leqslant 1$ and $T > 0$ implies the estimate

$$\Delta_n \leqslant I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$I_1 = \frac{2}{T} \int\limits_0^{t_0 T} \left| K\left(\frac{t}{T}\right) \right| r_n(t)\, dt,$$

$$I_2 = \frac{1.0253}{\pi} \int\limits_{t_0}^{t_1} |\overline{f}_n(Tt)| \frac{dt}{t},$$

$$I_3 = 2 \int\limits_{t_1}^{1} |K(t)| \cdot |\overline{f}_n(Tt)| dt,$$

$$I_4 = \int\limits_0^{t_0} \left(1 - t + \frac{\pi^2 t^2}{18}\right) e^{-T^2 t^2/2} dt,$$

$$I_5 = \frac{1}{\pi} \int\limits_{t_0}^{\infty} e^{-T^2 t^2/2} \frac{dt}{t}.$$

We will estimate the integrals I_2, I_3, I_4, I_5 in the same way as it was done in [25, 11]. We have

$$\begin{aligned} I_4 + I_5 &= \int\limits_0^{\infty} \left(1 - t + \frac{\pi^2 t^2}{18}\right) e^{-T^2 t^2/2}\, dt + \int\limits_{t_0}^{\infty} \left(\frac{1}{\pi t} - 1 + t - \frac{\pi^2 t^2}{18}\right) e^{-T^2 t^2/2}\, dt \\ &= \sqrt{\frac{\pi}{2}} \cdot \frac{1}{T} - \frac{1}{T^2} + \frac{\pi^{5/2}}{18\sqrt{2}} \cdot \frac{1}{T^3} + \frac{\widetilde{I}_4(T, t_0)}{T^2}, \end{aligned}$$

where

$$\widetilde{I}_4(T, t_0) = T^2 \int\limits_{t_0}^{\infty} g(t) t e^{-T^2 t^2/2}\, dt, \quad g(t) = \frac{1}{t}\left(\frac{1}{\pi t} - 1 + t - \frac{\pi^2 t^2}{18}\right), \quad t > 0.$$

Since

$$\sup_{t>0} g'(t) = \sup_{t>0}\left(-\frac{2}{\pi t^3}+\frac{1}{t^2}-\frac{\pi^2}{18}\right) = \left(-\frac{2}{\pi t^3}+\frac{1}{t^2}-\frac{\pi^2}{18}\right)\bigg|_{t=3/\pi} = -\frac{\pi^2}{54} < 0,$$

the function $g(t)$ decreases monotonically for $t > 0$ and hence,

$$\begin{aligned}\widetilde{I}_4(T,t_0) &\leqslant (g(t_0)\vee 0)T^2\int\limits_{t_0}^{\infty} te^{-T^2t^2/2}\,dt \\ &= \frac{1}{t_0}\left(\frac{1}{\pi t_0}-1+t_0-\frac{\pi^2t_0^2}{18}\right)e^{-T^2t_0^2/2}\vee 0 \equiv J_4(T,t_0).\end{aligned}$$

The function $J_4(T,t_0)$ of $T>0$ is obviously in $\mathcal{D}$ for each fixed $t_0\in(0,1]$.

Now choose the values of the parameters T and $t_1\in(0,\,1]$. It is clear that for the efficient estimation of I_2 and I_3 we should use the upper bounds of $|\overline{f}_n(Tt)|$ which are almost everywhere strictly less than one. These upper bounds are given by theorem 2.2, but for their applicability we should assume that $T(\nu_n\ell_n)^{1/\delta}\leqslant 2\pi$. On the other hand, taking into account the term of the form $1/T$ in the estimate for I_4+I_5, we come to the conclusion that T should be taken as large as possible. Therefore finally we set

$$T = 2\pi\left(\nu_n\ell_n\right)^{-1/\delta},\quad t_1=t_1(\delta)=\frac{\theta_0(\delta)}{T(\nu_n\ell_n)^{1/\delta}}=\frac{\theta_0(\delta)}{2\pi}. \tag{4.4}$$

As it follows from the definition, $\theta_0(\delta)\in(\pi,\,2\pi)$, so that $t_1(\delta)\in(1/2,\,1)$ for all $0<\delta\leqslant 1$. Moreover, since $\nu_n\leqslant 2$, the quantities T and ℓ_n are linked by the inequalities

$$T\geqslant 2\pi(2\ell_n)^{-1/\delta},\quad \ell_n\leqslant\left(\frac{2\pi}{T}\right)^{\delta}.$$

So, for the specified T and t_1 the estimates from theorem 2.2 and lemma 2.7 take the form

$$|\overline{f}_n(Tt)|\leqslant\exp\left\{-\frac{T^2t^2}{2}\Big(1-2\varkappa_\delta(2\pi|t|)^{\delta}\Big)\right\},\quad t\in\mathbf{R}, \tag{4.5}$$

$$|\overline{f}_n(Tt)|\leqslant\exp\left\{-T^2\frac{1-\cos 2\pi t}{4\pi^2}\right\},\quad t_1(\delta)\leqslant|t|\leqslant 1, \tag{4.6}$$

$$\begin{aligned} r_n(t)\leqslant\sum_{j=1}^{n}&\left|f_j(t/B_n)-e^{-\sigma_j^2t^2/(2B_n^2)}\right|\times \\ &\times\exp\left\{-\frac{t^2}{2}\left(1-\frac{\sigma_j^2}{B_n^2}-2\varkappa_\delta\Big(\frac{2\pi|t|}{T}\Big)^{\delta}\right)\right\},\quad t\in\mathbf{R}.\end{aligned} \tag{4.7}$$

Using the estimate (4.5) in the integral I_2 and the estimate (4.6) in the integral I_3, for any $t_0 \leqslant t_1(\delta)$ we obtain

$$I_2 \leqslant \frac{1.0253}{\pi} \int_{t_0}^{t_1} \exp\left\{-\frac{T^2t^2}{2}\Big(1 - 2\varkappa_\delta(2\pi t)^\delta\Big)\right\} \frac{dt}{t},$$

$$\begin{aligned} I_3 &\leqslant 2\int_{t_1}^{1} |K(t)| \exp\left\{-T^2\frac{1-\cos 2\pi t}{4\pi^2}\right\} dt \\ &= 2\int_0^{1-t_1} |K(1-t)| \exp\left\{-T^2\frac{1-\cos 2\pi t}{4\pi^2}\right\} dt \\ &= \int_0^{1-t_1} t\sqrt{1+\Big(\frac{1}{\pi t} - \cot \pi t\Big)^2} \exp\left\{-T^2\frac{1-\cos 2\pi t}{4\pi^2}\right\} dt. \end{aligned}$$

As is known (see, e. g., [1, 4.3.91]), the cotangent can be expanded into simple fractions as follows:

$$f(x) \equiv \frac{1}{x} - \cot x = 2x\sum_{k=1}^{\infty} \frac{1}{\pi^2k^2 - x^2}, \quad x \neq 0, \pm\pi, \pm 2\pi, \ldots,$$

whence it follows that the function $f(x)$ is nonnegative and increases monotonically for all $0 < x < \pi$ and hence, for any $0 < \epsilon < 1$ we have

$$I_3 \leqslant \int_0^{1-t_1} t\sqrt{1+\Big(\frac{1}{\pi(t\vee\epsilon)} - \cot \pi(t\vee\epsilon)\Big)^2} \exp\left\{-T^2\frac{1-\cos 2\pi t}{4\pi^2}\right\} dt \equiv J_3(T)/T^2.$$

Set

$$g(t) = \frac{1-\cos 2\pi t}{4\pi^2}, \quad dG(t) = t\sqrt{1+\Big(\frac{1}{\pi(t\vee\epsilon)} - \cot \pi(t\vee\epsilon)\Big)^2} dt, \quad 0 \leqslant t \leqslant 1-t_1.$$

Obviously, $g(0) = 0$ and $g(t)$ increases monotonically for $0 \leqslant t \leqslant 1/2 > 1 - t_1$ (recall that $t_1 > 1/2$). Moreover, it can be made sure that the function $\sin t/t$ decreases for $0 < t \leqslant \pi$ and hence, on the interval $0 \leqslant t \leqslant 1-t_1 \leqslant 1/2$ the function

$$\frac{G'(t)}{g'(t)} = \frac{4\pi^2|K(1-t)|}{(1-\cos 2\pi t)'} = \frac{\pi t}{\sin 2\pi t}\sqrt{1+\Big(\frac{1}{\pi(t\vee\epsilon)} - \cot \pi(t\vee\epsilon)\Big)^2}$$

increases as the product of two monotonically increasing nonnegative functions. So, according to lemma 4.2, $J_3 \in \mathcal{D}$ for any $0 < \epsilon < 1$. Everywhere in what follows we use the value $\epsilon = 10^{-4}$.

Consider the upper bound for I_2 obtained above. It is easy to see that the function $t^2\left(1-2\varkappa_\delta(2\pi t)^\delta\right)$ is positive for $t\in(0,\,t_1]$, since, as it has been already mentioned, $t_1>1/2$, $\varkappa_\delta\leqslant\pi^{-\delta}/2$, and has a unique maximum at the point $t=t_{max}(\delta)=\left((2\pi)^\delta(2+\delta)\varkappa_\delta\right)^{-1/\delta}\in(0,t_1)$, and hence, there exists a unique root

$$t_2=t_2(\delta)\in(0,t_{max}(\delta))$$

of the equation

$$t^2\left(1-2\varkappa_\delta(2\pi t)^\delta\right)=t_1^2\left(1-2\varkappa_\delta(2\pi t_1)^\delta\right),\quad 0<t<t_1(\delta),$$

so that for all $t\in(t_2,t_1)$ we have

$$t^2\left(1-2\varkappa_\delta(2\pi t)^\delta\right)>t_1^2\left(1-2\varkappa_\delta(2\pi t_1)^\delta\right).$$

Splitting the integration domain in the upper bound for I_2 in two parts by the point t_2 we obtain the estimate

$$I_2\leqslant\left(J_{21}(T,t_0)+I_{22}(T,t_0)\right)/T^2,$$

where $J_{21}(T,t_0)=0$, if $t_0\geqslant t_2$, and

$$J_{21}(T,t_0)=\frac{1.0253}{\pi}\,T^2\int\limits_{t_0}^{t_2}\exp\left\{-\frac{T^2t^2}{2}\left(1-2\varkappa_\delta(2\pi t)^\delta\right)\right\}\frac{dt}{t},\quad\text{if } t_0\leqslant t_2,$$

$$\begin{aligned}I_{22}(T,t_0)&=\frac{1.0253}{\pi}\,T^2\int\limits_{t_0\vee t_2}^{t_1}\exp\left\{-\frac{T^2t^2}{2}\left(1-2\varkappa_\delta(2\pi t)^\delta\right)\right\}\frac{dt}{t}\\&\leqslant\frac{1.0253}{\pi}\,T^2\exp\left\{-\frac{T^2t_1^2}{2}\left(1-2\varkappa_\delta(2\pi t_1)^\delta\right)\right\}\int\limits_{t_0\vee t_2}^{t_1}\frac{dt}{t}\\&=\frac{1.0253}{\pi}\,T^2\exp\left\{-\frac{T^2t_1^2}{2}\left(1-2\varkappa_\delta(2\pi t_1)^\delta\right)\right\}\ln\frac{t_1}{t_0\vee t_2}\equiv J_{22}(T,t_0).\end{aligned}$$

The function $J_{22}(T)$ obviously belongs to the class $\mathcal{D}$.

With a fixed $t_0\leqslant t_2(\delta)$, consider $J_{21}(T,t_0)$ as a function of $T>0$. As was mentioned above, on the interval $[t_0,t_2]$ the function $t^2\left(1-2\varkappa_\delta(2\pi t_1)^\delta\right)$ increases, therefore, according to lemma 4.2, for $J_{21}\in\mathcal{D}$ it suffices that the function

$$\frac{\ln t-\ln t_0}{t^2\left(1-K_\delta t^\delta\right)},\quad K_\delta=2\varkappa_\delta(2\pi)^\delta,$$

increases on $[t_0,\,t_2]$, which is equivalent to the inequality

$$\left(\frac{\ln t-\ln t_0}{t^2(1-K_\delta t^\delta)}\right)'=\frac{t(1-K_\delta t^\delta)-(\ln t-\ln t_0)(2t-(2+\delta)K_\delta t^{1+\delta})}{t^4(1-K_\delta t^\delta)^2}\geqslant 0,$$

$t_0 \leqslant t \leqslant t_2$. The last condition is satisfied, if t_0 satisfies the condition

$$\ln t_0 \geqslant \max_{t\in[t_0,\,t_2]} g(t), \quad g(t) = \ln t - \frac{1 - K_\delta t^\delta}{2 - (2+\delta)K_\delta t^\delta}.$$

Taking the derivative

$$g'(t) = \frac{(2+\delta)^2 K_\delta^2 t^{2\delta} - (4 + (2+\delta)^2)K_\delta t^\delta + 4}{t(2 - (2+\delta)K_\delta t^\delta)^2},$$

we find that $g'(t)$ changes its sign from positive to negative in the point

$$t^* = \left(\frac{4}{(2+\delta)^2 K_\delta}\right)^{1/\delta} = \frac{1}{2\pi}\left(\frac{2}{(2+\delta)^2 \varkappa_\delta}\right)^{1/\delta},$$

which maximizes the function $g(t)$ and

$$g(t^*) = \ln t^* - \frac{4+\delta}{2(2+\delta)},$$

and hence, for

$$\begin{aligned} t_0 &\geqslant \max_{t\in[t_0,\,t_2]} \exp\{g(t)\} = \exp\{g(t^*)\} \\ &= \frac{1}{2\pi}\left(\frac{2}{(2+\delta)^2\varkappa_\delta}\right)^{1/\delta} \exp\left\{-\frac{4+\delta}{2(2+\delta)}\right\} \equiv t_3(\delta) \end{aligned}$$

we have $J_{21} \in \mathcal{D}$. So,

$$I_2 + I_3 + I_4 + I_5 \leqslant \sqrt{\frac{\pi}{2}} \cdot \frac{1}{T} + \frac{J(T, t_0)}{T^2},$$

where

$$J(T, t_0) = 0 \vee \left(J_{21}(T, t_0) + J_{22}(T, t_0) + J_3(T) + J_4(T, t_0) - 1 + \frac{\pi^{5/2}}{18\sqrt{2}} \cdot \frac{1}{T}\right),$$

with the functions $J_{21}(T, t_0)$, $J_{22}(T, t_0)$, $J_3(T)$, $J_4(T, t_0)$ of $T > 0$ belonging to $\mathcal{D}$ for each fixed t_0.

Finally, consider I_1. Estimating $r_n(t)$ by (4.7) with T defined in (4.4) we obtain

$$\begin{aligned} I_1 = \frac{2}{T}\int_0^{t_0 T} \left|K\left(\frac{t}{T}\right)\right| r_n(t)dt \leqslant \frac{2}{T}\sum_{j=1}^{n}\int_0^{t_0 T} \left|K\left(\frac{t}{T}\right)\right| \cdot \left|f_j(t/B_n) - e^{-\sigma_j^2 t^2/(2B_n^2)}\right| \times \\ \times \exp\left\{-\frac{t^2}{2}\left(1 - \frac{\sigma_j^2}{B_n^2} - 2\varkappa_\delta\left(\frac{2\pi t}{T}\right)^\delta\right)\right\} dt \end{aligned}$$

$$
-\frac{2}{T}\sum_{j=1}^{n}\int\limits_{0}^{t_0T}\left|K\left(\frac{t}{T}\right)\right|\cdot\left|f_j(t/B_n)-e^{-\sigma_j^2t^2/(2B_n^2)}\right|e^{-t^2/2}\,dt
$$

$$
+\frac{2}{T}\sum_{j=1}^{n}\int\limits_{0}^{t_0T}\left|K\left(\frac{t}{T}\right)-\frac{iT}{2\pi t}\right|\cdot\left|f_j(t/B_n)-e^{-\sigma_j^2t^2/(2B_n^2)}\right|e^{-t^2/2}\,dt
$$

$$
+\frac{2}{T}\sum_{j=1}^{n}\int\limits_{0}^{t_0T}\left|\frac{iT}{2\pi t}\right|\cdot\left|f_j(t/B_n)-e^{-\sigma_j^2t^2/(2B_n^2)}\right|e^{-t^2/2}\,dt\leqslant I_{11}+I_{12}+I_{13},
$$

where

$$
\begin{aligned}
I_{11}=&\frac{2}{T}\sum_{j=1}^{n}\int\limits_{0}^{t_0T}\left|K\left(\frac{t}{T}\right)\right|\cdot\left|f_j(t/B_n)-e^{-\sigma_j^2t^2/(2B_n^2)}\right|e^{-t^2/2}\times\\
&\times\left(\exp\left\{\frac{\sigma_j^2t^2}{2B_n^2}+\varkappa_\delta\left(2\pi/T\right)^\delta t^{2+\delta}\right\}-1\right)dt,\\
I_{12}=&\frac{2}{T}\sum_{j=1}^{n}\int\limits_{0}^{t_0T}\left|K\left(\frac{t}{T}\right)-\frac{iT}{2\pi t}\right|\cdot\left|f_j(t/B_n)-e^{-\sigma_j^2t^2/(2B_n^2)}\right|e^{-t^2/2}\,dt,\\
I_{13}=&\frac{1}{\pi}\sum_{j=1}^{n}\int\limits_{0}^{\infty}\frac{1}{t}\left|f_j(t/B_n)-e^{-\sigma_j^2t^2/(2B_n^2)}\right|e^{-t^2/2}\,dt.
\end{aligned}
$$

Note that for all $j=1,\dots,n$ and $t\leqslant t_0T$ with $T=2\pi\left(\nu_n\ell_n\right)^{-1/\delta}$

$$
\begin{aligned}
&\exp\left\{\frac{\sigma_j^2t^2}{2B_n^2}+\varkappa_\delta\left(2\pi/T\right)^\delta t^{2+\delta}\right\}-1\\
&\leqslant\left(\frac{\sigma_j^2t^2}{2B_n^2}+\frac{\varkappa_\delta(2\pi)^\delta t^{2+\delta}}{T^\delta}\right)\exp\left\{\frac{\sigma_j^2t^2}{2B_n^2}+\frac{\varkappa_\delta(2\pi)^\delta t^{2+\delta}}{T^\delta}\right\}\\
&\leqslant\left(\frac{\sigma_j^2t^2}{2B_n^2}+\varkappa_\delta\nu_n\ell_nt^{2+\delta}\right)\exp\left\{\frac{t^2}{2}\left(\frac{\sigma_j^2}{B_n^2}+2\varkappa_\delta(2\pi t_0)^\delta\right)\right\},
\end{aligned}
$$

Taking into account the estimates for $K(t)$ given by lemma 4.1 and the estimates (2.4), (2.6) for the modulus of the difference of the ch.f.'s from lemma 2.8 for the integral I_{11} we obtain

$$
\begin{aligned}
I_{11}\leqslant&\frac{1.0253}{\pi}\sum_{j=1}^{n}\int\limits_{0}^{t_0T}\left(\frac{\gamma_\delta\beta_{2+\delta,j}\,t^{2+\delta}}{B_n^{2+\delta}}+\frac{\sigma_j^4t^4}{8B_n^4}\mathbf{1}(\delta<1)\right)\left(\frac{\sigma_j^2t^2}{2B_n^2}+\varkappa_\delta\nu_n\ell_nt^{2+\delta}\right)\times\\
&\times\exp\left\{-\frac{t^2}{2}\left(1-\frac{\sigma_j^2}{B_n^2}-2\varkappa_\delta(2\pi t_0)^\delta\right)\right\}dt
\end{aligned}
$$

$$= \frac{1.0253}{\pi} \sum_{j=1}^{n} \int_0^\infty \bigg[\gamma_\delta t^{4+\delta} \frac{\beta_{2+\delta,j}\sigma_j^2}{2B_n^{4+\delta}} + \gamma_\delta \varkappa_\delta \nu_n t^{4+2\delta} \ell_n \frac{\beta_{2+\delta,j}}{B_n^{2+\delta}}$$
$$+ \left(\frac{t^6 \sigma_j^6}{16 B_n^6} + \varkappa_\delta \nu_n t^{6+\delta} \ell_n \frac{\sigma_j^4}{8 B_n^4} \right) \mathbf{1}(\delta < 1) \bigg] \times$$
$$\times \exp\left\{ -\frac{t^2}{2} \left(1 - \frac{\sigma_j^2}{B_n^2} - 2\varkappa_\delta (2\pi t_0)^\delta \right) \right\} dt.$$

Estimate the exponent uniformly with respect to $j = 1, \ldots, n$ by the inequality

$$\max_{1 \leqslant j \leqslant n} \sigma_j^2 \leqslant \Big(\max_{1 \leqslant j \leqslant n} \sigma_j \Big)^2 \leqslant \Big(\max_{1 \leqslant j \leqslant n} \beta_{2+\delta,j} \Big)^{2/(2+\delta)}$$
$$\leqslant \Big(\sum_{j=1}^{n} \beta_{2+\delta,j} \Big)^{2/(2+\delta)} = B_n^2 \ell_n^{2/(2+\delta)}.$$

