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Pasting of two one-dimensional diffusion processes*

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Dedicated to Mátyás Arató on his eightieth birthday

Abstract

By the method of classical potential theory we obtain an integral representation of the two-parameter semigroup of operators that describes an inhomogeneous Feller process on a line that is a result of pasting together two diffusion processes with the nonlocal boundary condition of non-transversal type.

Keywords: Feller semigroup, diffusion process, boundary condition of Feller-Wentzell

MSC: Primary 60J60

1. Introduction

Let $D_i = \{x \in \mathbb{R} : (-1)^i x > 0\}$, $i = 1, 2$, be the two domains on the line $\mathbb{R}$ with the common boundary $S = \{0\}$ and the closures $\overline{D}_i = D_i \cup \{0\}$, and let $T > 0$ be fixed. If Γ is $\overline{D}_i$ or $\mathbb{R}$, then we denote by $C_b(\Gamma)$ a Banach space of all functions $\varphi(x)$, real-valued, bounded and continuous on Γ with the norm

$$\|\varphi\| = \sup_{x\in\Gamma} |\varphi(x)|,$$

and by $C_2(\Gamma)$ the set of all functions φ, bounded and uniformly continuous on Γ together with their first- and second-order derivatives. Let φ_i be the restriction of any function $\varphi \in C_b(\mathbb{R})$ to D_i.

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Assume that an inhomogeneous diffusion process is given in $D_i,\ i = 1, 2$, and it is generated by a second-order differential operator $A_s^{(i)},\ s \in [0, T]$, with the domain of definition $C_2(\overline{D}_i)$:

$$A_s^{(i)}\varphi_i(x) = \frac{1}{2}b_i(s,x)\frac{d^2\varphi_i(x)}{dx^2} + a_i(s,x)\frac{d\varphi_i(x)}{dx}, \quad i = 1, 2, \tag{1.1}$$

where the diffusion coefficient $b_i(s, x)$ and the drift coefficient $a_i(s, x)$ satisfy the conditions:

1) there exist the constants b and B such that $0 < b \le b_i(s, x) \le B$ for all $(s, x) \in [0, T] \times \overline{D}_i$;

2) the function $a_i(s, x)$ is bounded on $[0, T] \times \overline{D}_i$;

3) for all $s, s' \in [0, T],\ x, x' \in \overline{D}_i$ the next inequalities hold:

$$|b_i(s,x) - b_i(s',x')| \le c\left(|s - s'|^{\frac{\alpha}{2}} + |x - x'|^{\alpha}\right),$$

$$|a_i(s,x) - a_i(s',x')| \le c\left(|s - s'|^{\frac{\alpha}{2}} + |x - x'|^{\alpha}\right),$$

where c and α are the positive constants, $0 < \alpha < 1$.

Assume also that at the zero point the boundary operator L_s is defined by the formula

$$L_s\varphi(0) = \gamma(s)\varphi(0) + \int\limits_{D_1 \cup D_2} [\varphi(0) - \varphi(y)]\mu(s, dy), \quad s \in [0, T], \tag{1.2}$$

where the function γ and the measure μ satisfy the following conditions:

a) the function $\gamma(s)$ is nonegative and Hölder continuous, with exponent $\frac{1+\alpha}{2}$, on [0, T];

b) $\mu(s, \cdot)$ is a nonnegative measure on $D_1 \cup D_2$ such that $0 < \mu(s, D_1 \cup D_2) < \infty$, $s \in [0, T]$, and for all the functions f, bounded and measurable in $\mathbb{R}$, the integrals

$$G_f^{(i)}(s) = \int\limits_{D_i} f_i(y)\mu(s, dy), \quad i = 1, 2,$$

are Hölder continuous, with exponent $\frac{1+\alpha}{2}$, on $[0, T]$.

Note that the operator L_s is a particular case of Feller-Wentzell boundary operator ([1, 2]) which describes the behavior of a diffusion particle at the time when it reaches the origin. Its terms $\gamma(s)\varphi(0)$ and $\int\limits_{D_1 \cup D_2} [\varphi(0) - \varphi(y)]\mu(s, dy)$ are supposed to correspond to the absorption phenomenon, and the inward jump phenomenon from the boundary, respectively.

The problem is to clarify the question of existence of the two-parameter semigroup of operators $T_{st},\ 0 \le s < t \le T$, describing the inhomogeneous Feller process in $\mathbb{R}$ such that in the domains D_1 and D_2 it coincides with the given diffusion processes generated by $A_s^{(1)}$ and $A_s^{(2)}$, respectively, and its behavior at the point zero is determined by the boundary condition

$$L_s\varphi(0) = 0. \tag{1.3}$$

This problem is also often called a problem of pasting together two diffusion processes on a line or a problem of the mathematical modeling of the diffusion phenomenon on a line with a membrane placed in a fixed point (see [3, 4]).