Estimate the power-type multiplier by the Lyapounov inequality and relation (4.2) to obtain

$$I_{11} \leqslant \frac{1.0253}{16\pi} \int_0^\infty \bigg[(8\gamma_\delta \ell_n^{(4+\delta)/(2+\delta)} t^{4+\delta} + 16 \gamma_\delta \varkappa_\delta \nu_n \ell_n^2 t^{4+2\delta}$$
$$+ \left(\ell_n^{6/(2+\delta)} t^6 + 2\varkappa_\delta \nu_n \ell_n^{(6+\delta)/(2+\delta)} t^{6+\delta} \right) \mathbf{1}(\delta < 1) \bigg] \times$$
$$\times \exp\left\{ -\frac{t^2}{2} \left(1 - \ell_n^{2/(2+\delta)} - 2\varkappa_\delta (2\pi t_0)^\delta \right) \right\} dt.$$

For the case of identically distributed summands, since $\sigma_j^2 = B_n^2/n \leqslant B_n^2 \ell_n n^{-1+\delta/2}$ for all $j = 1, \ldots, n$, we obtain the estimate

$$I_{11} \leqslant \frac{1.0253}{16\pi} \ell_n^2 \int_0^\infty \left(8\gamma_\delta t^{2+\delta} + \frac{t^4}{n^{1-\delta/2}} \mathbf{1}(\delta < 1) \right) \left(\frac{t^2}{n^{1-\delta/2}} + 2\varkappa_\delta \nu_n t^{2+\delta} \right) \times$$
$$\times \exp\left\{ -\frac{t^2}{2} \left(1 - n^{-1} - 2\varkappa_\delta (2\pi t_0)^\delta \right) \right\} dt.$$

Assume that

$$t_0 < \frac{1}{2\pi} \left(\frac{1 - \ell_n^{2/(2+\delta)}}{2\varkappa_\delta} \right)^{1/\delta} \equiv t_4(\delta, \ell_n).$$

The domain $t_3(\delta) \leqslant t_0 < t_4(\delta, \ell_n)$ is non-empty, if

$$\ell_n^{2/(2+\delta)} < 1 - \frac{4}{(2+\delta)^2} \exp\left\{ -\frac{\delta(4+\delta)}{2(2+\delta)} \right\} > 0, \quad 0 < \delta \leqslant 1.$$

For t_0 and ℓ_n specified above introduce the function

$$Q(\ell_n, t_0, r) = \int_0^\infty t^r \exp\left\{-\frac{t^2}{2}\Big(1 - \ell_n^{2/(2+\delta)} - 2\varkappa_\delta(2\pi t_0)^\delta\Big)\right\}$$
$$= \frac{2^{(r-1)/2}\Gamma\big(\frac{r+1}{2}\big)}{\big(1 - \ell_n^{2/(2+\delta)} - 2\varkappa_\delta(2\pi t_0)^\delta\big)^{(r+1)/2}}, \quad r > 0.$$

It is obvious that $Q(\ell_n, t_0, r)$ increases monotonically in ℓ_n with fixed t_0 and r. So, for I_{11} for all $t_0 \leqslant t_4(\delta, \ell_n)$ we obtain

$$I_{11} \leqslant \frac{1.0253}{16\pi}\,\ell_n^{(4+\delta)/(2+\delta)}\Big(8\gamma_\delta Q\left(\ell_n, t_0, 4+\delta\right)$$
$$+ 16\nu_n\varkappa_\delta\gamma_\delta\ell_n^{\delta/(2+\delta)}Q\left(\ell_n, t_0, 4+2\delta\right) + \ell_n^{(2-\delta)/(2+\delta)}Q\left(\ell_n, t_0, 6\right)\mathbf{1}(\delta<1)$$
$$+ 2\nu_n\varkappa_\delta\ell_n^{2/(2+\delta)}Q\left(\ell_n, t_0, 6+\delta\right)\mathbf{1}(\delta<1)\Big) \equiv \ell_n^{(4+\delta)/(2+\delta)}J_{11}(\ell_n, \nu_n, t_0)$$

in the general case, whereas for

$$\frac{1}{n} < 1 - \frac{4\exp\big\{-\frac{\delta(4+\delta)}{2(2+\delta)}\big\}}{(2+\delta)^2} \equiv (\bar\ell(\delta))^{2/(2+\delta)},\ t_0 < \frac{1}{2\pi}\left(\frac{1-n^{-1}}{2\varkappa_\delta}\right)^{1/\delta} \equiv t_4\Big(\delta, n^{-1-\delta/2}\Big),$$

for identically distributed summands

$$I_{11} \leqslant \frac{1.0253}{16\pi}\,\ell_n^2\bigg[\frac{8\gamma_\delta}{n^{1-\delta/2}}Q\Big(\frac{1}{n^{1+\delta/2}}, t_0, 4+\delta\Big) + 16\nu_n\varkappa_\delta\gamma_\delta Q\Big(\frac{1}{n^{1+\delta/2}}, t_0, 4+2\delta\Big) +$$
$$+\Big(n^{-2+\delta}Q\Big(\frac{1}{n^{1+\delta/2}}, t_0, 6\Big) + \frac{2\nu_n\varkappa_\delta}{n^{1-\delta/2}}\,Q\Big(\frac{1}{n^{1+\delta/2}}, t_0, 6+\delta\Big)\Big)\mathbf{1}(\delta<1)\bigg] \equiv \ell_n^2\widehat{J}_{11}(n, t_0).$$

Similarly, for I_{12} with the account of the definition of $T = 2\pi(\nu_n\ell_n)^{-1/\delta}$ we obtain

$$I_{12} \leqslant \frac{1}{T}\sum_{j=1}^n\int_0^{t_0T}\Big(1 - \frac{t}{T} + \frac{\pi^2t^2}{18T^2}\Big)\Big(\frac{\gamma_\delta\beta_{2+\delta,j}\,t^{2+\delta}}{B_n^{2+\delta}} + \frac{\sigma_j^4t^4}{8B_n^4}\,\mathbf{1}(\delta<1)\Big)e^{-t^2/2}\,dt$$
$$\leqslant \frac{1}{T}\int_0^\infty\Big(1 + \frac{\pi^2t^2}{18T^2}\Big)\Big(\gamma_\delta\ell_nt^{2+\delta} + \frac{t^4}{8B_n^4}\sum_{j=1}^n\sigma_j^4\mathbf{1}(\delta<1)\Big)e^{-t^2/2}\,dt$$
$$= \frac{2^{(\delta-1)/2}\gamma_\delta}{\pi}\,\nu_n^{1/\delta}\ell_n^{(1+\delta)/\delta}\Gamma\Big(\frac{3+\delta}{2}\Big)\Big(1 + \frac{3+\delta}{72}(\nu_n\ell_n)^{2/\delta}\Big)$$
$$+ \frac{3(\nu_n\ell_n)^{1/\delta}}{16\sqrt{2\pi}B_n^4}\sum_{j=1}^n\sigma_j^4\Big(1 + \frac{5}{72}(\nu_n\ell_n)^{2/\delta}\Big)\mathbf{1}(\delta<1),$$

whence by the Lyapounov inequality and (4.2) it follows that in the general case

$$I_{12} \leqslant \ell_n^{(1+\delta)/\delta}\nu_n^{1/\delta}\bigg[\frac{2^{(\delta-1)/2}\gamma_\delta}{\pi}\,\Gamma\Big(\frac{3+\delta}{2}\Big)\Big(1 + \frac{3+\delta}{72}(\nu_n\ell_n)^{2/\delta}\Big)$$

$$+\frac{3\ell_n^{(2-\delta)/(2+\delta)}}{16\sqrt{2\pi}}\left(1+\frac{5}{72}(\nu_n\ell_n)^{2/\delta}\right)\mathbf{1}(\delta<1)\Big]\equiv \ell_n^{(1+\delta)/\delta}J_{12}(\ell_n,\nu_n),$$

and in the case of identically distributed summands

$$I_{12}\leqslant \ell_n^{(1+\delta)/\delta}\nu_n^{1/\delta}\left[\frac{2^{(\delta-1)/2}\gamma_\delta}{\pi}\,\Gamma\Big(\frac{3+\delta}{2}\Big)\left(1+\frac{3+\delta}{72}(\nu_n\ell_n)^{2/\delta}\right)\right.$$
$$\left.+\frac{3n^{-1+\delta/2}}{16\sqrt{2\pi}}\left(1+\frac{5}{72}(\nu_n\ell_n)^{2/\delta}\right)\mathbf{1}(\delta<1)\right]\equiv \ell_n^{(1+\delta)/\delta}\widehat{J}_{12}(\ell_n,\nu_n,n).$$

Summarize the above reasoning as a lemma.

Lemma 4.3. *For $0<\delta\leqslant 1$ by $\theta_0(\delta)$ denote the unique root of the equation*

$$\delta\theta^2+2\theta\sin\theta+2(2+\delta)(\cos\theta-1)=0,\quad \pi<\theta<2\pi,$$

$$\varkappa_\delta=\sup_{x>0}\frac{\left|\cos x-1+x^2/2\right|}{x^{2+\delta}}=\frac{\cos\theta_0(\delta)-1+\theta_0^2(\delta)/2}{\theta_0^{2+\delta}(\delta)}=\frac{\theta_0(\delta)-\sin\theta_0(\delta)}{(2+\delta)\theta_0^{1+\delta}(\delta)},$$

$$\gamma_\delta=\sup_{x>0}\sqrt{\Big(\frac{\cos x-1+x^2/2}{x^{2+\delta}}\Big)^2+\Big(\frac{\sin x-x}{x^{2+\delta}}\Big)^2},$$

$t_1(\delta)=\theta_0(\delta)/(2\pi)$, let $t_2=t_2(\delta)$ be the unique root of the equation

$$t^2\left(1-2\varkappa_\delta(2\pi t)^\delta\right)=t_1^2(\delta)\left(1-2\varkappa_\delta(2\pi t_1(\delta))^\delta\right)$$

on the interval $(0,t_1(\delta))$. Let

$$\overline{\ell}(\delta)=\left(1-\frac{4}{(2+\delta)^2}\,\exp\left\{-\frac{\delta(4+\delta)}{2(2+\delta)}\right\}\right)^{1+\delta/2},$$

$$t_4(\delta,\ell)=\frac{1}{2\pi}\left(\frac{1-\ell^{2/(2+\delta)}}{2\varkappa_\delta}\right)^{1/\delta},\quad 0<\ell<\overline{\ell}(\delta).$$

$$t_3(\delta)=\frac{1}{2\pi}\left(\frac{2}{(2+\delta)^2\varkappa_\delta}\right)^{1/\delta}\exp\left\{-\frac{4+\delta}{2(2+\delta)}\right\}=t_4\left(\delta,\overline{\ell}(\delta)\right).$$

Then for all $n\geqslant 1$ and $F_1,\dots,F_n\in\mathcal{F}_{2+\delta}$ such that $\ell_n<\overline{\ell}(\delta)$ and for any t_0 from the interval

$$t_3(\delta)\leqslant t_0<\min\{t_1(\delta),t_4(\delta,\ell_n)\}$$

there holds the estimate

$$\Delta_n\leqslant\frac{1}{\pi}\sum_{j=1}^n\int_0^\infty\frac{1}{t}\left|f_j(t/B_n)-e^{-\sigma_j^2t^2/(2B_n^2)}\right|e^{-t^2/2}\,dt+\frac{(\nu_n\ell_n)^{1/\delta}}{2\sqrt{2\pi}}+$$
$$+\ell_n^{(4+\delta)/(2+\delta)}J_{11}(\ell_n,\nu_n,t_0)+\ell_n^{(1+\delta)/\delta}J_{12}(\ell_n,\nu_n)+\frac{(\nu_n\ell_n)^{2/\delta}}{4\pi^2}\,J\left(\frac{2\pi}{(\nu_n\ell_n)^{1/\delta}},t_0\right),$$

where

$$J_{11}(\ell,\nu,t_0)=\frac{1.0253}{16\pi}\Big(8\gamma_\delta Q\left(\ell,t_0,4+\delta\right)+16\nu\varkappa_\delta\gamma_\delta\ell^{\delta/(2+\delta)}Q\left(\ell,t_0,4+2\delta\right)$$
$$+\Big(\ell^{(2-\delta)/(2+\delta)}Q\left(\ell,t_0,6\right)+2\nu\varkappa_\delta\ell^{2/(2+\delta)}Q\left(\ell,t_0,6+\delta\right)\Big)\mathbf{1}(\delta<1)\Big),$$
$$1\leqslant\nu\leqslant2,\ \ell>0,$$

$$Q(\ell,t_0,r)=\frac{2^{(r-1)/2}\Gamma\left(\frac{r+1}{2}\right)}{\left(1-\ell^{2/(2+\delta)}-2\varkappa_\delta(2\pi t_0)^\delta\right)^{(r+1)/2}},\quad 0<\ell<\overline{\ell}(\delta),\ r>0,$$

$$J_{12}(\ell,\nu)=\nu^{1/\delta}\left[\frac{2^{(\delta-1)/2}\gamma_\delta}{\pi}\,\Gamma\Big(\frac{3+\delta}{2}\Big)\left(1+\frac{3+\delta}{72}(\nu\ell)^{2/\delta}\right)\right.$$
$$\left.+\frac{3\ell^{(2-\delta)/(2+\delta)}}{16\sqrt{2\pi}}\left(1+\frac{5}{72}(\nu\ell)^{2/\delta}\right)\mathbf{1}(\delta<1)\right],\quad 1\leqslant\nu\leqslant2,\ \ell>0,$$

$$J(T,t_0)=0\vee\left(J_{21}(T,t_0)+J_{22}(T,t_0)+J_3(T)+J_4(T,t_0)-1+\frac{\pi^{5/2}}{18\sqrt{2}}\cdot\frac{1}{T}\right),$$

$$J_{21}(T,t_0)=\frac{1.0253}{\pi}\,T^2\int\limits_{t_0\wedge t_2(\delta)}^{t_2(\delta)}\exp\left\{-\frac{T^2t^2}{2}\left(1-2\varkappa_\delta(2\pi t)^\delta\right)\right\}\frac{dt}{t},$$

$$J_{22}(T,t_0)=\frac{1.0253}{\pi}\,T^2\exp\left\{-\frac{T^2t_1^2(\delta)}{2}\left(1-2\varkappa_\delta(2\pi t_1(\delta))^\delta\right)\right\}\ln\frac{t_1(\delta)}{t_0\vee t_2(\delta)},$$

$$J_3(T)=T^2\int\limits_0^{1-t_1(\delta)}t\sqrt{1+\Big(\frac{1}{\pi(t\vee10^{-4})}-\cot\pi(t\vee10^{-4})\Big)^2}\times$$
$$\times\exp\left\{-T^2\frac{1-\cos2\pi t}{4\pi^2}\right\}dt,$$

$$J_4(T,t_0)=0\vee\frac{1}{t_0}\left(\frac{1}{\pi t_0}-1+t_0-\frac{\pi^2t_0^2}{18}\right)e^{-T^2t_0^2/2},\quad T>0.$$

If $F_1=\ldots=F_n\in\mathcal{F}_{2+\delta}$*, then for all* $n\geqslant(\overline{\ell}(\delta))^{-2/(2+\delta)}$ *and* t_0 *such that*

$$t_3(\delta)\leqslant t_0<\min\left\{t_1(\delta),\ t_4\left(\delta,n^{-1-\delta/2}\right)\right\},$$

there holds the estimate

$$\Delta_n\leqslant\frac{1}{\pi}\sum_{j=1}^n\int\limits_0^\infty\frac{1}{t}\left|f_j(t/B_n)-e^{-\sigma_j^2t^2/(2B_n^2)}\right|e^{-t^2/2}\,dt+\frac{1}{2\sqrt{2\pi n}}\left(\frac{\beta_{2+\delta}}{\sigma^{2+\delta}}+\frac{\beta_\delta}{\sigma^\delta}\right)^{1/\delta}$$
$$+\ell_n^2\widehat{J}_{11}(n,\nu_n,t_0)+\ell_n^{(1+\delta)/\delta}\widehat{J}_{12}(\ell_n,\nu_n,n)+\frac{(\nu_n\ell_n)^{2/\delta}}{4\pi^2}\,J\left(\frac{2\pi}{(\nu_n\ell_n)^{1/\delta}},t_0\right),$$

where

$$\widehat{J}_{11}(n,\nu,t_0)=\frac{1.0253}{16\pi}\left[\frac{8\gamma_\delta}{n^{1-\delta/2}}\,Q\Big(\frac{1}{n^{1+\delta/2}},t_0,4+\delta\Big)\right.$$

$$+ 16\nu\varkappa_\delta\gamma_\delta Q\Big(\frac{1}{n^{1+\delta/2}}, t_0, 4+2\delta\Big) + \Big(n^{-2+\delta} Q\Big(\frac{1}{n^{1+\delta/2}}, t_0, 6\Big)$$

$$+ \frac{2\nu\varkappa_\delta}{n^{1-\delta/2}}\, Q\Big(\frac{1}{n^{1+\delta/2}}, t_0, 6+\delta\Big)\Big)\mathbf{1}(\delta<1)\Big],$$

$$\widehat{J}_{12}(\ell,\nu,n) = \nu^{1/\delta}\Big[\frac{2^{(\delta-1)/2}\gamma_\delta}{\pi}\,\Gamma\Big(\frac{3+\delta}{2}\Big)\Big(1+\frac{3+\delta}{72}(\nu\ell)^{2/\delta}\Big)$$

$$+ \frac{3n^{-1+\delta/2}}{16\sqrt{2\pi}}\Big(1+\frac{5}{72}(\nu\ell)^{2/\delta}\Big)\mathbf{1}(\delta<1)\Big],\ 1\leqslant\nu\leqslant 2,\ \ell>0,\ n\geqslant 1.$$

With t_0 fixed, the functions $J_{21}(T,t_0)$, $J_{22}(T)$, $J_3(T)$, $J_4(T,t_0)$ of T for $T>0$ have at most one maximum and have no minima; $\widehat{J}_{11}(n,\nu,t_0)$, $\widehat{J}_{12}(\ell,\nu,n)$ decrease monotonically in $n\geqslant 1$ with ℓ and ν fixed; $t_4(\delta,\ell)$ decreases monotonically in ℓ; $J_{11}(\ell,\nu,t_0)$, $J_{12}(\ell,\nu)$, $\widehat{J}_{11}(\ell^{-2/\delta},\nu,t_0)$, $\widehat{J}_{12}(\ell,\nu,\ell^{-2/\delta})$ increase monotonically in ℓ; $J_{11}(\ell,\nu,t_0)$, $J_{12}(\ell,\nu)$, $\widehat{J}_{11}(n,\nu,t_0)$, $\widehat{J}_{12}(\ell,\nu,n)$ increase monotonically in $\nu\in[1,\,2]$, and

$$\lim_{n\to\infty}\widehat{J}_{11}(n,\nu,t_0) = \frac{1.0253\cdot 2^{3/2+\delta}\nu\varkappa_\delta\gamma_\delta\Gamma(5/2+\delta)}{\pi(1-2\varkappa_\delta(2\pi t_0)^\delta)^{5/2+\delta}},\ 1\leqslant\nu\leqslant 2,\ t_3(\delta)\leqslant t_0\leqslant t_1(\delta),$$

$$\lim_{\ell\to 0}\sup_{n\geqslant\ell^{-2/\delta}}\widehat{J}_{12}(\ell,\nu,n) = \nu^{1/\delta}2^{(\delta-1)/2}\pi^{-1}\gamma_\delta\Gamma((3+\delta)/2),\quad 1\leqslant\nu\leqslant 2,$$

$$\lim_{T\to\infty} J(T,t_0)=0,\quad t_3(\delta)\leqslant t_0\leqslant t_1(\delta).$$

The values of γ_δ, $\varkappa_\delta$, $t_1(\delta)$, $t_2(\delta)$, $t_3(\delta)$, $t_4(\delta,\ell)$, $\overline{\ell}(\delta)$ and

$$N(\delta) = \inf\Big\{n\in\mathbf{N}\colon n>(\overline{\ell}(\delta))^{-2/(2+\delta)}\Big\} = 1+\Big\lfloor(\overline{\ell}(\delta))^{-2/(2+\delta)}\Big\rfloor$$

for some $0<\delta\leqslant 1$ and $\ell=0.1,\,0.01$ calculated with the accuracy to the fourth decimal digit are given in table 3.

Remark 4.4. On the right-hand sides of the inequalities in lemma 4.3 the "leading" terms are two first summands: the integral

$$I_{13} = \frac{1}{\pi}\sum_{j=1}^n\int_0^\infty\frac{1}{t}\Big|f_j(t/B_n)-e^{-\sigma_j^2t^2/(2B_n^2)}\Big|\,e^{-t^2/2}\,dt$$

and

$$\frac{(\nu_n\ell_n)^{1/\delta}}{2\sqrt{2\pi}} = \sqrt{\frac{\pi}{2}}\cdot\frac{1}{T},$$

appearing when the sum of the integrals I_4 and I_5 is estimated. It is interesting to clarify the nature of these summands and their contribution into the constants at the leading terms in the resulting estimates. For simplicity consider the case of identically distributed summands. As we will see below, the integral I_{13} contains the information concerning the "heavy-tailedness" of the distribution: the order of

δ	γ_δ	$\varkappa_\delta$	$t_1(\delta)$	$t_2(\delta)$	$t_3(\delta)$	$t_4(\delta, 0.1)$	$t_4(\delta, 0.01)$	$\overline{\ell}(\delta)$	$N(\delta)$
0.01	0.5225	0.4909	0.9950	0.0261	0.1356	0.0000	0.3566	0.0193	51
0.05	0.4885	0.4563	0.9761	0.0673	0.1370	0.1055	0.7887	0.0886	11
0.10	0.4498	0.4170	0.9539	0.1019	0.1386	0.2990	0.8613	0.1626	6
0.15	0.4149	0.3815	0.9331	0.1302	0.1401	0.4197	0.8798	0.2265	4
0.20	0.3833	0.3494	0.9132	0.1551	0.1416	0.4944	0.8841	0.2827	4
0.25	0.3548	0.3203	0.8941	0.1778	0.1431	0.5429	0.8826	0.3327	3
0.30	0.3290	0.2940	0.8756	0.1989	0.1444	0.5758	0.8784	0.3776	3
0.35	0.3058	0.2701	0.8576	0.2187	0.1457	0.5987	0.8725	0.4181	3
0.40	0.2847	0.2484	0.8399	0.2375	0.1469	0.6147	0.8658	0.4549	2
0.45	0.2657	0.2287	0.8226	0.2556	0.1480	0.6260	0.8584	0.4884	2
0.50	0.2486	0.2108	0.8054	0.2729	0.1490	0.6338	0.8507	0.5191	2
0.55	0.2331	0.1945	0.7884	0.2896	0.1500	0.6390	0.8427	0.5474	2
0.60	0.2193	0.1796	0.7716	0.3058	0.1509	0.6422	0.8345	0.5734	2
0.65	0.2070	0.1661	0.7548	0.3214	0.1517	0.6439	0.8262	0.5975	2
0.70	0.1960	0.1537	0.7380	0.3366	0.1524	0.6442	0.8177	0.6198	2
0.75	0.1865	0.1424	0.7212	0.3514	0.1530	0.6435	0.8091	0.6405	2
0.80	0.1783	0.1321	0.7044	0.3657	0.1536	0.6420	0.8005	0.6597	2
0.85	0.1715	0.1227	0.6875	0.3797	0.1540	0.6397	0.7918	0.6776	2
0.90	0.1665	0.1142	0.6705	0.3932	0.1544	0.6369	0.7830	0.6944	2
0.95	0.1637	0.1063	0.6533	0.4064	0.1547	0.6334	0.7741	0.7100	2
1.00	0.1666	0.0991	0.6359	0.4191	0.1550	0.6296	0.7652	0.7247	2

Table 3: The values of γ_δ, $\varkappa_\delta$, $t_1(\delta)$, $t_2(\delta)$, $t_3(\delta)$, $t_4(\delta,\ell)$, $\overline{\ell}(\delta)$ and $N(\delta) = 1 + \left\lfloor (\overline{\ell}(\delta))^{-2/(2+\delta)} \right\rfloor$ for some $0 < \delta \leqslant 1$ and $\ell = 0.1$, 0.01.

its decrease is completely determined by the maximum order of the finite moment of a summand (in our case $I_{13} = O(n^{-\delta/2})$) whereas the role of the corresponding characteristic of the distribution is played by the normalized moment of the maximum order $\beta_{2+\delta}/\sigma^{2+\delta}$. In other words, there exists such an *absolute* positive finite constant C that

$$I_{13} \leqslant C \cdot \frac{\beta_{2+\delta}}{\sigma^{2+\delta} n^{\delta/2}},$$

moreover, as is illustrated by the corresponding examples in [29], the order of this estimate is exact, if it is meant uniformly in $F \in \mathcal{F}_{2+\delta}$. The importance of the remark concerning the exactness of the order is conditioned by the fact that $\Delta_n(F) = o(n^{-\delta/2})$ for any fixed $F \in \mathcal{F}_{2+\delta}$ (see also [22]). But, on the other hand, if a distribution $F \in \mathcal{F}_{2+\delta}$ depends on n and the moment-type characteristic $\beta_{2+\delta}/\sigma^{2+\delta}$ is included in the estimate, then $\beta_{2+\delta}/\sigma^{2+\delta} n^{-\delta/2}$ is an exact characteristic of the rate of convergence.

Now consider the second term $\sqrt{\pi/2}/T$. Here the coefficient $\sqrt{\pi/2}$ is determined by the limit distribution which is normal in the case under consideration. The value of T chosen in the process of estimation of the integral I_3 is determined by the maximum length of a zero-left-ended interval on which it is possible to bound the absolute value of the ch.f. by a number less than one (see remark 2.5). So, the term under consideration contains the information concerning the smoothness of the pre-limit distribution. Moreover, since the sum of random variables is normalized by $\sqrt{n}$, the length of the interval on which the absolute value of the ch.f. is bounded by a number less than one is proportional to $\sqrt{n}$, that is, for $\delta < 1$ the effects due to the smoothness or discreteness of the original distribution disappear making no influence on the constant at the leading term of the estimate having the order $n^{-\delta/2}$. At the same time,for $\delta = 1$ the order of normalization of the sum of r.v.'s coincides with the order of the maximum length of the interval on which the absolute value of the ch.f. is bounded by a number less than one, therefore, the effects of "heavy-tailedness" revealing themselves in the integral I_{13} are added with the effects of "non-smoothness" which leads to abrupt increase (discontinuity) of the constant at the leading term of order $1/\sqrt{n}$ in the point $\delta = 1$.

Remark 4.5. Let $\nu \in [1, 2]$ and $\ell > 0$ be arbitrary numbers. For the purpose of construction of estimates of the function $J\left(2\pi(\nu_n\ell_n)^{-1/\delta}, t_0\right)$ with fixed t_0 uniform in $\ell_n \leqslant \ell$ and $\nu_n \in [1, \nu]$ consider the behavior of the functions $J_{21}(T, t_0)$, $J_{22}(T, t_0)$, $J_3(T)$, $J_4(T, t_0)$ of $T = 2\pi(\nu_n\ell_n)^{-1/\delta} \geqslant 2\pi(\nu\ell)^{-1/\delta} > 0$, which are components of $J(T, t_0)$. Obviously, the function $J_4(T, t_0)$ decreases monotonically in $T > 0$. Noticing that the function xe^{-ax} decreases monotonically for $x > 1/a > 0$ we conclude that $J_{22}(T, t_0)$ decreases monotonically for

$$T \geqslant \frac{\sqrt{2}}{t_1(\delta)\sqrt{1 - 2\varkappa_\delta(2\pi t_1(\delta))^\delta}} \equiv T_{22}(\delta).$$

If $t_3(\delta) \geqslant t_2(\delta)$, then $J_{21}(T, t_0) = 0$ for all $t_0 \geqslant t_3(\delta)$. And if $t_3(\delta) < t_2(\delta)$, then using the property of monotonic increase of the function $t^2(1 - 2\varkappa_\delta(2\pi t)^\delta)$ for $t \in (0, t_2(\delta))$ established in the proof of lemma 4.3 we similarly conclude that $J_{21}(T, t_0)$ decreases monotonically for

$$T \geqslant \frac{\sqrt{2}}{t_3(\delta)\sqrt{1 - 2\varkappa_\delta(2\pi t_3(\delta))^\delta}} \equiv T_{21}(\delta)$$

for each fixed $t_0 \geqslant t_3(\delta)$. Finally, for each fixed δ it is possible to find numerically the unique point $T_3(\delta)$ of the maximum of the function $J_3(T) \in \mathcal{D}$ such that $J_3(T)$ decreases monotonically for $T \geqslant T_3(\delta)$. So, if the numbers $\nu \in [1, 2]$ and $\ell > 0$ satisfy the inequality

$$\nu\ell \leqslant \left(\frac{2\pi}{\max\{T_{21}(\delta), T_{22}(\delta), T_3(\delta)\}}\right)^\delta \equiv \overline{\varepsilon}(\delta),$$

then

$$\max_{\ell_n \leqslant \ell,\, \nu_n \in [1,\nu]} J\left(\frac{2\pi}{(\nu_n\ell_n)^{1/\delta}}, t_0\right) \leqslant J\left(\frac{2\pi}{(\nu\ell)^{1/\delta}}, t_0\right).$$

The values of $T_{21}(\delta)$, $T_{22}(\delta)$, $T_3(\delta)$ and $\overline{\varepsilon}(\delta)$ are given in table 4. From this table it can be seen, in particular, that for $\ell_n \leqslant 0.3$ the monotonicity takes place for all $1 \leqslant \nu_n \leqslant 2$ and $0.01 \leqslant \delta \leqslant 1$ given in table 4.

Depending on whether $\delta = 1$ or not, to estimate the integral

$$I_{13} = \frac{1}{\pi}\sum_{j=1}^{n}\int_0^\infty \frac{1}{t}\left|f_j(t/B_n) - e^{-\sigma_j^2 t^2/(2B_n^2)}\right| e^{-t^2/2}\,dt$$

δ	$T_{21}(\delta) \leqslant$	$T_{22}(\delta) \leqslant$	$T_3(\delta) \leqslant$	$\overline{\varepsilon}(\delta) \geqslant$
0.01	74.1670	285.6369	1065.6543	0.9498
0.05	33.6579	59.2429	188.6696	0.8434
0.10	24.2258	30.8361	89.8283	0.7663
0.15	20.1242	21.3082	58.3999	0.7156
0.20	17.7237	16.5114	43.1128	0.6802
0.25	16.1158	13.6136	34.1103	0.6550
0.30	14.9517	11.6694	28.1896	0.6373
0.35	14.0650	10.2731	24.0043	0.6254
0.40	13.3653	9.2211	20.8912	0.6183
0.45	12.7987	8.4003	18.4862	0.6152
0.50	12.3308	7.7426	16.5734	0.6156
0.55	11.9386	7.2046	15.0164	0.6191
0.60	11.6060	6.7573	13.7250	0.6256
0.65	11.3215	6.3806	12.6372	0.6348
0.70	11.0764	6.0602	11.7091	0.6466
0.75	10.8643	5.7855	10.9083	0.6610
0.80	10.6802	5.5487	10.2111	0.6540
0.85	10.5202	5.3440	9.5992	0.6451
0.90	10.3813	5.1668	9.0585	0.6363
0.95	10.2609	5.0135	8.5779	0.6274
1.00	10.1571	4.8815	8.1488	0.6185

Table 4: The values of $T_{21}(\delta)$, $T_{22}(\delta)$, $T_3(\delta)$ and $\overline{\varepsilon}(\delta)$ for some δ.

we will use principally different techniques. The thing is that, as was mentioned above, for $\delta < 1$ the quantity

$$\frac{(\nu_n \ell_n)^{1/\delta}}{2\sqrt{2\pi}} = \frac{1}{2\sqrt{2\pi}}\Big(\ell_n + \frac{1}{B_n^{2+\delta}}\sum_{j=1}^{n}\beta_{\delta,j}\,\sigma_j^2\Big)^{1/\delta} \leqslant \frac{(2\ell_n)^{1/\delta}}{2\sqrt{2\pi}},$$

appearing in the estimate for Δ_n from lemma 4.3, is an infinitesimal of higher order of decrease than ℓ_n as $\ell_n \to 0$. Therefore, to estimate I_{13} it suffices to use traditional techniques. For $\delta = 1$ this quantity has the same order of decrease as the Lyapounov fraction ℓ_n and, as we will see below, makes the main contribution in the corresponding constant. The use of the same method as for $\delta < 1$ to estimate I_{13} makes it possible to obtain a new moment-type estimate whose structure is in some sense asymptotically optimal. But if this new estimate is used for the construction of the classical estimate with a single term, the Lyapounov fraction, then the coefficient $7/(6\sqrt{2\pi}) = 0.4654\ldots$ at the Lyapounov fraction in this classical estimate will be noticeably greater than its "exact" value $(\sqrt{10}+3)/6/\sqrt{2\pi} = 0.4097\ldots$. So, the new estimate with the asymptotically exact structure is too rough for the solution of the problem in the classical setting. Therefore, to estimate the integral I_{13} in the case $\delta = 1$ we will use another technique which is more delicate and is based on inequality (2.5) from lemma 2.8. This technique develops and sharpens the method used by G. P. Chistyakov in [7].

First consider the general case $\delta \leqslant 1$. With the account of estimates (2.4), (2.6) from lemma 2.8, for the integral I_{13} we obtain

$$I_{13} \leqslant \frac{1}{\pi}\sum_{j=1}^{n}\int_0^\infty \frac{1}{t}\left(\frac{\gamma_\delta \beta_{2+\delta,j}\, t^{2+\delta}}{B_n^{2+\delta}} + \frac{\sigma_j^4 t^4}{8B_n^4}\,\mathbf{1}(\delta<1)\right)e^{-t^2/2}\,dt$$

$$= C(\delta)\ell_n + \frac{1}{4\pi B_n^4}\sum_{j=1}^n \sigma_j^4 \mathbf{1}(\delta<1),$$

where $C(\delta) = \gamma_\delta 2^{\delta/2}\Gamma(1+\delta/2)/\pi$. Further, by virtue of the Lyapounov inequality and (4.2) we conclude that

$$I_{13} \leqslant \begin{cases} C(\delta)\ell_n + \ell_n^{4/(2+\delta)}/(4\pi)\mathbf{1}(\delta<1), & \text{in the general case,} \\ C(\delta)\ell_n + (4\pi n)^{-1}\mathbf{1}(\delta<1), & \text{if } F_1 = \ldots = F_n. \end{cases}$$

So, from lemma 4.3 we obtain that for all $n \geqslant 1$ and $F_1,\ldots,F_n \in \mathcal{F}_{2+\delta}$ such that $\ell_n \leqslant \overline{\ell}(\delta)$ the estimate

$$\Delta_n \leqslant C(\delta)\cdot \ell_n + \frac{1}{2\sqrt{2\pi}}\left(\ell_n + \frac{1}{B_n^{2+\delta}}\sum_{j=1}^n \beta_{\delta,j}\,\sigma_j^2\right)^{1/\delta} + \begin{cases} \widetilde{C}_\delta(\ell_n)\cdot \ell_n^{4/(2+\delta)}, & 0<\delta<1, \\ \widetilde{C}_1(\ell_n)\cdot \ell_n^{5/3}, & \delta = 1, \end{cases} \tag{4.8}$$

holds, where

$$\widetilde{C}_\delta(\ell) = \frac{1}{4\pi} + \ell^{\frac{2-\delta(1-\delta)}{\delta(2+\delta)}} J_{12}(\ell,2) + \inf\Big\{\ell^{\delta/(2+\delta)} J_{11}(\ell,2,t_0) + \ell^{\frac{2(2-\delta)}{\delta(2+\delta)}}\cdot \frac{2^{2/\delta}}{4\pi^2}\sup_{0<\varepsilon\leqslant 2\ell} J\left(2\pi\varepsilon^{-1/\delta}, t_0\right) : t_3(\delta)\leqslant t_0 \leqslant t_1(\delta)\wedge t_4(\delta,\ell)\Big\},$$
$$\ell \in \big(0,\,\overline{\ell}(\delta)\big),\ \delta\in(0,1),$$

$$\widetilde{C}_1(\ell) = \ell^{1/3} J_{12}(\ell,2) + \inf\Big\{J_{11}(\ell,2,t_0) + \ell^{1/3}/\pi^2 \sup_{0<\varepsilon\leqslant 2\ell} J\left(2\pi\varepsilon^{-1/\delta}, t_0\right) : t_3(1)\leqslant t_0\leqslant t_1(1)\wedge t_4(1,\ell)\Big\},$$
$$\ell\in\big(0,\,\overline{\ell}(1)\big),$$

and for all $n > \big(\overline{\ell}(\delta)\big)^{-2/(2+\delta)}$, $F_1 = \ldots = F_n \in \mathcal{F}_{2+\delta}$ and

$$t_0 \in \Big[t_3(\delta),\, t_1(\delta)\wedge t_4\left(\delta, n^{-1-\delta/2}\right)\Big)$$

we have

$$\Delta_n \leqslant C(\delta)\cdot\frac{\beta_{2+\delta}}{\sigma^{2+\delta}n^{\delta/2}} + \frac{1}{2\sqrt{2\pi n}}\left(\frac{\beta_{2+\delta}}{\sigma^{2+\delta}} + \frac{\beta_\delta}{\sigma^\delta}\right)^{1/\delta} + \frac{1}{4\pi n}\mathbf{1}(\delta<1) + \ell_n^2\Big(\widehat{J}_{11}(n,\nu_n,t_0)$$

$$+\,\ell_n^{(1-\delta)/\delta}\widehat{J}_{12}(\ell_n,\nu_n,n)+\ell_n^{2(1-\delta)/\delta}\cdot\frac{\nu_n^{2/\delta}}{4\pi^2}\,J\left(\frac{2\pi}{(\nu_n\ell_n)^{1/\delta}},t_0\right)\Bigg). \tag{4.9}$$

From (4.9) with the account of relations $n\geqslant\ell_n^{-2/\delta}$, $1\leqslant\nu_n\leqslant 2$ and the properties of the functions $\widehat{J}_{11}(n,\nu_n,t_0)$, $\widehat{J}_{12}(\ell_n,\nu_n,n)$, $t_4(\delta,n^{-1-\delta/2})$ described in lemma 4.3 it follows that, uniformly in n and ν_n,

$$\Delta_n\leqslant C(\delta)\cdot\frac{\beta_{2+\delta}}{\sigma^{2+\delta}n^{\delta/2}}+\frac{1}{2\sqrt{2\pi n}}\left(\frac{\beta_{2+\delta}}{\sigma^{2+\delta}}+\frac{\beta_\delta}{\sigma^\delta}\right)^{1/\delta}+\ell_n^2\cdot\widehat{C}_\delta(\ell_n),\quad \ell_n\leqslant\left(\overline{\ell}(\delta)\right)^{\delta/(2+\delta)},$$

where

$$\begin{aligned}\widehat{C}_\delta(\ell)&=\frac{1}{4\pi}\,\mathbf{1}(\delta<1)+\ell^{(1-\delta)/\delta}\widehat{J}_{12}(\ell,2,\ell^{-2/\delta})\\&\quad+\inf\Big\{\ell^{2(1-\delta)/\delta}\cdot\frac{2^{2/\delta}}{4\pi^2}\sup_{0<\varepsilon\leqslant 2\ell}J\left(2\pi\varepsilon^{-1/\delta},t_0\right)\\&\quad+\widehat{J}_{11}(\ell^{-2/\delta},2,t_0)\colon t_3(\delta)\leqslant t_0\leqslant t_1(\delta)\wedge t_4\left(\delta,\ell^{1+2/\delta}\right)\Big\},\\&\quad 0<\ell\leqslant\left(\overline{\ell}(\delta)\right)^{\delta/(2+\delta)}.\end{aligned}$$

For the calculation of the least upper bound of $J(2\pi\varepsilon^{-1/\delta},t_0)$ over $0<\varepsilon\leqslant 2\ell$ see remark 4.5.

Note that for each $0<\delta\leqslant 1$ the functions $\widetilde{C}_\delta(\ell)$ and $\widehat{C}_\delta(\ell)$ increase monotonically varying within the limits

$$\widetilde{C}_\delta(0)\equiv\lim_{\ell\to 0}\widetilde{C}_\delta(\ell)<\widetilde{C}_\delta(\ell)<\lim_{\ell\to\overline{\ell}(\delta)}\widetilde{C}_\delta(\ell)=+\infty,\quad 0<\ell<\overline{\ell}(\delta),$$

$$\widehat{C}_\delta(0)\equiv\lim_{\ell\to 0}\widehat{C}_\delta(\ell)<\widehat{C}_\delta(\ell)<\lim_{\ell\to(\overline{\ell}(\delta))^{\delta/(2+\delta)}}\widehat{C}_\delta(\ell)=+\infty,\quad 0<\ell<(\overline{\ell}(\delta))^{\delta/(2+\delta)},$$

$$\widetilde{C}_\delta(0)=\begin{cases}(4\pi)^{-1}=0.0795\ldots, & 0<\delta<1,\\ \dfrac{2\cdot 1.0253}{3\pi(1-4/9e^{-5/6})^3}=0.4142\ldots, & \delta=1,\end{cases}$$

$$\widehat{C}_\delta(0)=\begin{cases}\dfrac{1.0253\cdot 2^{5/2+\delta}\varkappa_\delta\gamma_\delta\Gamma(5/2+\delta)}{\pi\left(1-2\varkappa_\delta(2\pi t_3(\delta))^\delta\right)^{5/2+\delta}}+\dfrac{1}{4\pi}, & 0<\delta<1,\\ \dfrac{1.0253\cdot 5\varkappa_1}{\sqrt{2\pi}(1-4/9e^{-5/6})^{7/2}}+\dfrac{1}{3\pi}=0.5359\ldots, & \delta=1,\end{cases}$$

infinitely large values of the functions $\widetilde{C}_\delta(\ell)$ and $\widehat{C}_\delta(\ell)$ appear since $t_4(\delta,\ell)\to t_3(\delta)$ as $\ell\to\overline{\ell}(\delta)$, and for all $r>0$

$$\lim_{\ell\to\overline{\ell}(\delta)}\ \inf_{t_3(\delta)\leqslant t_0\leqslant t_1(\delta)\wedge t_4(\delta,\ell)}Q(\ell,t_0,r)=\lim_{\ell\to\overline{\ell}(\delta)}Q(\ell,t_3(\delta),r)=+\infty.$$

The values of $\widetilde{C}_\delta(\ell)$ for some $0<\delta\leqslant 1$ and ℓ are given in table 6. The values of $\widehat{C}_\delta(0)$ and $\widehat{C}_\delta(\ell)$ are given in table 7.

From inequality (4.9) one can also obtain improved estimates in a special scheme of a double array of row-wise i.i.d. summands:

$$F_j(x)=F_{j,n}(x)=F_{1,n}(x),\quad j=1,\ldots,n,$$

$$\beta_{2+\delta}=\beta_{2+\delta,n},\quad \sigma=\sigma_n,\quad \ell_n=\frac{\beta_{2+\delta,n}}{\sigma_n^{2+\delta}n^{\delta/2}},\quad n\geqslant 1.$$

The double array scheme admits such a dependence of the distributions $F_1,\ldots,F_n$ within each row on the number of the row n that whatever large n is, the Lyapounov fraction ℓ_n may remain fixed and, in particular, may be arbitrarily far from zero. Such a situation occurs, for example, in the construction of estimates of the rate of convergence of the distributions of Poisson random sums of i.i.d. summands with the use of the property of infinite divisibility of the compound Poisson distribution. The success in solving these problems directly depends on the quality of estimates of

$$\limsup_{n\to\infty}\ \sup_{F\in\mathcal{F}_{2+\delta}\colon\,|\ell_n(F)-\ell|\leqslant\theta_n}\Delta_n(F),$$

with $\ell>0$ and $\{\theta_n\}_{n\geqslant 1}$ being some infinitesimal sequence, to the construction of which we proceed. Recall that $\Delta_n(F)$ denotes the uniform distance between the d.f. of the standard normal law and the d.f. of the standardized sum of i.i.d. r.v.'s with the common d.f. $F\in\mathcal{F}_{2+\delta}$.

First note that for any $\ell>0$ and arbitrary infinitesimal sequence of nonnegative numbers $\{\theta_n\}_{n\geqslant 1}$ by virtue of (4.1) we have

$$\begin{aligned}1\leqslant\limsup_{n\to\infty}\ &\sup_{F_1=\ldots=F_n\in\mathcal{F}_{2+\delta}\colon\,|\ell_n-\ell|\leqslant\theta_n}\nu_n(F)\leqslant 1+\limsup_{n\to\infty}\ \sup_{\ell_n\colon\,|\ell_n-\ell|\leqslant\theta_n}\frac{1}{n^{\delta/2}\ell_n}\\&\leqslant 1+\lim_{n\to\infty}\frac{1}{n^{\delta/2}(\ell-\theta_n)}=1,\end{aligned}$$

and with account of the inequality $\varkappa_\delta\leqslant(2\theta_0(\delta))^{-1/\delta}$ (see (2.2))

$$\lim_{n\to\infty}t_4\left(\delta,n^{-1-\delta/2}\right)=\frac{(2\varkappa_\delta)^{-1/\delta}}{2\pi}\geqslant\frac{\theta_0(\delta)}{2\pi}=t_1(\delta).$$

Further, it is easy to make sure that for any $\ell>0$ and $t_0\in\big[t_3(\delta),\,t_1(\delta)\big)$ the relations

$$\begin{aligned}\limsup_{n\to\infty}\ \sup_{|\ell_n-\ell|\leqslant\theta_n}\widehat{J}_{11}(n,\nu_n,t_0)&=1.0253\pi^{-1}\varkappa_\delta\gamma_\delta Q(0,t_0,4+2\delta)\\&=1.0253\frac{2^{3/2+\delta}\varkappa_\delta\gamma_\delta\Gamma(5/2+\delta)}{\pi\left(1-2\varkappa_\delta(2\pi t_0)^\delta\right)^{5/2+\delta}},\\ \limsup_{n\to\infty}\ \sup_{|\ell_n-\ell|\leqslant\theta_n}\widehat{J}_{12}(\ell_n,\nu_n,n)&=2^{(\delta-1)/2}\pi^{-1}\gamma_\delta\times\end{aligned}$$

$$\times\Gamma\Big(\frac{3+\delta}{2}\Big)\left(1+\frac{(3+\delta)\ell^{2/\delta}}{72}\right)\to\infty,\quad \ell\to\infty,$$

hold, where the least upper bounds are taken over all $F_1=\ldots=F_n\equiv F\in\mathcal{F}_{2+\delta}$ such that $|\ell_n(F)-\ell|\leqslant\theta_n$. So, from (4.9) for all $\ell>0$ follows the estimate

$$\limsup_{n\to\infty}\sup_{F\in\mathcal{F}_{2+\delta}\colon|\ell_n-\ell|\leqslant\theta_n}\Delta_n(F)\leqslant C(\delta)\cdot\ell+\frac{\ell^{1/\delta}}{2\sqrt{2\pi}}+C'_\delta(\ell)\cdot\ell^2,$$

where

$$C'_\delta(\ell)=\ell^{(1-\delta)/\delta}2^{(\delta-1)/2}\pi^{-1}\gamma_\delta\Gamma\Big(\frac{3+\delta}{2}\Big)\left(1+\frac{(3+\delta)\ell^{2/\delta}}{72}\right)$$
$$+\inf_{t_3(\delta)\leqslant t_0<t_1(\delta)}\left(1.0253\frac{2^{3/2+\delta}\varkappa_\delta\gamma_\delta\Gamma(5/2+\delta)}{\pi\left(1-2\varkappa_\delta(2\pi t_0)^\delta\right)^{5/2+\delta}}+\frac{\ell^{2(1-\delta)/\delta}}{4\pi^2}\sup_{0<\varepsilon\leqslant\ell}J\left(\frac{2\pi}{\varepsilon^{1/\delta}},t_0\right)\right).$$

For the calculation of the least upper bound of $J(2\pi\varepsilon^{-1/\delta},t_0)$ over $0<\varepsilon\leqslant\ell$ see remark 4.5. Note that for each $0<\delta\leqslant1$ the function $C'_\delta(\ell)$ increases monotonically varying within the limits

$$C'_\delta(0)\equiv\lim_{\ell\to0}C'_\delta(\ell)<C'_\delta(\ell)<\lim_{\ell\to\infty}C'_\delta(\ell)=+\infty,\quad 0<\delta\leqslant1,\ \ell>0,$$

$$C'_\delta(0)=\begin{cases}\frac{1.0253\cdot2^{3/2+\delta}\varkappa_\delta\gamma_\delta\Gamma(5/2+\delta)}{\pi(1-2\varkappa_\delta(2\pi t_3(\delta))^\delta)^{5/2+\delta}}=\frac{1.0253\cdot2^{3/2+\delta}\varkappa_\delta\gamma_\delta\Gamma(5/2+\delta)}{\pi\left(1-\frac{4}{(2+\delta)^2}\exp\left\{-\frac{\delta(4+\delta)}{2(2+\delta)}\right\}\right)^{5/2+\delta}}, & 0<\delta<1,\\[2ex] C'_{1-}(0)+\frac{1}{6\pi}=\frac{1.0253\cdot5\varkappa_1}{2\sqrt{2\pi}(1-4/9e^{-5/6})^{7/2}}+\frac{1}{6\pi}=0.2679\ldots, & \delta=1.\end{cases}$$

The values of $C'_\delta(0)$ and $C'_\delta(\ell)$ for some ℓ and $0<\delta\leqslant1$ are given in table 8.

To obtain estimates with constants $\widetilde{C}_\delta$, $\widehat{C}_\delta$, C'_δ at remainders bounded for all $\ell_n>0$, note that if $\ell_n\geqslant\ell$ for some $\ell>0$, then by virtue of (4.3) for any

$$A\geqslant\kappa-C(\delta)\cdot\ell-\frac{\ell^{1/\delta}}{2\sqrt{2\pi}}$$

the trivial estimate

$$C(\delta)\cdot\ell_n+\frac{1}{2\sqrt{2\pi}}\left(\ell_n+\frac{1}{B_n^{2+\delta}}\sum_{j=1}^n\beta_{\delta,j}\,\sigma_j^2\right)^{1/\delta}+A\geqslant C(\delta)\cdot\ell+\frac{\ell^{1/\delta}}{2\sqrt{2\pi}}+A\geqslant\kappa\geqslant\Delta_n,$$

holds so that the quantities $\widetilde{C}_\delta(\ell_n)\ell_n^{4/(2+\delta)}$ and $\widetilde{C}_1(\ell_n)\ell_n^{5/3}$ in (4.8) for $\ell_n\geqslant\ell$ can be respectively replaced by $\min\left\{\widetilde{C}_\delta(\ell_n)\ell_n^{4/(2+\delta)},\ \kappa-C(\delta)\cdot\ell-(2\sqrt{2\pi})^{-1}\ell^{1/\delta}\right\}$ and $\min\left\{\widetilde{C}_1(\ell_n)\ell_n^{5/3},\ \kappa-2/(3\sqrt{2\pi})\ell\right\}$ for any $\ell\in(0,\overline{\ell}(\delta))$. Note that

$$\kappa-C(\delta)\cdot\ell-\frac{\ell^{1/\delta}}{2\sqrt{2\pi}}\leqslant\kappa-\frac{\ell^{1/\delta}}{2\sqrt{2\pi}}\leqslant0\quad\text{for }\ell\geqslant(2\sqrt{2\pi}\kappa)^\delta.$$

Define $\widetilde{\ell}(\delta)$ as the unique root of the equation

$$\widetilde{C}_\delta(\ell)\cdot\ell^{4/(2+\delta)}=\kappa-C(\delta)\cdot\ell-\frac{\ell^{1/\delta}}{2\sqrt{2\pi}},\quad 0<\delta<1,$$

$$\widetilde{C}_1(\ell)\cdot\ell^{5/3}=\kappa-\frac{2\ell}{3\sqrt{2\pi}},\quad \delta=1,$$

on the interval $0<\ell<\overline{\ell}(\delta)\wedge(2\sqrt{2\pi}\kappa)^\delta=\overline{\ell}(\delta)$ (recall that, by definition, $\overline{\ell}(\delta)<1<(2\sqrt{2\pi}\kappa)^\delta$ for all $0<\delta\leqslant 1$, since $\kappa=0.54\ldots>1/2$, see (4.3)). The existence and uniqueness of $\widetilde{\ell}(\delta)$ follow from that on the interval under consideration the left-hand side of the equation is a continuous strictly monotonically increasing function taking *all* values from 0 to $+\infty$, and the right-hand side is a continuous strictly monotonically decreasing function taking positive values at small ℓ, that is, the graphs of these functions have a unique point of intersection on the interval $(0,\overline{\ell}(\delta))$. So, since the function $\widetilde{C}_\delta$ increases monotonically in ℓ, for any $\ell>0$ the estimate

$$\Delta_n\leqslant C(\delta)\cdot\ell_n+\frac{1}{2\sqrt{2\pi}}\left(\ell_n+\frac{1}{B_n^{2+\delta}}\sum_{j=1}^n\beta_{\delta,j}\,\sigma_j^2\right)^{1/\delta}+\begin{cases}\widetilde{C}_\delta\left(\ell\wedge\widetilde{\ell}(\delta)\right)\cdot\ell_n^{4/(2+\delta)}, & 0<\delta<1,\\ \widetilde{C}_1\left(\ell\wedge\widetilde{\ell}(1)\right)\cdot\ell_n^{5/3}, & \delta=1.\end{cases}$$

holds for all $n\geqslant 1$ and $F_1,\ldots,F_n\in\mathcal{F}_{2+\delta}$ such that $\ell_n\leqslant\ell$.

Similar reasoning also can be applied to the functions $\widehat{C}_\delta(\ell)$, $C'_\delta(\ell)$ with the only remark that for $C'_\delta(\ell)$ the root of the corresponding equation lies within the interval $\big(0,\,(2\sqrt{2\pi}\kappa)^\delta\big)$ which results in the following theorem.

Theorem 4.6. *For any $0<\delta\leqslant 1$ and $\ell>0$, for all $n\geqslant 1$ and $F_1,\ldots,F_n\in\mathcal{F}_{2+\delta}$ such that $\ell_n\leqslant\ell$, the following estimates hold: in the general case*

$$\Delta_n\leqslant C(\delta)\cdot\ell_n+\frac{1}{2\sqrt{2\pi}}\left(\ell_n+\frac{1}{B_n^{2+\delta}}\sum_{j=1}^n\beta_{\delta,j}\,\sigma_j^2\right)^{1/\delta}+\begin{cases}\widetilde{C}_\delta\left(\ell\wedge\widetilde{\ell}(\delta)\right)\cdot\ell_n^{4/(2+\delta)}, & 0<\delta<1,\\ \widetilde{C}_1\left(\ell\wedge\widetilde{\ell}(1)\right)\cdot\ell_n^{5/3}, & \delta=1,\end{cases}$$

in the case $F_1=\ldots=F_n$

$$\Delta_n\leqslant C(\delta)\cdot\frac{\beta_{2+\delta}}{\sigma^{2+\delta}n^{\delta/2}}+\frac{1}{2\sqrt{2\pi n}}\left(\frac{\beta_{2+\delta}}{\sigma^{2+\delta}}+\frac{\beta_\delta}{\sigma^\delta}\right)^{1/\delta}+\widehat{C}_\delta\left(\ell\wedge\widehat{\ell}(\delta)\right)\cdot\ell_n^2,$$

and also for any $\ell>0$ and arbitrary infinitesimal sequence of nonnegative numbers $\{\theta_n\}_{n\geqslant 1}$

$$\limsup_{n\to\infty}\sup_{F\in\mathcal{F}_{2+\delta}\colon|\ell_n-\ell|\leqslant\theta_n}\Delta_n(F)\leqslant C(\delta)\cdot\ell+\frac{\ell^{1/\delta}}{2\sqrt{2\pi}}+C'_\delta\left(\ell\wedge\ell'(\delta)\right)\cdot\ell^2,$$

where

$$C(\delta) = \frac{\gamma_\delta 2^{\delta/2}}{\pi}\,\Gamma\left(\frac{2+\delta}{2}\right),$$

$$\widetilde{C}_\delta(\ell) = \frac{1}{4\pi} + \ell^{\frac{2-\delta(1-\delta)}{\delta(2+\delta)}} J_{12}(\ell,2) + \inf\Big\{\ell^{\delta/(2+\delta)} J_{11}(\ell,2,t_0)$$
$$+\,\ell^{\frac{2(2-\delta)}{\delta(2+\delta)}}\cdot\frac{2^{2/\delta}}{4\pi^2}\sup_{0<\varepsilon\leqslant 2\ell} J\left(2\pi\varepsilon^{-1/\delta},t_0\right) : t_3(\delta)\leqslant t_0\leqslant t_1(\delta)\wedge t_4(\delta,\ell)\Big\},$$
$$0<\delta<1,$$

$$\widetilde{C}_1(\ell) = \inf\Big\{J_{11}(\ell,2,t_0) + \ell^{1/3}J_{12}(\ell,2)$$
$$+\,\ell^{1/3}\pi^{-2}\sup_{0<\varepsilon\leqslant 2\ell} J\left(2\pi\varepsilon^{-1/\delta},t_0\right) : t_3(1)\leqslant t_0\leqslant t_1(1)\wedge t_4(1,\ell)\Big\},$$

$$\widehat{C}_\delta(\ell) = \frac{\mathbf{1}(\delta<1)}{4\pi} + \ell^{(1-\delta)/\delta}\widehat{J}_{12}(\ell,2,\ell^{-2/\delta})$$
$$+\inf\Big\{\ell^{2(1-\delta)/\delta}\cdot\frac{2^{2/\delta}}{4\pi^2}\sup_{0<\varepsilon\leqslant 2\ell} J\left(2\pi\varepsilon^{-1/\delta},t_0\right)$$
$$+\,\widehat{J}_{11}(\ell^{-2/\delta},2,t_0)\colon t_3(\delta)\leqslant t_0\leqslant t_1(\delta)\wedge t_4\left(\delta,\ell^{1+2/\delta}\right)\Big\},$$

$$C'_\delta(\ell) = \ell^{(1-\delta)/\delta}2^{(\delta-1)/2}\pi^{-1}\gamma_\delta\Gamma\Big(\frac{3+\delta}{2}\Big)\left(1+\frac{(3+\delta)\ell^{2/\delta}}{72}\right)$$
$$+\inf_{t_3(\delta)\leqslant t_0<t_1(\delta)}\left(1.0253\frac{2^{3/2+\delta}\varkappa_\delta\gamma_\delta\Gamma(5/2+\delta)}{\pi\left(1-2\varkappa_\delta(2\pi t_0)^\delta\right)^{5/2+\delta}} + \frac{\ell^{2(1-\delta)/\delta}}{4\pi^2}\sup_{0<\varepsilon\leqslant\ell} J\left(\frac{2\pi}{\varepsilon^{1/\delta}},t_0\right)\right),$$

$\widetilde{\ell}(1)$ *is the unique root of the equation* $\widetilde{C}_1(\ell)\cdot\ell^{5/3} = \kappa - 2\ell/(3\sqrt{2\pi})$ *on the interval* $0<\ell<\overline{\ell}(1)$, $\widetilde{\ell}(\delta)$, $\widehat{\ell}(\delta)$, $\ell'(\delta)$ *are respectively the unique roots of the equations*

$$\widetilde{C}_\delta(\ell)\cdot\ell^{4/(2+\delta)} = \kappa - C(\delta)\cdot\ell - \frac{\ell^{1/\delta}}{2\sqrt{2\pi}},\quad 0<\ell<\overline{\ell}(\delta),\ 0<\delta<1,$$

$$\widehat{C}_\delta(\ell)\cdot\ell^2 = \kappa - C(\delta)\cdot\ell - \frac{\ell^{1/\delta}}{2\sqrt{2\pi}},\quad 0<\ell<\left(\overline{\ell}(\delta)\right)^{\delta/(2+\delta)},\ 0<\delta\leqslant 1,$$

$$C'_\delta(\ell)\cdot\ell^2 = \kappa - C(\delta)\cdot\ell - \frac{\ell^{1/\delta}}{2\sqrt{2\pi}},\quad 0<\ell<(2\sqrt{2\pi}\kappa)^\delta,\ 0<\delta\leqslant 1,$$

on the intervals specified above; $\kappa = 0.5409\ldots$ *is defined in* (4.3)*;* γ_δ, $\varkappa_\delta$, $t_1(\delta)$, $t_3(\delta)$, $t_4(\delta,\ell)$, $\overline{\ell}(\delta)$, $J_{11}(\ell,\nu,t_0)$, $\widehat{J}_{11}(n,\nu,t_0)$, $J_{12}(\ell,\nu)$, $\widehat{J}_{12}(\ell,\nu,n)$, $J(T,t_0)$, $T>0$, *are defined in lemma* 4.3.

$\delta =$	$C(\delta) \leqslant$	$\underline{C_{\text{AE}}}(\delta) \geqslant$	$\delta =$	$C(\delta) \leqslant$	$\underline{C_{\text{AE}}}(\delta) \geqslant$	$\delta =$	$C(\delta) \leqslant$	$\underline{C_{\text{AE}}}(\delta) \geqslant$
0+	0.1693	0.0883	0.35	0.1017	0.0422	0.70	0.0709	0.0253
0.05	0.1561	0.0759	0.40	0.0956	0.0390	0.75	0.0685	0.0237
0.10	0.1444	0.0674	0.45	0.0902	0.0361	0.80	0.0665	0.0223
0.15	0.1339	0.0606	0.50	0.0854	0.0334	0.85	0.0650	0.0210
0.20	0.1245	0.0550	0.55	0.0810	0.0311	0.90	0.0642	0.0198
0.25	0.1161	0.0501	0.60	0.0772	0.0290	0.95	0.0642	0.0187
0.30	0.1085	0.0459	0.65	0.0738	0.0271	1−	0.0665	0.0177

Table 5: The values of $C(\delta)$ from theorem 4.6 which bounds above the asymptotically exact constant $C_{\text{AE}}(\delta)$ (see theorem 4.12) rounded up to the fourth decimal digit and the corresponding values of the lower bound for the lower asymptotically exact constant $\underline{C_{\text{AE}}}(\delta)$ (see (4.10)) for some $0 < \delta \leqslant 1$. By definition, $\underline{C_{\text{AE}}}(\delta) \leqslant C_{\text{AE}}(\delta) \leqslant C(\delta)$ for all $0 < \delta \leqslant 1$.

$\delta =$	$\widetilde{\ell}(\delta) \leqslant$	$\widetilde{C}_\delta\big(\widetilde{\ell}(\delta)\big) \leqslant$	$\widetilde{C}_\delta(0.1) \leqslant$	$\widetilde{C}_\delta(0.01) \leqslant$	$\widetilde{C}_\delta\left(10^{-3}\right) \leqslant$	$\widetilde{C}_\delta\left(10^{-4}\right) \leqslant$
0.05	0.0218	943.5902	943.5902	492.0103	290.6531	253.8418
0.10	0.0437	208.2037	208.2037	67.6270	43.7421	35.7650
0.15	0.0635	89.9006	89.9006	21.7830	13.7457	10.5124
0.20	0.0812	51.0184	51.0184	9.7720	5.8904	4.2460
0.25	0.0969	33.5946	33.5946	5.2585	3.0192	2.0712
0.30	0.1108	24.2825	20.0463	3.1846	1.7473	1.1531
0.35	0.1230	18.7024	12.7760	2.0993	1.1074	0.7110
0.40	0.1337	15.0778	8.8825	1.4770	0.7546	0.4767
0.45	0.1430	12.5785	6.5814	1.0951	0.5460	0.3431
0.50	0.1511	10.7742	5.1210	0.8479	0.4157	0.2623
0.55	0.1580	9.4240	4.1429	0.6812	0.3306	0.2111
0.60	0.1639	8.3842	3.4602	0.5648	0.2730	0.1773
0.65	0.1688	7.5649	2.9681	0.4814	0.2328	0.1543
0.70	0.1728	6.9071	2.6044	0.4203	0.2040	0.1381
0.75	0.1761	6.3715	2.3308	0.3748	0.1830	0.1266
0.80	0.1786	5.9306	2.1226	0.3405	0.1675	0.1182
0.85	0.1804	5.5650	1.9638	0.3146	0.1559	0.1119
0.90	0.1814	5.2610	1.8442	0.2953	0.1473	0.1073
0.95	0.1818	5.0102	1.7588	0.2816	0.1411	0.1040
1.00	0.2325	5.4527	1.6948	0.6317	0.4856	0.4427

Table 6: The values of $\widetilde{\ell}(\delta)$ and $\widetilde{C}_\delta(\ell)$ from theorem 4.6 for $\ell = \widetilde{\ell}(\delta)$, $0.1 \wedge \widetilde{\ell}(\delta)$, 0.01, 0.001, 0.0001 and some $0 < \delta \leqslant 1$; the fourth column contains the values of $\widetilde{C}_\delta\big(0.1 \wedge \widetilde{\ell}(\delta)\big)$. The optimal values of t_0 coincide with $t_3(\delta)$ (see table 3).

The values of $C(\delta)$, $\widetilde{\ell}(\delta)$, $\widehat{\ell}(\delta)$, $\ell'(\delta)$, $\widetilde{C}_\delta\big(\ell\big)$, $\widehat{C}_\delta\big(\ell\big)$, $C'_\delta(\ell)$ rounded above up to the fourth decimal digit are given in tables 5, 6, 7, 8 for some $0 < \delta \leqslant 1$ and $\ell \geqslant 0$. The computations were carried out in the Matlab R2011a environment.

Since $C(1) = 1/(6\sqrt{2\pi})$, from theorem 4.6 for $\delta = 1$ we obtain

Corollary 4.7. *For all $n \geqslant 1$ and $F_1, \ldots, F_n \in \mathcal{F}_3$*

$$\Delta_n \leqslant \frac{2\ell_n}{3\sqrt{2\pi}} + \frac{1}{2\sqrt{2\pi}B_n^3}\sum_{j=1}^n \beta_{1,j}\,\sigma_j^2 + 5.4527 \cdot \ell_n^{5/3}$$

$\delta =$	$\widehat{\ell}(\delta) \leqslant$	$t_0 =$	$\widehat{C}_\delta(\widehat{\ell}(\delta)) \leqslant$	$\widehat{C}_\delta(0.1) \leqslant$	$\widehat{C}_\delta(0.01) \leqslant$	$\widehat{C}_\delta\left(10^{-3}\right) \leqslant$	$\widehat{C}_\delta(0+) \leqslant$
0.05	0.0468	0.1370	243.6690	243.6690	243.6690	243.6690	243.6690
0.10	0.1050	0.1386	47.7282	47.7282	47.7282	47.7282	47.7282
0.15	0.1662	0.1401	18.7976	18.7973	18.7973	18.7973	18.7973
0.20	0.2283	0.1416	9.8319	9.8249	9.8246	9.8246	9.8246
0.25	0.2897	0.1431	6.0285	5.9929	5.9916	5.9916	5.9916
0.30	0.3407	0.1444	4.2951	4.0322	4.0288	4.0287	4.0287
0.35	0.3652	0.1457	3.6948	2.9060	2.8988	2.8987	2.8987
0.40	0.3795	0.1469	3.3818	2.2057	2.1932	2.1928	2.1928
0.45	0.3889	0.1480	3.1837	1.7448	1.7256	1.7246	1.7245
0.50	0.3950	0.1490	3.0525	1.4292	1.4018	1.3996	1.3994
0.55	0.3987	0.1525	2.9657	1.2069	1.1702	1.1661	1.1654
0.60	0.4005	0.1563	2.9104	1.0480	1.0007	0.9941	0.9923
0.65	0.4007	0.1588	2.8812	0.9338	0.8749	0.8652	0.8614
0.70	0.3996	0.1603	2.8742	0.8526	0.7811	0.7682	0.7608
0.75	0.3973	0.1613	2.8863	0.7968	0.7117	0.6957	0.6826
0.80	0.3940	0.1618	2.9157	0.7614	0.6618	0.6432	0.6216
0.85	0.3898	0.1622	2.9611	0.7435	0.6283	0.6081	0.5744
0.90	0.3847	0.1623	3.0228	0.7417	0.6097	0.5895	0.5389
0.95	0.3787	0.1623	3.1027	0.7571	0.6069	0.5886	0.5148
1.00	0.4180	0.1770	2.4606	0.6023	0.5403	0.5364	0.5360

Table 7: The values of $\widehat{\ell}(\delta)$ and $\widehat{C}_\delta(\ell)$ from theorem 4.6 for $\ell = \widehat{\ell}(\delta)$, $0.1 \wedge \widehat{\ell}(\delta)$, 0.01, 0.001 and $\ell \to 0+$ for some $0 < \delta \leqslant 1$. The third column contains the optimal values of t_0 delivering the infimum in $\widehat{C}_\delta(\widehat{\ell}(\delta))$, for other ℓ the optimal values of t_0 coincide with $t_3(\delta)$ (see table 3).

$\delta =$	$\ell'(\delta) \leqslant$	$t_0 =$	$C'_\delta(\ell'(\delta)) \leqslant$	$C'_\delta(0.5) \leqslant$	$C'_\delta(0.1) \leqslant$	$C'_\delta(0+) \leqslant$
0.05	0.0661	0.1370	121.7947	121.7947	121.7947	121.7947
0.10	0.1477	0.1386	23.8244	23.8244	23.8244	23.8244
0.15	0.2334	0.1401	9.3589	9.3589	9.3589	9.3589
0.20	0.3205	0.1416	4.8734	4.8734	4.8726	4.8726
0.25	0.4062	0.1431	2.9613	2.9613	2.9561	2.9561
0.30	0.4863	0.1444	1.9884	1.9884	1.9750	1.9746
0.35	0.5581	0.1457	1.4339	1.4291	1.4106	1.4096
0.40	0.6170	0.1469	1.1095	1.0805	1.0588	1.0566
0.45	0.6577	0.1480	0.9320	0.8508	0.8263	0.8225
0.50	0.6867	0.1490	0.8237	0.6935	0.6660	0.6599
0.55	0.7094	0.1500	0.7485	0.5833	0.5519	0.5429
0.60	0.7283	0.1513	0.6924	0.5051	0.4686	0.4564
0.65	0.7457	0.1621	0.6456	0.4497	0.4068	0.3909
0.70	0.7628	0.1717	0.6040	0.4110	0.3605	0.3406
0.75	0.7794	0.1801	0.5673	0.3847	0.3256	0.3015
0.80	0.7950	0.1874	0.5356	0.3680	0.2997	0.2710
0.85	0.8091	0.1937	0.5087	0.3565	0.2811	0.2474
0.90	0.8209	0.1988	0.4870	0.3492	0.2687	0.2297
0.95	0.8291	0.2026	0.4714	0.3469	0.2628	0.2176
1.00	0.8280	0.2044	0.4679	0.3559	0.2684	0.2680

Table 8: The values of $\ell'(\delta)$ and $C'_\delta(\ell)$ from theorem 4.6 for $\ell = \ell'(\delta)$, $0.5 \wedge \ell'(\delta)$, $0.1 \wedge \ell'(\delta)$ and $\ell \to 0+$ for some $0 < \delta \leqslant 1$. The third column contains the optimal values of t_0 delivering the infimum in $C'_\delta(\ell'(\delta))$, for other ℓ the optimal values of t_0 coincide with $t_3(\delta)$ (see table 3).

in the general case and

$$\Delta_n \leqslant \frac{2}{3\sqrt{2\pi}} \cdot \frac{\beta_3}{\sigma^3\sqrt{n}} + \frac{1}{2\sqrt{2\pi}} \cdot \frac{\beta_1}{\sigma\sqrt{n}} + 2.4606 \cdot \ell_n^2,$$

if $F_1 = \ldots = F_n$.

Remark 4.8. Corollary 4.7 improves the inequalities of Prawitz (1.9)

$$\Delta_n \leqslant \frac{2}{3\sqrt{2\pi}} \cdot \frac{\beta_3}{\sigma^3\sqrt{n-1}} + \frac{1}{2\sqrt{2\pi(n-1)}} + A_3 \cdot \ell_{n-1}^2, \quad n \geqslant 1,\ F_1 = \ldots = F_n \in \mathcal{F}_3,$$

and Bentkus (1.10)

$$\begin{aligned}\Delta_n &\leqslant \frac{2\ell_n}{3\sqrt{2\pi}} + \frac{1}{2\sqrt{2\pi}B_n^3}\sum_{j=1}^n \sigma_j^3 + A_4 \cdot \ell_n^{4/3} \\ &\leqslant \frac{7\ell_n}{6\sqrt{2\pi}} + A_4 \cdot \ell_n^{4/3}, \quad n \geqslant 1,\ F_1, \ldots, F_n \in \mathcal{F}_3,\end{aligned}$$

first, with respect to the second term, since $\beta_{1,j} \leqslant \sigma_j$, $j = 1, \ldots, n$, by the Lyapounov inequality, and second, with respect to the remainder, since it gives concrete values of the constants A_3 and A_4. And as regards the general case, corollary 4.7 also improves the order of decrease of the remainder to $\ell_n^{5/3}$ as compared with $\ell_n^{4/3}$ in Bentkus' inequality.

Remark 4.9. The values of the coefficients $2/(3\sqrt{2\pi})$ and $\bigl(2\sqrt{2\pi}\bigr)^{-1}$ in the estimates given in corollary 4.7 are optimal in the sense that whatever the coefficient at the second term is, the coefficient $2/(3\sqrt{2\pi})$ at the first term cannot be made less and for the given value $2/(3\sqrt{2\pi})$ of the coefficient at the first term, the coefficient at the second term cannot be made less than $\bigl(2\sqrt{2\pi}\bigr)^{-1}$. To make this sure it suffices to consider the estimates of the form

$$\Delta_n \leqslant C \cdot \frac{\beta_3}{\sigma^3\sqrt{n}} + K \cdot \frac{\beta_1}{\sigma\sqrt{n}} + A \cdot \ell_n^{1+\theta},$$

with some constants C, K, $A \in \mathbf{R}$ and $\theta > 0$ assuming that they hold for all (or at least for large enough) values of n and all $F_1 = \ldots = F_n \in \mathcal{F}_3$, and notice that by virtue of these estimates

$$\begin{aligned}\underline{C_{\text{AE}}} &= \limsup_{\ell\to 0}\limsup_{n\to\infty} \sup_{F\colon \beta_3=\sigma^3\ell\sqrt{n}} \frac{\Delta_n(F)}{\ell} \\ &\leqslant C + \limsup_{\ell\to 0}\limsup_{n\to\infty} \sup_{F\colon \beta_3=\sigma^3\ell\sqrt{n}} K \cdot \frac{\beta_1}{\sigma\sqrt{n}\ell} \leqslant C,\end{aligned}$$

since $K\beta_1/(\sigma\sqrt{n}\ell) \leqslant 0$ for $K \leqslant 0$, and for $K > 0$ by virtue of the Lyapounov inequality

$$K \cdot \frac{\beta_1}{\sigma\sqrt{n}\ell} \leqslant K \cdot \frac{1}{\ell\sqrt{n}} \to 0, \quad n \to \infty,$$

for any $\ell > 0$. So, with the account of the equality $\underline{C_{\text{AE}}} = 2/(3\sqrt{2\pi})$ [27] we conclude that for any $K \in \mathbf{R}$

$$C \geqslant \underline{C_{\text{AE}}} = \frac{2}{3\sqrt{2\pi}}.$$

Now let $C = 2/(3\sqrt{2\pi})$. Show that in this case K is no less than $\left(2\sqrt{2\pi}\right)^{-1}$. Indeed, by virtue of (1.4) we have

$$K \geqslant \sup_{X_1\in\mathcal{F}_3}\limsup_{n\to\infty}\frac{3\sqrt{2\pi n}\Delta_n(\mathsf{E}X_1^2)^{3/2}-2\mathsf{E}|X_1|^3}{3\sqrt{2\pi}\mathsf{E}|X_1|\mathsf{E}X_1^2}$$
$$=\sup_{h>0}\sup_{X\in\mathcal{F}_3^h}\frac{|\mathsf{E}X^3|+3h\mathsf{E}X^2-4\mathsf{E}|X|^3}{6\sqrt{2\pi}\mathsf{E}|X|\mathsf{E}X^2}.$$

Now letting $\mathsf{P}(X=-\sqrt{p/q})=q$, $\mathsf{P}(X=\sqrt{q/p})=p=1-q$, $0<p\leqslant 1/2$, we arrive at

$$\mathsf{E}X=0,\ \mathsf{E}X^2=1,\ \mathsf{E}X^3=\frac{q-p}{\sqrt{pq}},\ \mathsf{E}|X|=2\sqrt{pq},\ \mathsf{E}|X|^3=\frac{p^2+q^2}{\sqrt{pq}},\ h=\frac{1}{\sqrt{pq}},$$

and hence,

$$K\geqslant\sup_{0<p<1/2,\ q=1-p}\frac{q-p+3-4(p^2+q^2)}{12\sqrt{2\pi}pq}=\frac{1}{6\sqrt{2\pi}}\lim_{p\to0+}\frac{3-4p}{1-p}=\frac{1}{2\sqrt{2\pi}}.$$

Remark 4.10. The estimate given in corollary 4.7, for summands with the common symmetric distribution $\mathsf{P}(X=\pm1)=1/2$ with the moments $\beta_1=\sigma^2=\beta_3=1$, takes the form

$$\Delta_n\leqslant\frac{7}{6\sqrt{2\pi n}}+2.4606\ell_n^2=\frac{7\ell_n}{6\sqrt{2\pi}}+2.4606\ell_n^2.$$

On the other hand, for the distribution under consideration it follows from Esseen's asymptotic expansion (1.3) (see [9, 10]) that

$$\Delta_n=\frac{1}{\sqrt{2\pi n}}+o\left(\frac{1}{\sqrt{n}}\right)=\frac{\ell_n}{\sqrt{2\pi}}+o\left(\ell_n\right),\quad n\to\infty,$$

that is, the "exact" constant at the Lyapounov fraction ℓ_n is $7/6\approx1.17$ times less than that given by the "optimal" estimate from corollary 4.7. Actually there is no paradox, since the obtained estimate is optimal in another sense, but the remark reveals the fact that to obtain estimates with "exact" coefficients at the Lyapounov fraction, the information concerning all first three *absolute* moments is not enough and it is required to use also the information concerning the *original* moments, the only informative of which is the third, since the summands are assumed centered.

Corollary 4.11. *For any $\ell>0$ and arbitrary infinitesimal sequence of nonnegative numbers $\{\theta_n\}_{n\geqslant1}$*

$$\limsup_{n\to\infty}\sup_{F\in\mathcal{F}_3:\,|\ell_n-\ell|\leqslant\theta_n}\Delta_n(F)\leqslant\frac{2\ell}{3\sqrt{2\pi}}+C_1'\left(\ell\wedge0.8280\right)\cdot\ell^2\leqslant0.2660\cdot\ell+0.4679\cdot\ell^2,$$

where $C_1'(\ell)$ *is defined in theorem* 4.6. *In particular,* $C_1'(0.1) \leqslant 0.2684$*, and for all* $0 < \ell \leqslant 0.1$

$$\limsup_{n\to\infty} \sup_{F\in\mathcal{F}_3\colon |\ell_n-\ell|\leqslant\theta_n} \Delta_n(F) \leqslant \frac{2\ell}{3\sqrt{2\pi}} + 0.2684\cdot\ell^2 < \begin{cases} 0.2929\cdot\ell, & \ell\leqslant 0.1,\\ 0.2687\cdot\ell, & \ell\leqslant 10^{-2},\\ 0.2663\cdot\ell, & \ell\leqslant 10^{-3},\\ 0.2660\cdot\ell, & \ell\leqslant 10^{-4}.\end{cases}$$

Letting $\ell\to 0$, from theorem 4.6 one can obtain an upper bound for the asymptotically exact constant

$$C_{\text{AE}}(\delta) = \limsup_{\ell\to 0} \sup_{n\geqslant 1,\, F_1,\ldots,F_n\in\mathcal{F}_{2+\delta}\colon \ell_n=\ell} \Delta_n(F_1,\ldots,F_n)/\ell, \quad 0<\delta\leqslant 1.$$

Theorem 4.12. *For all* $0<\delta<1$ *the estimate* $C_{\text{AE}}(\delta)\leqslant C(\delta)$ *holds with* $C(\delta)$ *defined in theorem* 4.6. *In particular,*

$$\lim_{\delta\to 1-} C_{\text{AE}}(\delta) \leqslant \frac{1}{6\sqrt{2\pi}} < 0.0665, \qquad \lim_{\delta\to 0+} C_{\text{AE}}(\delta) \leqslant \frac{\gamma_0}{\pi} < 0.1693.$$

The values of $C(\delta)$ *for other* $0<\delta<1$ *are given in table* 5.

For the scheme of summation of identically distributed r.v.'s theorem 4.12 was proved in [11].

The lower bounds for the asymptotically exact constant $C_{\text{AE}}(\delta)$ for $0<\delta<1$ were obtained in [29] in terms of the so-called lower asymptotically exact constant

$$\underline{C_{\text{AE}}}(\delta) = \limsup_{\ell\to 0}\limsup_{n\to\infty} \sup_{F\in\mathcal{F}_{2+\delta}\colon \ell_n=\ell} \Delta_n(F)/\ell$$

and have the form

$$C_{\text{AE}}(\delta) \geqslant \underline{C_{\text{AE}}}(\delta) \geqslant \sup_{a\geqslant 0,\, b>0} \frac{\frac{4}{\sqrt{2+b^2}}\exp\left\{-\frac{a^2}{2(2+b^2)}\right\} + \frac{a^2+b^2}{\sqrt{2}} - 2\sqrt{2}}{8M_{2+\delta}b^{2+\delta}e^{-a^2/(2b^2)}{}_1F_1\left(\frac{3+\delta}{2},\frac{1}{2},\frac{a^2}{2b^2}\right)}, \tag{4.10}$$

where $\Gamma(\cdot)$ is the Euler's gamma-function, ${}_1F_1$ is the generalized hypergeometric function (the degenerate Meijer function), $M_{2+\delta}$ is the absolute moment of order $2+\delta$ of the standard normal law. The values of the lower bound mentioned above, as well as those of the corresponding upper bound, are given in table 5.

For $\delta=1$ from theorem 4.6 one can obtain only the estimate

$$C_{\text{AE}}(1) \leqslant \frac{7}{6\sqrt{2\pi}} = 0.4654\ldots,$$

whereas Chistyakov [7] showed that actually

$$C_{\text{AE}}(1) = \frac{\sqrt{10}+3}{6\sqrt{2\pi}} = 0.4097\ldots,$$

that is, the technique used above is too rough for the construction of asymptotically exact estimates in the classical setting in the case $\delta = 1$, and, as it has been noted in remark 4.10, the only way of sharpening of this technique is the use of the information concerning the third *original* moments. This information can be taken into account, if to estimate the absolute value of the difference of ch.f.'s in the integral

$$I_{13} = \frac{1}{\pi}\sum_{j=1}^{n}\int_0^\infty \frac{1}{t}\left|f_j(t/B_n) - e^{-\sigma_j^2 t^2/(2B_n^2)}\right| e^{-t^2/2}\,dt$$

in lemma 4.3, inequality (2.5) given in lemma 2.8 is used. Taking into consideration that $\mathsf{E}|X_j|^4\mathbf{1}(|X_j| \leqslant U) \leqslant U\beta_{3,j}$, $j = 1, \ldots, n$, for any $U > 0$ we obtain

$$I_{13} \leqslant$$

$$\frac{1}{\pi}\sum_{j=1}^{n}\int_0^\infty\!\left(\frac{t^2}{6B_n^3}\left(\left|\mathsf{E}X_j^3\mathbf{1}(|X_j|\leqslant U)\right| + \mathsf{E}|X_j|^3\mathbf{1}(|X_j|>U)\right) + \frac{U\beta_{3,j}\,t^3}{24B_n^4} + \frac{\sigma_j^4 t^3}{8B_n^4}\right)e^{-t^2/2}dt$$

$$= \frac{1}{6\sqrt{2\pi}B_n^3}\sum_{j=1}^{n}\left(\left|\mathsf{E}X_j^3\mathbf{1}(|X_j|\leqslant U)\right| + \mathsf{E}|X_j|^3\mathbf{1}(|X_j|>U)\right) + \frac{U\ell_n}{12\pi B_n} + \frac{1}{4\pi B_n^4}\sum_{j=1}^{n}\sigma_j^4,$$

so that

$$I_{13} + \frac{\nu_n\ell_n}{2\sqrt{2\pi}} = I_{13} + \frac{\ell_n}{2\sqrt{2\pi}} + \frac{1}{2\sqrt{2\pi}B_n^3}\sum_{j=1}^{n}\beta_{1,j}\,\sigma_j^2 \leqslant \frac{I_{14}}{\sqrt{2\pi}} + \frac{\ell_n}{2\sqrt{2\pi}},$$

where

$$I_{14} = \frac{1}{B_n^3}\sum_{j=1}^{n}\left(\frac{1}{6}\left|\mathsf{E}X_j^3\mathbf{1}(|X_j| \leqslant U)\right| + \frac{1}{2}\mathsf{E}|X_j|\mathsf{E}X_j^2\right) + \frac{U\ell_n}{6\sqrt{2\pi}B_n} + \frac{1}{2\sqrt{2\pi}B_n^4}\sum_{j=1}^{n}\sigma_j^4$$
$$+ \frac{1}{6B_n^3}\sum_{j=1}^{n}\mathsf{E}|X_j|^3\mathbf{1}(|X_j| > U), \quad U > 0.$$

The quantity I_{14} will be estimated in two steps.

1. Truncation. Denote $Y_j = X_j\mathbf{1}(|X_j| \leqslant U)$, $j = 1, \ldots, n$, $U > 0$. Then $X_j^k = Y_j^k + X_j^k\mathbf{1}(|X_j| \geqslant U)$ almost surely, $\mathsf{E}|Y_j|^k \leqslant \mathsf{E}|X_j|^k$, $k = 1, 2, 3$, and for all $j = 1, \ldots, n$

$$\mathsf{E}|X_j|\mathsf{E}X_j^2 \leqslant \mathsf{E}|Y_j|\mathsf{E}Y_j^2 + \mathsf{E}|Y_j|\mathsf{E}X_j^2\mathbf{1}(|X_j| > U) + \mathsf{E}|X_j|\mathbf{1}(|X_j| > U)\mathsf{E}X_j^2$$
$$\leqslant \mathsf{E}|Y_j|\mathsf{E}Y_j^2 + U^{-1}\left(\mathsf{E}|Y_j|^3\right)^{1/3}\mathsf{E}|X_j|^3\mathbf{1}(|X_j| > U)$$
$$+ U^{-2}\mathsf{E}|X_j|^3\mathbf{1}(|X_j| > U)\left(\mathsf{E}|X_j|^3\right)^{2/3} \leqslant \mathsf{E}|Y_j|\mathsf{E}Y_j^2 + \beta_{3,j}^{4/3}/U + \beta_{3,j}^{5/3}/U^2,$$

whence with the account of the relation

$$\sum_{j=1}^{n}\beta_{3,j}^r \leqslant \left(\sum_{j=1}^{n}\beta_{3,j}\right)^r = (B_n^3\ell_n)^r, \quad r \geqslant 1,$$

(see (4.2)) in the general case we obtain

$$\sum_{j=1}^{n} \mathsf{E}|X_j|\mathsf{E}X_j^2 \leqslant \sum_{j=1}^{n} \mathsf{E}|Y_j|\mathsf{E}Y_j^2 + \frac{B_n^4\ell_n^{4/3}}{U} + \frac{B_n^5\ell_n^{5/3}}{U^2}.$$

And if $X_1,\dots,X_n$ are identically distributed, then $\beta_{3,j}=\beta_3=B_n^3\ell_n/n$, $B_n=\sigma\sqrt{n}$ and hence,

$$\sum_{j=1}^{n} \mathsf{E}|X_j|\mathsf{E}X_j^2 \leqslant \sum_{j=1}^{n} \mathsf{E}|Y_j|\mathsf{E}Y_j^2 + \frac{B_n^4\ell_n^{4/3}}{Un^{1/3}} + \frac{B_n^5\ell_n^{5/3}}{U^2n^{2/3}}.$$

So, by the Lyapounov inequality and (4.2), for I_{14} we obtain

$$I_{14} \leqslant I_{15} + \frac{B_n\ell_n^{4/3}}{2U} + \frac{B_n^2\ell_n^{5/3}}{2U^2} + \frac{U\ell_n}{6\sqrt{2\pi}B_n} + \frac{\ell_n^{4/3}}{2\sqrt{2\pi}}, \quad U>0,$$

in the general case and

$$I_{14} \leqslant I_{15} + \frac{B_n\ell_n^{4/3}}{2Un^{1/3}} + \frac{B_n^2\ell_n^{5/3}}{2U^2n^{2/3}} + \frac{U\ell_n}{6\sqrt{2\pi}B_n} + \frac{1}{2\sqrt{2\pi}n}, \quad U>0,$$

in the case of identically distributed summands, where

$$I_{15} = \frac{1}{6B_n^3}\sum_{j=1}^{n}\bigl|\mathsf{E}Y_j^3\bigr| + \frac{1}{2B_n^3}\sum_{j=1}^{n}\mathsf{E}|Y_j|\mathsf{E}Y_j^2 + \frac{1}{6B_n^3}\sum_{j=1}^{n}\mathsf{E}|X_j|^3\mathbf{1}(|X_j|>U).$$

Now choose the parameter U for the reason of equality of the orders of the "worst" terms in the obtained estimates for I_{14}, that is, so that for some free parameter $u>0$ in the general case $U\ell_n/B_n = u^2B_n\ell_n^{4/3}/U$, and hence, $U=uB_n\ell_n^{1/6}$, and in the case of identically distributed summands $U\ell_n/B_n = u^2B_n\ell_n^{4/3}/(Un^{1/3})$, and hence, $U=uB_n(\ell_n/n)^{1/6}$, the parameter u being evaluated later. Then we obtain the estimates: in the general case

$$I_{14} \leqslant I_{15} + \ell_n^{7/6}\left(\frac{1}{2u} + \frac{u}{6\sqrt{2\pi}} + \frac{\ell_n^{1/6}}{2}\left(\frac{1}{u^2} + \frac{1}{\sqrt{2\pi}}\right)\right), \quad u>0,$$

and, since

$$\frac{1}{n} \leqslant \frac{\ell_n^{4/3}}{n^{1/3}} = \frac{\ell_n^{7/6}}{n^{1/6}}\cdot\left(\frac{\ell_n}{n}\right)^{1/6},$$

in the case of identically distributed summands

$$I_{14} \leqslant I_{15} + \frac{\ell_n^{7/6}}{n^{1/6}}\left(\frac{1}{2u} + \frac{u}{6\sqrt{2\pi}} + \frac{1}{2}\Bigl(\frac{\ell_n}{n}\Bigr)^{1/6}\left(\frac{1}{u^2} + \frac{1}{\sqrt{2\pi}}\right)\right), \quad u>0.$$

2. Centering. Since $\mathsf{E}X_j=0$ for all $1\leqslant j\leqslant n$, we have

$$|\mathsf{E}Y_j| = |\mathsf{E}X_j\mathbf{1}(|X_j|>U)| \leqslant \mathsf{E}|X_j|\mathbf{1}(|X_j|>U)$$

$$\leqslant U^{-2}\mathsf{E}|X_j|^3\mathbf{1}(|X_j|>U)\leqslant U^{-2}\beta_{3,j}, \tag{4.11}$$

and hence,

$$|\mathsf{E}Y_j^3-\mathsf{E}(Y_j-\mathsf{E}Y_j)^3|=|3\mathsf{D}Y_j\mathsf{E}Y_j+(\mathsf{E}Y_j)^3|$$
$$\leqslant 3\mathsf{E}Y_j^2|\mathsf{E}Y_j|+|\mathsf{E}Y_j|^3\leqslant 3\beta_{3,j}^{5/3}/U^2+\beta_{3,j}^3/U^6,$$

whence for U chosen above, with the account of (4.2), we obtain

$$\sum_{j=1}^n\left|\mathsf{E}Y_j^3\right|\leqslant\sum_{j=1}^n\left|\mathsf{E}(Y_j-\mathsf{E}Y_j)^3\right|+\frac{3B_n^3\ell_n^{4/3}}{u^2}+\frac{B_n^3\ell_n^2}{u^6},\quad u>0,$$

in the general case and

$$\sum_{j=1}^n\left|\mathsf{E}Y_j^3\right|\leqslant\sum_{j=1}^n\left|\mathsf{E}(Y_j-\mathsf{E}Y_j)^3\right|+\frac{3B_n^3\ell_n^{4/3}}{u^2n^{1/3}}+\frac{B_n^3\ell_n^2}{u^6n},\quad u>0,$$

in the case of identically distributed summands. Similarly, for the terms of the second group in I_{15} we obtain

$$\mathsf{E}|Y_j|\mathsf{E}Y_j^2=\mathsf{E}|Y_j|\mathsf{D}Y_j+\mathsf{E}|Y_j|(\mathsf{E}Y_j)^2\leqslant\mathsf{E}|Y_j-\mathsf{E}Y_j|\mathsf{D}Y_j+|\mathsf{E}Y_j|\mathsf{D}Y_j+\mathsf{E}|Y_j|(\mathsf{E}Y_j)^2$$
$$\leqslant\mathsf{E}|Y_j-\mathsf{E}Y_j|\mathsf{D}Y_j+\left(U^{-2}(\mathsf{E}|Y_j|^3)^{2/3}+U^{-4}\beta_{3,j}(\mathsf{E}|Y_j|^3)^{1/3}\right)\mathsf{E}|X_j|^3\mathbf{1}(|X_j|>U)$$
$$\leqslant\mathsf{E}|Y_j-\mathsf{E}Y_j|\mathsf{D}Y_j+\beta_{3,j}^{5/3}/U^2+\beta_{3,j}^{7/3}/U^4,$$

so that

$$\sum_{j=1}^n\mathsf{E}|Y_j|\mathsf{E}Y_j^2\leqslant\sum_{j=1}^n\mathsf{E}|Y_j-\mathsf{E}Y_j|\mathsf{D}Y_j+\frac{B_n^3\ell_n^{4/3}}{u^2}+\frac{B_n^3\ell_n^{5/3}}{u^4},\quad u>0,$$

in the general case and

$$\sum_{j=1}^n\mathsf{E}|Y_j|\mathsf{E}Y_j^2\leqslant\sum_{j=1}^n\mathsf{E}|Y_j-\mathsf{E}Y_j|\mathsf{D}Y_j+\frac{B_n^3\ell_n^{4/3}}{u^2n^{1/3}}+\frac{B_n^3\ell_n^{5/3}}{u^4n^{2/3}},\quad u>0,$$

in the case of identically distributed summands. With the parameter U specified above, denote

$$I_{16}=\frac{1}{6B_n^3}\sum_{j=1}^n\left|\mathsf{E}(Y_j-\mathsf{E}Y_j)^3\right|+\frac{1}{2B_n^3}\sum_{j=1}^n\mathsf{E}|Y_j-\mathsf{E}Y_j|\mathsf{D}Y_j+\frac{1}{6B_n^3}\sum_{j=1}^n\mathsf{E}|X_j|^3\mathbf{1}(|X_j|>U).$$

Then we have: in the general case

$$I_{15}=\frac{1}{6B_n^3}\sum_{j=1}^n\left|\mathsf{E}Y_j^3\right|+\frac{1}{2B_n^3}\sum_{j=1}^n\mathsf{E}|Y_j|\mathsf{E}Y_j^2$$

$$\leqslant I_{16} + \frac{\ell_n^{4/3}}{u^2} + \frac{\ell_n^{5/3}}{2u^4} + \frac{\ell_n^2}{6u^6} = I_{16} + \ell_n^{7/6}\bigg(\frac{\ell_n^{1/6}}{u^2} + \frac{\ell_n^{1/2}}{2u^4} + \frac{\ell_n^{5/6}}{6u^6}\bigg), \quad u > 0,$$

and in the case of identically distributed summands

$$I_{15} \leqslant I_{16} + \frac{\ell_n^{7/6}}{n^{1/6}}\bigg(\frac{1}{u^2}\Big(\frac{\ell_n}{n}\Big)^{1/6} + \frac{1}{2u^4}\Big(\frac{\ell_n}{n}\Big)^{1/2} + \frac{1}{6u^6}\Big(\frac{\ell_n}{n}\Big)^{5/6}\bigg), \quad u > 0.$$

The application of the moment inequality of theorem 3.1 to the r.v.'s Y_j leads to the estimate

$$I_{16} \leqslant \frac{\lambda}{6B_n^3}\sum_{j=1}^n \mathsf{E}|Y_j - \mathsf{E}Y_j|^3 + \frac{M(p(\lambda),\lambda)}{6B_n^3}\sum_{j=1}^n (\mathsf{D}Y_j)^{3/2} + \frac{1}{6B_n^3}\sum_{j=1}^n \mathsf{E}|X_j|^3\mathbf{1}(|X_j|>U)$$
$$= \frac{\lambda\ell_n}{6} + \frac{M(p(\lambda),\lambda)}{6B_n^3}\sum_{j=1}^n \sigma_j^3 + I_{17} - I_{18}, \quad \lambda \geqslant 1,$$

where

$$I_{17} = \frac{\lambda}{6B_n^3}\sum_{j=1}^n (\mathsf{E}|Y_j - \mathsf{E}Y_j|^3 - \mathsf{E}|Y_j|^3) - \frac{M(p(\lambda),\lambda)}{6B_n^3}\sum_{j=1}^n \Big(\sigma_j^3 - (\mathsf{D}Y_j)^{3/2}\Big), \tag{4.12}$$

$$I_{18} = \frac{\lambda-1}{6B_n^3}\sum_{j=1}^n \mathsf{E}|X_j|^3\mathbf{1}(|X_j| > U), \tag{4.13}$$

with $p(\lambda)$ and $M(p,\lambda)$ defined in theorem 3.1. With the account of (4.11) we obtain

$$\mathsf{E}|Y_j - \mathsf{E}Y_j|^3 - \mathsf{E}|Y_j|^3 \leqslant 3|\mathsf{E}Y_j|\mathsf{E}Y_j^2 + |\mathsf{E}Y_j|^2\mathsf{E}|Y_j|$$
$$\leqslant \big(3\beta_{3,j}^{2/3}/U^2 + \beta_{3,j}^{4/3}/U^4\big)\mathsf{E}|X_j|^3\mathbf{1}(|X_j| > U).$$

By virtue of the inequality $(1-x)^\alpha \geqslant 1 - \alpha x$ which holds for all $0 \leqslant x \leqslant 1$, $\alpha \geqslant 1$, we have

$$0 \leqslant \sigma_j^3 - (\mathsf{D}Y_j)^{3/2} = \sigma_j^3 - \sigma_j^3\left(1 - \frac{\mathsf{E}X_j^2\mathbf{1}(|X_j| > U) + (\mathsf{E}Y_j)^2}{\sigma_j^2}\right)^{3/2}$$
$$\leqslant \frac{3\sigma_j}{2}\left(\mathsf{E}X_j^2\mathbf{1}(|X_j| > U) + (\mathsf{E}Y_j)^2\right) \leqslant \frac{3}{2}\left(\beta_{3,j}^{1/3}/U + \beta_{3,j}^{4/3}/U^4\right)\mathsf{E}|X_j|^3\mathbf{1}(|X_j| > U).$$

Noting that $M(p(\lambda),\lambda) \geqslant 3 - \lambda \geqslant 1 - \lambda$ for all $\lambda \geqslant 1$ (see (3.2)) and using the estimates for the difference between the third moments and variances and denoting $b_j = \beta_{3,j}^{1/3}/U$ we obtain

$$I_{17} \leqslant \frac{\lambda}{6B_n^3}\sum_{j=1}^n \big(\mathsf{E}|Y_j - \mathsf{E}Y_j|^3 - \mathsf{E}|Y_j|^3\big) + \frac{\lambda-1}{6B_n^3}\sum_{j=1}^n \Big(\sigma_j^3 - (\mathsf{D}Y_j)^{3/2}\Big)$$

$$\leqslant \frac{1}{6B_n^3}\sum_{j=1}^{n}\Big(\lambda\left(3b_j^2+b_j^4\right)+\frac{3}{2}(\lambda-1)(b_j+b_j^4)\Big)\mathsf{E}|X_j|^3\mathbf{1}(|X_j|>U)$$

$$=\frac{1}{6B_n^3}\sum_{j=1}^{n}\Big((\lambda-1)\Big(\frac{3}{2}\,b_j+3b_j^2+\frac{5}{2}\,b_j^4\Big)+3b_j^2+b_j^4\Big)\mathsf{E}|X_j|^3\mathbf{1}(|X_j|>U), \quad (4.14)$$

so that

$$I_{17}-I_{18}\leqslant$$

$$\frac{\lambda-1}{6B_n^3}\sum_{j=1}^{n}\Big(\frac{3}{2}\,b_j+3b_j^2+\frac{5}{2}\,b_j^4-1\Big)\mathsf{E}|X_j|^3\mathbf{1}(|X_j|>U)+\frac{1}{6B_n^3}\sum_{j=1}^{n}\bigg(\frac{3\beta_{3,j}^{5/3}}{U^2}+\frac{\beta_{3,j}^{7/3}}{U^4}\bigg).$$

Let $b_0=0.36701\ldots$ be the unique root of the equation $1-\frac{3}{2}b-3b^2-\frac{5}{2}b^4=0$, $b>0$. Then we can guarantee that the first term in the estimate for $I_{17}-I_{18}$ is non-positive if $b_j\equiv\beta_{3,j}^{1/3}/U\leqslant b_0$, i.e. if $U\geqslant\beta_{3,j}^{1/3}/b_0$ for all $j=1,\ldots,n$. In the i.i.d. case, the condition $U\geqslant\beta_{3,j}^{1/3}/b_0$ is equivalent to $u\geqslant(\ell_n/n)^{1/6}/b_0$ and may be strengthened to $u\geqslant\ell_n^{1/2}/b_0$, since $n\geqslant1/\ell_n^2$, while in the general case it follows from the condition $u\geqslant\ell_n^{1/6}/b_0$, since $\beta_{3,j}^{1/3}\leqslant B_n\ell_n^{1/3}$. Thus, we finally arrive at the estimate

$$I_{17}\leqslant\frac{\ell_n^{4/3}}{2u^2}+\frac{\ell_n^{5/3}}{6u^4}=\ell_n^{7/6}\bigg(\frac{\ell_n^{1/6}}{2u^2}+\frac{\ell_n^{1/2}}{6u^4}\bigg),\quad u\geqslant\ell_n^{1/6}/b_0=2.7246\ldots\ell_n^{1/6},$$

in the general case, and

$$I_{17}\leqslant\frac{\ell_n^{4/3}}{2u^2n^{1/3}}+\frac{\ell_n^{5/3}}{6u^4n^{2/3}}=\frac{\ell_n^{7/6}}{n^{1/6}}\bigg(\frac{1}{2u^2}\Big(\frac{\ell_n}{n}\Big)^{1/6}+\frac{1}{6u^4}\Big(\frac{\ell_n}{n}\Big)^{1/2},\quad u\geqslant\frac{1}{b_0}\Big(\frac{\ell_n}{n}\Big)^{1/6},$$

in the i.i.d. case.

Gathering the estimates for I_{14}, I_{15}, I_{16}, and I_{17}, in the general case for all $u\geqslant\ell_n^{1/6}/b_0$ we obtain

$$I_{14}\leqslant\frac{\lambda\ell_n}{6}+\frac{1}{6}M(p(\lambda),\lambda)\tau_n+\ell_n^{7/6}\bigg(\frac{1}{2u}+\frac{u}{6\sqrt{2\pi}}+\frac{\ell_n^{1/6}}{2}\left(\frac{4}{u^2}+\frac{1}{\sqrt{2\pi}}\right)+\frac{2\ell_n^{1/2}}{3u^4}+\frac{\ell_n^{5/6}}{6u^6}\bigg),$$

where

$$\tau_n=\frac{1}{B_n^3}\sum_{j=1}^{n}\sigma_j^3.$$

For $\ell>0$ denote

$$J_{13}(\ell)=\frac{1}{\sqrt{2\pi}}\inf\bigg\{\frac{1}{2u}+\frac{u}{6\sqrt{2\pi}}+\frac{\ell^{1/6}}{2}\left(\frac{4}{u^2}+\frac{1}{\sqrt{2\pi}}\right)+\frac{2\ell^{1/2}}{3u^4}+\frac{\ell^{5/6}}{6u^6}:u\geqslant\ell^{1/6}/b_0\bigg\}.$$

It is obvious that the function $J_{13}(\ell)$ monotonically and infinitely increases in $\ell > 0$ and

$$\lim_{\ell\to 0} J_{13}(\ell) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{2u}+\frac{u}{6\sqrt{2\pi}}\right)\bigg|_{u=\sqrt{3\sqrt{2\pi}}} = \frac{\sqrt{3}}{3(2\pi)^{3/4}} = 0.1454\ldots\,.$$

So, with the account of what has been said for arbitrary $\lambda \geqslant 1$ we obtain

$$I_{14} \leqslant \frac{\lambda\ell_n}{6}+\frac{1}{6}M(p(\lambda),\lambda)\tau_n+\begin{cases} \ell_n^{7/6}\cdot\sqrt{2\pi}J_{13}(\ell_n), & \text{in the general case,}\\ \ell_n^{7/6}n^{-1/6}\cdot\sqrt{2\pi}J_{13}(\ell_n/n), & \text{in the i.i.d. case,}\end{cases}$$

and, since $n \geqslant 1/\ell_n^2$,

$$I_{13}+\frac{\nu_n\ell_n}{2\sqrt{2\pi}} \leqslant \frac{I_{14}}{\sqrt{2\pi}}+\frac{\ell_n}{2\sqrt{2\pi}} \leqslant c\ell_n+K(c)\tau_n+\begin{cases} \ell_n^{7/6}\cdot J_{13}(\ell_n), & \text{in the general case,}\\ \ell_n^{3/2}\cdot J_{13}(\ell_n^3), & \text{in the i.i.d. case,}\end{cases}$$

where

$$c = \frac{\lambda+3}{6\sqrt{2\pi}} \geqslant \frac{2}{3\sqrt{2\pi}} = 0.2659\ldots, \quad K(c) = \frac{M(p(\lambda),\lambda)}{6\sqrt{2\pi}}\bigg|_{\lambda=6\sqrt{2\pi}c-3}.$$

So, from lemma 4.3 with the account of the estimates for $I_{13}+\nu_n\ell_n/(2\sqrt{2\pi})$ established above we finally obtain

$$\Delta_n \leqslant c\ell_n + K(c)\tau_n + R(\ell_n), \tag{4.15}$$

where $R(\ell) = \widetilde{C}(\ell)\cdot\ell^{7/6}$ in the general case and $R(\ell) = \widehat{C}(\ell)\cdot\ell^{3/2}$ in the i.i.d. case,

$$\widetilde{C}(\ell) = J_{13}(\ell)+\ell^{5/6}J_{12}(\ell,2)+\min_{t_3\leqslant t_0\leqslant t_1\wedge t_4}\Big\{\ell^{1/2}J_{11}(\ell,2,t_0)+\ell^{5/6}\pi^{-2}\max_{T\geqslant\pi/\ell}J(T,t_0)\Big\},$$

$$\widehat{C}(\ell) = J_{13}\big(\ell^3\big)+\ell^{1/2}\Big(\widehat{J}_{12}(\ell,2,\ell^{-2/\delta})+\min_{t_3\leqslant t_0\leqslant t_1\wedge t_4}\Big\{\widehat{J}_{11}\big(\ell^{-2},2,t_0\big)+\pi^{-2}\max_{T\geqslant\pi/\ell}J(T,t_0)\Big\}\Big),$$

$t_3 = t_3(1) = 0.3566\ldots$, $t_1 = t_1(1) = 0.6359\ldots$, $t_4 = t_4(1,\ell) = (1-\ell^{2/3})/(4\pi\varkappa_1)$. Moreover, the functions $\widetilde{C}(\ell)$, $\widehat{C}(\ell)$ monotonically and infinitely increase on the intervals $0 < \ell < \overline{\ell}$ and $0 < \ell < (\overline{\ell})^{1/3}$ correspondingly, where $\overline{\ell} = \overline{\ell}(1) = \big(1-4/9e^{-5/6}\big)^{3/2} = 0.7247\ldots$.

Let us note that in (4.15) the "constants" $\widetilde{C}(\ell)$ and $\widehat{C}(\ell)$ in the remainder $R(\ell_n)$ do not depend on the choice of the coefficient c at the main term ℓ_n. But this "universality" contains a lack: the rate of decrease of the remainder $R(\ell_n)$ is too low than it could be for $c > 2/(3\sqrt{2\pi})$ if the remainder could depend on c. Indeed, in the final estimate for I_{17} we bounded $(1-\lambda)\mathsf{E}|X_j|^3\mathbf{1}(|X_j| > U)$ above by zero (recall that $c = (\lambda + 3)/(6\sqrt{2\pi})$). The price for this operation is extremely high for $\lambda > 1$ (i.e. for $c > 2/(3\sqrt{2\pi})$), since the cubic tail $\mathsf{E}|X_j|^3\mathbf{1}(|X_j| > U)$ due to the truncation determines the rate of decrease of the remainder $R(\ell_n)$ in the final estimate (4.15), instead of being added to the main term and accurately estimated in a sum with $(1-\lambda)\mathsf{E}|X_j|^3\mathbf{1}(|X_j| > U)$. So, if this cubic tail is "transferred" to the main term, the remainder becomes better. Let us accomplish this transfer.

Estimating the integral I_{13} in the same way as above, for all $U \geqslant 0$ we obtain

$$I_{13} + \frac{\nu_n \ell_n}{2\sqrt{2\pi}} \leqslant \frac{U\ell_n}{12\pi B_n} + \frac{1}{4\pi B_n^4}\sum_{j=1}^n \sigma_j^4 + \frac{\ell_n}{2\sqrt{2\pi}} + \frac{I'_{14}}{\sqrt{2\pi}},$$

where

$$I'_{14} = \frac{1}{B_n^3}\sum_{j=1}^n\left(\frac{1}{6}\left|\mathsf{E}X_j^3\mathbf{1}(|X_j| \leqslant U)\right| + \frac{1}{2}\mathsf{E}|X_j|\mathsf{E}X_j^2 + \frac{1}{6}\,\mathsf{E}|X_j|^3\mathbf{1}(|X_j| > U)\right).$$

Truncating the moments $\mathsf{E}|X_j|$ and $\mathsf{E}X_j^2$ in the same way as in the integral I_{14} we obtain (recall that $Y_j = X_j\mathbf{1}(|X_j| \leqslant U)$)

$$\mathsf{E}|X_j|\mathsf{E}X_j^2 \leqslant \mathsf{E}|Y_j|\mathsf{E}Y_j^2 + \left(\beta_{3,j}^{1/3}/U + \beta_{3,j}^{2/3}/U^2\right)\mathsf{E}|X_j|^3\mathbf{1}(|X_j| > U), \quad j = 1, \ldots, n,$$

while the centering with the account of (4.11) leads to the estimates

$$\mathsf{E}|Y_j|\mathsf{E}Y_j^2 \leqslant \mathsf{E}|Y_j - \mathsf{E}Y_j|\mathsf{D}Y_j + \left(\beta_{3,j}^{2/3}/U^2 + \beta_{3,j}^{4/3}/U^4\right)\mathsf{E}|X_j|^3\mathbf{1}(|X_j| > U),$$

$$|\mathsf{E}Y_j^3| \leqslant |\mathsf{E}(Y_j - \mathsf{E}Y_j)^3| + \left(3\beta_{3,j}^{2/3}/U^2 + \beta_{3,j}^2/U^6\right)\mathsf{E}|X_j|^3\mathbf{1}(|X_j| > U), \quad j = 1, \ldots, n.$$

Summarizing the above estimates we obtain

$$I'_{14} \leqslant \frac{1}{B_n^3}\sum_{j=1}^n\left(\frac{1}{6}\left|\mathsf{E}(Y_j - \mathsf{E}Y_j)^3\right| + \frac{1}{2}\mathsf{E}|Y_j - \mathsf{E}Y_j|\mathsf{D}Y_j\right) + I_{19},$$

where

$$I_{19} = \frac{1}{6B_n^3}\sum_{j=1}^n\left(1 + 3b_j + 9b_j^2 + 3b_j^4 + b_j^6\right)\mathsf{E}|X_j|^3\mathbf{1}(|X_j| > U), \quad b_j = \beta_{3,j}^{1/3}/U.$$

Applying the moment inequality from theorem 3.1 to the r.v.'s Y_j we obtain

$$\begin{aligned} I'_{14} &\leqslant \frac{\lambda}{6B_n^3}\sum_{j=1}^n \mathsf{E}|Y_j - \mathsf{E}Y_j|^3 + \frac{M(p(\lambda),\lambda)}{6B_n^3}\sum_{j=1}^n(\mathsf{D}Y_j)^{3/2} + I_{19} \\ &= \frac{\lambda\ell_n}{6} + \frac{M(p(\lambda),\lambda)}{6B_n^3}\sum_{j=1}^n \sigma_j^3 + I_{17} - \frac{\lambda}{\lambda-1}I_{18} + I_{19}, \end{aligned}$$

with I_{17}, I_{18} defined in (4.12), (4.13) correspondingly. By virtue of the estimate (4.14) we have

$$I_{17} - \frac{\lambda}{\lambda-1} I_{18} + I_{19} \leqslant \frac{1}{6B_n^3} \sum_{j=1}^n \mathsf{E}|X_j|^3 \mathbf{1}(|X_j| > U) \times$$
$$\times \Big((1-\lambda)\Big(1 - \frac{3}{2}\, b_j - 3b_j^2 - \frac{5}{2}\, b_j^4\Big) + 3b_j + 12b_j^2 + 4b_j^4 + b_j^6 \Big).$$

As it was noticed above, $1 - \frac{3}{2}b - 3b^2 - \frac{5}{2}b^4 > 0$ for $0 \leqslant b < b_0$, where $b_0 = 0.36701\ldots$ is the unique root of the equation $1 - \frac{3}{2}b - 3b^2 - \frac{5}{2}b^4 = 0$, $b > 0$. Introduce the function

$$g(b) = \frac{3b + 12b^2 + 4b^4 + b^6}{1 - 3b/2 - 3b^2 - 5b^4/2}, \quad 0 \leqslant b < b_0.$$

Evidently, $g(b)$ increases monotonically varying within the limits

$$0 = \lim_{b\to 0} g(b) \leqslant g(b) < \lim_{b \to b_0} g(b) = +\infty, \quad 0 \leqslant b < b_0,$$

and therefore for each $\lambda > 1$ there exists a unique root of the equation $g(1/u) = \lambda - 1$ in the interval $u > 1/b_0 = 2.7246\ldots$. For $c = (\lambda+3)/(6\sqrt{2\pi}) \geqslant 2/(3\sqrt{2\pi})$ let u_c be the unique root of the equation $g(1/u) = 6\sqrt{2\pi}c - 4$, $u > 1/b_0$. It can easily be made sure that u_c decreases monotonically varying within the limits

$$2.7246\ldots = 1/b_0 = \lim_{c\to\infty} u_c \leqslant u_c \leqslant \lim_{c \to 2/(3\sqrt{2\pi})} u_c = +\infty, \quad c > 2/(3\sqrt{2\pi}).$$

If $b_j \equiv \beta_{3,j}^{1/3}/U \leqslant u_c^{-1}$, i.e. $U \geqslant u_c \beta_{3,j}^{1/3}$, for all $j = 1, \ldots, n$, then $I_{17} - \frac{\lambda}{\lambda-1} I_{18} + I_{19} \leqslant 0$, and thus we obtain the estimate

$$I_{13} + \frac{\nu_n \ell_n}{2\sqrt{2\pi}} \leqslant c\ell_n + K(c)\tau_n + R,$$

with c and $K(c)$ defined above, provided that $U \geqslant u_c \beta_{3,j}^{1/3}$ for all $j = 1, \ldots, n$, where

$$R = \frac{1}{4\pi B_n^4} \sum_{j=1}^n \sigma_j^4 + \frac{U\ell_n}{12\pi B_n} \leqslant \begin{cases} \dfrac{\ell_n}{12\pi}\big(3\ell_n^{1/3} + U/B_n\big), & \text{in the general case,} \\ \dfrac{\ell_n}{12\pi}\left(3\Big(\dfrac{\ell_n}{n}\Big)^{1/3} + \dfrac{U}{B_n}\right), & \text{in the i.i.d. case,} \end{cases}$$

Now choose the parameter U so that the orders of both terms in the above estimate for R coincide, i.e. let

$$U = \begin{cases} uB_n \ell_n^{1/3}, & \text{in the general case,} \\ uB_n (\ell_n/n)^{1/3}, & \text{in the i.i.d. case,} \end{cases}$$

$u \geqslant u_c$ being a free parameter. Then the condition $U \geqslant u_c \beta_{3,j}^{1/3}$ is satisfied for all $j = 1, \ldots, n$, since in the general case $\beta_{3,j}^{1/3}/U \leqslant B_n \ell_n^{1/3}/U = u^{-1} \leqslant u_c^{-1}$, as well as in the i.i.d. case $\beta_{3,j}^{1/3}/U = B_n(\ell_n/n)^{1/3}/U = u^{-1} \leqslant u_c^{-1}$ for all $j = 1, \dots, n$. So, we have

$$I_{13} + \frac{\nu_n \ell_n}{2\sqrt{2\pi}} \leqslant c\ell_n + K(c)\tau_n + \begin{cases} \ell_n^{4/3}(3+u)/(12\pi), & \text{in the general case,} \\ \ell_n^{4/3} n^{-1/3}(3+u)/(12\pi), & \text{in the i.i.d. case,} \end{cases}$$

for arbitrary $u \geqslant u_c$. Evidently the value $u = u_c$ minimizes the right-hand side of the obtained estimate. Gathering the estimates from lemma 4.3 and theorem 4.6 we finally obtain

$$\Delta_n \leqslant c\ell_n + K(c)\tau_n + R(\ell_n, c),$$

with

$$R(\ell, c) = \begin{cases} \left(\frac{3+u_c}{12\pi} + \widetilde{C}_1(\ell)\ell^{1/3}\right)\ell^{4/3}, & \text{in the general case,} \\ \frac{(3+u_c)\ell^{4/3}}{12\pi n^{1/3}} + \widehat{C}_1(\ell)\ell^2 \leqslant \left(\frac{3+u_c}{12\pi} + \widehat{C}_1(\ell)\right)\ell^2, & \text{in the i.i.d. case,} \end{cases}$$

$\widetilde{C}_1(\ell)$, $\widehat{C}_1(\ell)$ defined in theorem 4.6.

As it follows from remark 3.2,

$$\inf_{c \geqslant 2/(3\sqrt{2\pi})} (c + K(c)) = \lim_{\lambda \to \infty} \frac{\lambda + 3 + M(p(\lambda), \lambda)}{6\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}},$$

and also $K(c) > 0$ if and only if $\lambda < \sqrt{10}$, that is, $c < (\sqrt{10}+3)/(6\sqrt{2\pi}) = 0.4097\dots$, whence it follows that for all c such that $2/(3\sqrt{2\pi}) \leqslant c \leqslant 1/\sqrt{2\pi} = 0.3989\dots$ the estimates $K(c) > 0$ and

$$c\ell_n + K(c)\tau_n \geqslant c\ell_n \geqslant \frac{2\ell_n}{3\sqrt{2\pi}},$$

hold and for $c > 1/\sqrt{2\pi}$

$$\begin{aligned} c\ell_n + K(c)\tau_n &= \frac{\ell_n}{\sqrt{2\pi}} + \left(c - \frac{1}{\sqrt{2\pi}}\right)\ell_n + K(c)\tau_n \\ &\geqslant \frac{\ell_n}{\sqrt{2\pi}} + \left(c + K(c) - \frac{1}{\sqrt{2\pi}}\right)\tau_n \geqslant \frac{\ell_n}{\sqrt{2\pi}} \geqslant \frac{2\ell_n}{3\sqrt{2\pi}}, \end{aligned}$$

since $\ell_n \geqslant \tau_n$ by the Lyapounov inequality. So,

$$\inf_{c \geqslant 2/(3\sqrt{2\pi})} (c\ell_n + K(c)\tau_n) \geqslant \frac{2\ell_n}{3\sqrt{2\pi}}.$$

For the purpose of lowering the right bound of the interval of the values of ℓ under consideration and thus bound the range of the constants $\widetilde{C}(\ell)$, $\widehat{C}(\ell)$ above, note that if $\ell_n \geqslant \ell$ for some $\ell > 0$, then by virtue of (4.3) for any

$$A \geqslant \kappa - \frac{2\ell}{3\sqrt{2\pi}},$$

where $\kappa = 0.5409\ldots$ (see (4.3)), the trivial estimate

$$\inf_{c\geqslant 2/(3\sqrt{2\pi})}(c\ell_n + K(c)\tau_n) + A \geqslant \frac{2\ell}{3\sqrt{2\pi}} + A \geqslant \kappa \geqslant \Delta_n,$$

holds so that by virtue of the monotonicity of $R(\ell)$, in (4.15) for $\ell_n \geqslant \ell$ the quantity $R(\ell_n)$ can be replaced by

$$\min\left\{R(\ell),\, \kappa - \frac{2\ell}{3\sqrt{2\pi}}\right\} = R(\ell_R \wedge \ell),$$

where ℓ_R is the unique root of the equation

$$R(\ell) = \kappa - \frac{2\ell}{3\sqrt{2\pi}}$$

on the interval $(0,\, \overline{\ell}\,)$ in the general case and on the interval $\big(0, (\overline{\ell})^{1/3}\big)$ in the case of identically distributed summands. The existence of ℓ_R and its uniqueness follow from that on the interval under consideration the left-hand side of the equation is a continuous function which increases strictly monotonically and takes *all* values from 0 to $+\infty$, and the right-hand side is a continuous function which decreases strictly monotonically and takes positive values at small ℓ, that is, the graphs of these functions intersect in a single point. The same reasoning concerns $R(\ell_n, c)$ and is summarized in the following two theorems.

Theorem 4.13. *For any $\ell > 0$, for all $n \geqslant 1$ and $F_1, \ldots, F_n \in \mathcal{F}_3$ such that $\ell_n \leqslant \ell$ there hold the estimates:*

$$\Delta_n \leqslant \inf_{c\geqslant 2/(3\sqrt{2\pi})}\left\{c\ell_n + \frac{K(c)}{B_n^3}\sum_{j=1}^n \sigma_j^3\right\} + \widetilde{C}\big(\widetilde{\ell}\wedge\ell\big)\ell_n^{7/6}, \tag{4.16}$$

in the general case and

$$\Delta_n \leqslant \inf_{c\geqslant 2/(3\sqrt{2\pi})}\left\{c\ell_n + \frac{K(c)}{\sqrt{n}}\right\} + \widehat{C}\big(\widehat{\ell}\wedge\ell\big)\ell_n^{3/2}, \tag{4.17}$$

in the case of identically distributed summands, where

$$K(c) = \frac{M(p(\lambda),\lambda)}{6\sqrt{2\pi}}\bigg|_{\lambda=6\sqrt{2\pi}c-3},$$

$$M(p,\lambda) = \frac{1-\lambda+2(\lambda+2)p-2(\lambda+3)p^2}{\sqrt{p(1-p)}}, \quad 0 < p \leqslant \frac{1}{2},\ \lambda \geqslant 1,$$

$$p(\lambda) = \frac{1}{2} - \sqrt{\frac{\lambda+1}{\lambda+3}}\sin\left(\frac{\pi}{6} - \frac{1}{3}\arctan\sqrt{\lambda^2 + 2\frac{\lambda-1}{\lambda+3}}\right), \quad \lambda \geqslant 1;$$

$$\widetilde{C}(\ell) = J_{13}(\ell) + \ell^{5/6}J_{12}(\ell,2) + \min_{t_3\leqslant t_0\leqslant t_1\wedge t_4(\ell)}\Big\{\ell^{1/2}J_{11}(\ell,2,t_0) + \ell^{5/6}\pi^{-2}\max_{T\geqslant \pi/\ell} J(T,t_0)\Big\};$$

$$\widehat{C}(\ell) =$$

$$J_{13}\big(\ell^3\big)+\ell^{1/2}\Big(\widehat{J}_{12}(\ell,2,\ell^{-2})+\min_{t_3\leqslant t_0\leqslant t_1\wedge t_4(\ell^3)}\Big\{\widehat{J}_{11}\big(\ell^{-2},2,t_0\big)+\pi^{-2}\max_{T\geqslant\pi/\ell} J(T,t_0)\Big\}\Big);$$

$\widetilde{\ell} = 0.226547\ldots$, $\widehat{\ell} = 0.402361\ldots$ *are respectively the unique roots of the equations*

$$\widetilde{C}(\ell)\cdot\ell^{7/6} = \kappa - \frac{2\ell}{3\sqrt{2\pi}}, \quad 0<\ell<\overline{\ell} = \Big(1-4/9e^{-5/6}\Big)^{3/2} = 0.7247\ldots,$$

$$\widehat{C}(\ell)\cdot\ell^{3/2} = \kappa - \frac{2\ell}{3\sqrt{2\pi}}, \quad 0<\ell<(\overline{\ell}\,)^{1/3} = \sqrt{1-4/9e^{-5/6}} = 0.8982\ldots,$$

on the intervals specified above; $\kappa = 0.5409\ldots$ *is defined in* (4.3)*;*

$$J_{13}(\ell) = \frac{1}{\sqrt{2\pi}}\inf\left\{\frac{1}{2u}+\frac{u}{6\sqrt{2\pi}}+\frac{\ell^{1/6}}{2}\left(\frac{4}{u^2}+\frac{1}{\sqrt{2\pi}}\right)+\frac{2\ell^{1/2}}{3u^4}+\frac{\ell^{5/6}}{6u^6} : u\geqslant u_0\ell^{1/6}\right\},$$

$$t_3 = \frac{e^{-5/6}}{9\pi\varkappa_1} = 0.1550\ldots, \quad t_1 = \frac{\theta_0(1)}{2\pi} = 0.6359\ldots, \quad t_4(\ell) = \frac{1-\ell^{2/3}}{4\pi\varkappa_1},$$

$u_0 = 2.7246\ldots$ *is the unique root of the equation* $1-3/2u^{-1}-3u^{-2}-5/2u^{-4} = 0$*,* $u>0$*;* $J_{11}(\ell,\nu,t_0)$*,* $J_{12}(\ell,\nu)$*,* $\widehat{J}_{11}(n,\nu,t_0)$*,* $\widehat{J}_{12}(\ell,\nu,n)$*,* $J(T,t_0)$*,* $\theta_0(1)$*,* $\varkappa_1 = 0.0991\ldots$ *are defined in lemma* 4.3*. In particular,*

$$\widetilde{C}(0.226548)\leqslant 2.7176 \ (\textit{with } t_0 = t_3 = 0.1550\ldots,\ u = 4.3173\ldots),$$

$$\widehat{C}(0.402362)\leqslant 1.7002 \ (\textit{with } t_0 = 0.1802\ldots,\ u = 4.1157\ldots),$$

$$\widetilde{C}(0+) = \widehat{C}(0+) = \frac{\sqrt{3}}{3(2\pi)^{3/4}} = 0.1454\ldots \ (\textit{with } t_0 = t_3,\ u = \sqrt{3\sqrt{2\pi}} = 2.7422\ldots).$$

The values of $\widehat{C}(\ell)$*,* $\widetilde{C}(\ell)$ *for other* ℓ *are given in table* 9*, the functions* $\widetilde{C}(\ell)$*,* $\widehat{C}(\ell)$ *being monotonically increasing.*

ℓ	0.1	0.01	10^{-3}	10^{-4}	10^{-7}	10^{-20}
$\widetilde{u}(\ell) =$	4.1825	3.8521	3.5852	3.3724	2.9823	2.7440
$\widehat{u}(\ell) =$	3.5852	3.0782	2.8609	2.7813	2.7435	2.7422
$\widetilde{C}(\ell)\leqslant$	0.7802	0.2792	0.2110	0.1854	0.1577	0.1456
$\widehat{C}(\ell)\leqslant$	0.3861	0.2169	0.1682	0.1527	0.1458	0.1455

Table 9: The values of $\widetilde{C}(\ell)$, $\widehat{C}(\ell)$ from theorem 4.13 for some ℓ together with the optimal values of u from J_{13} which are denoted by $\widetilde{u}(\ell)$ for $\widetilde{C}(\ell)$ and $\widehat{u}(\ell)$ for $\widehat{C}(\ell)$. The optimal values of t_0 coincide with $t_3 = 0.1550\ldots$ in both cases.

Theorem 4.14. *For any $\ell > 0$, for all $n \geqslant 1$ and $F_1, \ldots, F_n \in \mathcal{F}_3$ such that $\ell_n \leqslant \ell$ there hold the estimates:*

$$\Delta_n \leqslant \inf_{c>2/(3\sqrt{2\pi})} \left\{ c\ell_n + \frac{K(c)}{B_n^3} \sum_{j=1}^n \sigma_j^3 + \widetilde{A}_c\big(\ell \wedge \widetilde{\ell}_c\big)\ell_n^{4/3} \right\}, \tag{4.18}$$

in the general case and

$$\Delta_n \leqslant \inf_{c>2/(3\sqrt{2\pi})} \left\{ c\ell_n + \frac{K(c)}{\sqrt{n}} + \widehat{A}_c\big(\ell \wedge \widehat{\ell}_c\big)\ell_n^2 \right\}, \tag{4.19}$$

in the case of identically distributed summands, where $K(c)$ is defined in theorem 4.13;

$$\widetilde{A}_c(\ell) = \frac{3+u_c}{12\pi} + \widetilde{C}_1(\ell)\cdot \ell^{1/3}, \quad \widehat{A}_c(\ell) \ = \ \frac{3+u_c}{12\pi} + \widehat{C}_1(\ell),$$

$\widetilde{C}_1(\ell)$, $\widehat{C}_1(\ell)$ are defined in theorem 4.6, *u_c is the unique root of the equation*

$$\frac{3u^{-1}+12u^{-2}+4u^{-4}+u^{-6}}{1-3/2u^{-1}-3u^{-2}-5/2u^{-4}} = 6\sqrt{2\pi}c - 4, \quad c > \frac{2}{3\sqrt{2\pi}},$$

in the interval $u > u_\infty$ with $u_\infty = 2.7246\ldots$ being the unique root of the equation $1 - 3/2u^{-1} - 3u^{-2} - 5/2u^{-4} = 0$, $u > 0$; $\widetilde{\ell}_c$, $\widehat{\ell}_c$ are respectively the unique roots of the equations

$$\widetilde{A}_c\big(\ell\big)\cdot \ell^{4/3} = \kappa - \frac{2\ell}{3\sqrt{2\pi}}, \quad 0 < \ell < \overline{\ell} = \Big(1 - 4/9e^{-5/6}\Big)^{3/2} = 0.7247\ldots,$$

$$\widehat{A}_c\big(\ell\big)\cdot \ell^2 = \kappa - \frac{2\ell}{3\sqrt{2\pi}}, \quad 0 < \ell < (\overline{\ell}\,)^{1/3} = \sqrt{1-4/9e^{-5/6}} = 0.8982\ldots,$$

on the intervals specified above; $\kappa = 0.5409\ldots$ is defined in (4.3). *The functions $\widetilde{A}_c\big(\ell\big)$, $\widehat{A}_c\big(\ell\big)$ increase monotonically in $\ell > 0$ and decrease monotonically in c, moreover*

$$\lim_{c\to 2/(3\sqrt{2\pi})} \inf_{\ell>0} \widetilde{A}_c(\ell) = \lim_{c\to 2/(3\sqrt{2\pi})} \inf_{\ell>0} \widehat{A}_c(\ell) = +\infty.$$

$$\lim_{c\to\infty} \widetilde{A}_c(\ell) - \widetilde{C}_1(\ell)\cdot \ell^{1/3} = \lim_{c\to\infty} \widehat{A}_c(\ell) - \widehat{C}_1(\ell) = \frac{3+u_\infty}{12\pi} = 0.1518\ldots, \quad \ell > 0,$$

$$\lim_{\ell\to 0} \widetilde{A}_c(\ell) = \frac{3+u_c}{12\pi},$$

$$\lim_{\ell\to 0} \widehat{A}_c(\ell) - \frac{3+u_c}{12\pi} = \widehat{C}_1(0) = \frac{1.0253\cdot 5\varkappa_1}{\sqrt{2\pi}(1-4/9e^{-5/6})^{7/2}} + \frac{1}{3\pi} = 0.5359\ldots\ .$$

The values of $\widetilde{A}_c\big(\ell\big)$, $\widehat{A}_c\big(\ell\big)$ for some ℓ and c are given in table 10.

c	0.27	0.28	0.29	0.30	$\frac{\sqrt{10}+3}{6\sqrt{2\pi}}$	∞
$K(c)$	0.1521	0.1402	0.1287	0.1174	0.0000	$-\infty$
u_c	54.5687	18.8812	12.6629	10.0115	4.7345	2.7247
$\widetilde{\ell}_c$	0.2048	0.2220	0.2250	0.2263	0.2288	0.2298
$\widetilde{A}_c(\widetilde{\ell}_c)$	4.0313	3.5851	3.5160	3.4872	3.4314	3.4106
$\widetilde{A}_c(0.01)$	1.6632	0.7165	0.5516	0.4813	0.3413	0.2880
$\widetilde{A}_c(10^{-3})$	1.5757	0.6290	0.4641	0.3937	0.2538	0.2005
$\widetilde{A}_c(0+)$	1.5271	0.5805	0.4155	0.3452	0.2052	0.1519
$\widehat{\ell}_c$	0.3596	0.3942	0.4008	0.4036	0.4094	0.4116
$\widehat{A}_c(\widehat{\ell}_c)$	3.4449	2.8068	2.7046	2.6619	2.5786	2.5475
$\widehat{A}_c(0.07)$	2.1042	1.1576	0.9927	0.9223	0.7823	0.7290
$\widehat{A}_c(0.05)$	2.0902	1.1435	0.9786	0.9083	0.7683	0.7150
$\widehat{A}_c(0.03)$	2.0780	1.1314	0.9664	0.8961	0.7561	0.7028
$\widehat{A}_c(0.01)$	2.0674	1.1207	0.9558	0.8855	0.7455	0.6922
$\widehat{A}_c(0+)$	2.0631	1.1164	0.9515	0.8811	0.7412	0.6878

Table 10: Upper bounds of $K(c)$, u_c, $\widetilde{\ell}_c$, $\widetilde{A}_c(\ell)$, $\widehat{\ell}_c$, $\widehat{A}_c(\ell)$ from Theorem 4.14 for some ℓ and c.

Remark 4.15. Taking into account the properties of the functions $M(p(\lambda),\lambda)$ and $\lambda+M(p(\lambda),\lambda)$, $\lambda\geqslant 1$ described in remark 3.2 it can be made sure that the functions $K(c)$ and $c+K(c)$ decrease monotonically for all $c\geqslant 2/(3\sqrt{2\pi})$ varying within the limits

$$-\infty=\lim_{c\to\infty}K(c)<K(c)\leqslant K\left(\frac{2}{3\sqrt{2\pi}}\right)=\sqrt{\frac{2\sqrt{3}-3}{6\pi}}=0.1569\ldots,$$

$$0.3989\ldots=\frac{1}{\sqrt{2\pi}}=\lim_{c\to\infty}(c+K(c))<c+K(c)\leqslant\frac{2}{3\sqrt{2\pi}}+\sqrt{\frac{2\sqrt{3}-3}{6\pi}}=0.4228\ldots,$$

moreover the function $K(c)$ changes its sign at the unique point $c=(\sqrt{10}+3)/(6\sqrt{2\pi})=0.4097\ldots$.

Remark 4.16. As it was shown in [27, 29], the least possible value of the coefficient c at ℓ_n in estimates (4.16), (4.17) cannot be made less than $\underline{C_{\mathrm{AE}}}=2/(3\sqrt{2\pi})$. Furthermore, the estimates obtained in theorem 4.13 for each $c\geqslant\underline{C_{\mathrm{AE}}}$ are optimal in the sense that the value of the coefficient $K(c)$ cannot be made less. Indeed, even in the case of identically distributed summands, for all $c\geqslant\underline{C_{\mathrm{AE}}}$, obviously, $K(c)$ can be estimated as

$$K(c)\geqslant\sup_{X_1\in\mathcal{F}_3}\limsup_{n\to\infty}\frac{\sqrt{n}\Delta_n(\mathsf{E}X_1^2)^{3/2}-c\mathsf{E}|X_1|^3}{(\mathsf{E}X_1^2)^{3/2}}.$$

On the other hand, with the account of (1.4) we obtain

$$K(c) \geqslant \sup_{h>0} \sup_{X\in\mathcal{F}_3^h} \frac{|\mathsf{E}X^3| + 3h\mathsf{E}X^2 - 6\sqrt{2\pi}c\mathsf{E}|X|^3}{6\sqrt{2\pi}\,(\mathsf{E}X^2)^{3/2}}, \quad c \geqslant \underline{C}_{\text{AE}}.$$

Now letting $\mathsf{P}(X = -\sqrt{p/q}) = q$, $\mathsf{P}(X = \sqrt{q/p}) = p = 1-q$, $0 < p \leqslant 1/2$, we arrive at

$$\mathsf{E}X = 0, \quad \mathsf{E}X^2 = 1, \quad \mathsf{E}X^3 = \frac{q-p}{\sqrt{pq}}, \quad \mathsf{E}|X|^3 = \frac{p^2+q^2}{\sqrt{pq}}, \quad h = \frac{1}{\sqrt{pq}},$$

and hence, for all $c \geqslant \underline{C}_{\text{AE}}$

$$\begin{aligned} K(c) &\geqslant \frac{1}{6\sqrt{2\pi}} \sup\left\{ \frac{q-p+3-6\sqrt{2\pi}c(p^2+q^2)}{\sqrt{pq}} : 0 < p \leqslant 1/2,\ q = 1-p \right\} \\ &= \frac{M(p(\lambda),\lambda)}{6\sqrt{2\pi}}\bigg|_{\lambda = 6\sqrt{2\pi}c-3} \end{aligned}$$

by virtue of representation (3.1), which coincides with the definition of $K(c)$ (see theorem 4.13).

From theorems 4.13 and 4.14 with concrete c we can obtain some corollaries. For example, with $c = \underline{C}_{\text{AE}} = 2/(3\sqrt{2\pi})$ we have $K(c) = \sqrt{(2\sqrt{3}-3)/(6\pi)}$, and hence, theorem 4.13 implies

Corollary 4.17. *For all $n \geqslant 1$ and $F_1, \ldots, F_n \in \mathcal{F}_3$ there hold the estimates*

$$\Delta_n \leqslant \frac{2\ell_n}{3\sqrt{2\pi}} + \sqrt{\frac{2\sqrt{3}-3}{6\pi} \sum_{j=1}^n \sigma_j^3/B_n^3} + 2.7176 \cdot \ell_n^{7/6}$$

in the general case and

$$\Delta_n \leqslant \frac{2\ell_n}{3\sqrt{2\pi}} + \sqrt{\frac{2\sqrt{3}-3}{6\pi n}} + 1.7002 \cdot \ell_n^{3/2}$$

in the case $F_1 = \ldots = F_n$, moreover, in each of the estimates the constant $\sqrt{(2\sqrt{3}-3)/(6\pi)} = 0.1569\ldots$ at the second term cannot be made less under the condition that the coefficient at the first term is fixed and equals $2/(3\sqrt{2\pi})$.

Corollary 4.17 sharpens the inequalities of Prawitz (1.9) and Bentkus (1.10) with respect to the second term by virtue of the smaller value of the constant

$$\sqrt{(2\sqrt{3}-3)/(6\pi)} = 0.1569\ldots$$

as compared to $\left(2\sqrt{2\pi}\right)^{-1} = 0.1994\ldots$ in (1.9), (1.10), but the "expense" of using the unimprovable constant at the second term is a worse order of decrease of the remainder, namely, $O(\ell_n^{3/2})$ and $O(\ell_n^{7/6})$ as compared with $O(\ell_n^2)$ in (1.9) and $O(\ell_n^{4/3})$ in (1.10) respectively. However, here we specify concrete values of the constants.

With $c = C_{\text{AE}} = (\sqrt{10}+3)/(6\sqrt{2\pi})$ we have $K(c) = 0$, and hence, theorem 4.14 implies

Corollary 4.18. *For all $n \geqslant 1$ and $F_1, \ldots, F_n \in \mathcal{F}_3$ there hold the estimates*

$$\begin{aligned}
&\Delta_n \leqslant C_{\text{AE}} \cdot \ell_n + 3.4314 \cdot \ell_n^{4/3}, && \textit{for any } \ell_n,\\
&\Delta_n \leqslant C_{\text{AE}} \cdot \ell_n + 0.3413 \cdot \ell_n^{4/3} < 0.4833 \cdot \ell_n, && \ell_n \leqslant 0.01,\\
&\Delta_n \leqslant C_{\text{AE}} \cdot \ell_n + 0.2538 \cdot \ell_n^{4/3} < 0.4352 \cdot \ell_n, && \ell_n \leqslant 10^{-3},\\
&\Delta_n \leqslant C_{\text{AE}} \cdot \ell_n + 0.2053 \cdot \ell_n^{4/3} < 0.4098 \cdot \ell_n, && \ell_n \leqslant 10^{-11},
\end{aligned}$$

in the general case, and

$$\begin{aligned}
&\Delta_n \leqslant C_{\text{AE}} \cdot \ell_n + 2.5786 \cdot \ell_n^2, && \textit{for any } \ell_n,\\
&\Delta_n \leqslant C_{\text{AE}} \cdot \ell_n + 0.7683 \cdot \ell_n^2 < 0.4482 \cdot \ell_n, && \ell_n \leqslant 0.05,\\
&\Delta_n \leqslant C_{\text{AE}} \cdot \ell_n + 0.7455 \cdot \ell_n^2 < 0.4172 \cdot \ell_n, && \ell_n \leqslant 0.01,\\
&\Delta_n \leqslant C_{\text{AE}} \cdot \ell_n + 0.7412 \cdot \ell_n^2 < 0.4098 \cdot \ell_n, && \ell_n \leqslant 10^{-5},
\end{aligned}$$

in the case $F_1 = \ldots = F_n$, where

$$C_{\text{AE}} = \frac{\sqrt{10}+3}{6\sqrt{2\pi}} = 0.4097\ldots\ .$$

This corollary improves Chistyakov's inequality (1.11)

$$\Delta_n \leqslant \frac{\sqrt{10}+3}{6\sqrt{2\pi}} \cdot \ell_n + A_5 \cdot \ell_n^{40/39} |\ln \ell_n|^{7/6},$$

with respect to the remainder: the order is improved, the value of the constant is explicitly specified. Moreover, comparing the leading term of Chistyakov's estimate (1.11)

$$\psi_1(F_1, \ldots, F_n) = \frac{\sqrt{10}+3}{6\sqrt{2\pi} B_n^3} \sum_{j=1}^n \beta_{3,\,j}$$

with those in theorems 4.13 and 4.14

$$\psi_2(F_1, \ldots, F_n) = \inf_{c \geqslant 2/(3\sqrt{2\pi})} \left(\frac{c}{B_n^3} \sum_{j=1}^n \beta_{3,\,j} + \frac{K(c)}{B_n^3} \sum_{j=1}^n \sigma_j^3 \right),$$

we notice that their values coincide if and only if

$$\sum_{j=1}^n \beta_{3,j} \Big/ \sum_{j=1}^n \sigma_j^3 = \sqrt{20(\sqrt{10}-3)/3} = 1.0401\ldots,$$

whereas in all the rest of the cases the strict inequality $\psi_1 > \psi_2$ holds, that is, the estimates in theorems 4.13 and 4.14 are more accurate. The optimal values of c delivering the infimum in ψ_2 can be found in the fifth column of table 2 for some values of the ratio $\ell_n/\tau_n = \sum_{j=1}^n \beta_{3,j}/\sum_{j=1}^n \sigma_j^3$, which is specified in the first column named β_3.

If the value of the Lyapounov fraction $\ell_n = B_n^{-3}\sum_{j=1}^n \beta_{3,j}$ coincides with that of $B_n^{-3}\sum_{j=1}^n \sigma_j^3$ (it is easy to see that this can be if and only if $\beta_{3,j} = \sigma_j^3$ for all $j = 1,\ldots,n$, that is, when the random summands have symmetric Bernoulli distributions $\mathsf{P}(X_j = \sigma_j) = \mathsf{P}(X_j = -\sigma_j) = 1/2$), then, as it follows from remark 3.2, the greatest lower bound in the estimates of theorem 4.14 is delivered as $c \to \infty$. So, one more corollary is valid.

Corollary 4.19. *For any $n \geqslant 1$ and $F_1,\ldots,F_n \in \mathcal{F}_3$ such that $\beta_{3,j} = \sigma_j^3$ for all $j = 1,\ldots,n$, the estimate*

$$\Delta_n \leqslant \frac{\ell_n}{\sqrt{2\pi}} + 3.4106\cdot \ell_n^{4/3}$$

holds. If $F_1 = \ldots = F_n \in \mathcal{F}_3$ and $\mathsf{E}|X_1|^3 = (\mathsf{E}X_1^2)^{3/2}$, then for all $n \geqslant 1$

$$\Delta_n \leqslant \frac{1}{\sqrt{2\pi n}} + \frac{2.5475}{n} = \frac{\ell_n}{\sqrt{2\pi}} + 2.5475\cdot \ell_n^2.$$

Corollary 4.19 completely agrees with the results of V. Bentkus [2, 3], G. P. Chistyakov [6, 7] and Ch. Hipp and L. Mattner [14] obtained for symmetric distributions. For the case of symmetric summands, in papers [2, 3] the estimate

$$\Delta_n \leqslant \frac{\ell_n}{\sqrt{2\pi}} + A_6\cdot \ell_n^{4/3}, \tag{4.20}$$

was announced with the same rate of decrease of the remainder, but unknown constant A_6. In [7], Chistyakov proved an analog of (4.20) with a slightly heavier remainder of the order $O\big(\ell_n^{40/39}|\ln \ell_n|^{7/6}\big)$. Corollary 4.19 improves these results of Bentkus and Chistyakov for symmetric Bernoulli distributions. The unimprovability of the constant $1/\sqrt{2\pi}$ at the Lyapounov fraction in estimates of type (4.20) for symmetric distributions was proved in 1945 by C.-G. Esseen [9] (see also [12]).

Ch. Hipp and L. Mattner in [14] considered the case where the random summands have identical symmetric Bernoulli distribution and established that

$$\Delta_n = \begin{cases} \Phi\left(\dfrac{1}{\sqrt{n}}\right) - \dfrac{1}{2}, & n \text{ odd},\\[2ex] \dfrac{n!}{2^{n+1}\big((n/2)!\big)^2}, & n \text{ even},\end{cases}$$

whence it follows that $\Delta_n < 1/\sqrt{2\pi n}$ for all $n \geqslant 1$. Unlike [14], corollary 4.19 gives computable estimate with the asymptotically exact constant $1/\sqrt{2\pi}$ not only for *identically* distributed summands, but for the case of *arbitrary* symmetric Bernoulli distributions as well.

However, it should be noted that for the symmetric case, actually, by the methods originally adjusted for that case, one can considerably improve all the results obtained above. These improvements will be published elsewhere.

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