The investigation of the problem formulated above is performed by the analytical methods. Such an approach ([3]–[10]) permits to determine the required operator family by means of the solution of the corresponding problem of conjugation for a linear parabolic equation of the second order with variable coefficients, discontinuous at the zero point. This problem is to find a function $u(s,x,t) = T_{st}\varphi(x)$ satisfying the following conditions:

$$\frac{\partial u(s,x,t)}{\partial s} + A_s^{(i)} u(s,x,t) = 0, \quad 0 \le s < t \le T,\ x \in D_i,\ i = 1, 2, \tag{1.4}$$

$$\lim_{s\uparrow t} u(s,x,t) = \varphi(x), \quad x \in D_1 \cup D_2, \tag{1.5}$$

$$u(s,0-,t) = u(s,0+,t), \quad 0 \le s < t \le T, \tag{1.6}$$

$$L_s u(s,0,t) = 0, \quad 0 \le s < t \le T, \tag{1.7}$$

where $\varphi \in C_b(\mathbb{R})$ is the given function. As we see, the condition (1.6) is the consequence of the Feller property of the required semigroup T_{st}, and the equality (1.7) corresponds to the non-transversal nonlocal boundary condition of Feller-Wentzell (1.2), (1.3). Note that in the transversal case (i.e., when the boundary condition of Feller-Wentzell includes the local terms of the orders higher than the order of the nonlocal one) the conjugation problem (1.4)–(1.7) was studied in [10] (cf. also [7, 8]).

A classical solvability of the problem (1.4)–(1.7) is established by the boundary integral equations method with the use of the ordinary parabolic simple-layer potentials that are constructed using the fundamental solutions of the uniformly parabolic operators. Application of this method permits us not only to prove the existence of the solution of the problem (1.4)–(1.7), but also to obtain its integral representation. It is necessary to note that we derived a generalization of the corresponding result obtained earlier in [6], where a similar problem was analyzed for the case of homogeneous diffusion processes. Furthermore, the boundary condition (1.3) considered there, had no term corresponding to the termination of the process at the zero point. The present paper can be also treated as a generalization of the work [9] devoted to construction of the two-parameter Feller semigroup that describes an inhomogeneous diffusion process on a half-line with the non-transversal nonlocal boundary condition of Feller-Wentzell.

We should also mention the works [11, 12, 13], where the problem of constructing of mathematical models of diffusion processes in mediums with membranes was studied by the methods of stochastic analysis.

2. Auxiliary propositions

Consider the Kolmogorov backward equations (1.4) ($i = 1, 2$). Assume that their coefficients $a_i(s, x)$ and $b_i(s, x)$ are defined on $[0, T] \times \mathbb{R}$ and in this domain they satisfy conditions 1)–3). These conditions imply the existence of the fundamental solutions of equations (1.4) in the domain $[0, T] \times \mathbb{R}$, i.e., the existence of the functions $G_i(s, x, t, y)$ defined for $0 \le s < t \le T,\ x, y \in \mathbb{R}$ such that:

- they are jointly continuous;
- for fixed $t \in (0, T],\ y \in \mathbb{R}$ they satisfy equations (1.4);
- for any function $\varphi \in C_b(\mathbb{R})$ and any $t \in (0, T],\ x \in \mathbb{R}$

$$\lim_{s \uparrow t} \int_{\mathbb{R}} G_i(s, x, t, y)\varphi(y)dy = \varphi(x).$$

Recall that (see [3, Ch. II, §2], [5, Addendum, §6], [14, Ch. IV, §§11–13]) the functions $G_i(s, x, t, y)$ are nonnegative, continuously differentiable with respect to s, twice continuously differentiable with respect to x and for $0 \le s < t \le T,\ x, y \in \mathbb{R}$ the following estimations hold:

$$|D_s^r D_x^p G_i(s, x, t, y)| \le c(t-s)^{-\frac{1+2r+p}{2}} \exp\left\{-h\frac{(y-x)^2}{t-s}\right\}, \tag{2.1}$$

where r and p are the nonnegative integers such that $2r + p \le 2$; D_s^r is the partial derivative with respect to s of order r; D_x^p is the partial derivative with respect to x of order p; c, h are positive constants[1]. Furthermore, $G_i(s, x, t, y)$ are represented as

$$G_i(s, x, t, y) = Z_{i0}(s, y - x, t, y) + Z_{i1}(s, x, t, y),$$

where

$$Z_{i0}(s, x, t, y) = [2\pi b_i(t, y)(t-s)]^{-\frac{1}{2}} \exp\left\{-\frac{(y-x)^2}{2b_i(t, y)(t-s)}\right\},$$

and the functions $Z_{i1}(s, x, t, y)$ satisfy the inequalities

$$|D_s^r D_x^p Z_{i1}(s, x, t, y)| \le c(t-s)^{-\frac{1+2r+p-\alpha}{2}} \exp\left\{-h\frac{(y-x)^2}{t-s}\right\}, \tag{2.2}$$

[1] We will subsequently denote various positive constants by the same symbol c (or h).

where $0 \le s < t \le T,\ x, y \in \mathbb{R},\ 2r + p \le 2,\ \alpha$ is the constant in 3).

Given the fundamental solutions $G_i,\ i = 1, 2$, we define the parabolic potentials that will be used to solve the problem (1.4)-(1.7), namely the Poisson potentials

$$u_{i0}(s,x,t) = \int\limits_{\mathbb{R}} G_i(s,x,t,y)\varphi(y)dy, \quad 0 \le s < t \le T,\ x \in \mathbb{R},$$

and the simple-layer potentials

$$u_{i1}(s,x,t) = \int\limits_{s}^{t} G_i(s,x,\tau,0)V_i(\tau,t,\varphi)d\tau, \quad 0 \le s < t \le T,\ x \in \mathbb{R}, \tag{2.3}$$

where φ is the function in (1.5), and $V_i(s,t,\varphi),\ i = 1, 2$, are arbitrary functions, continuous in $0 \le s < t \le T$ for which the integrals on the right side of (2.3) exist. Note that (see [3, Ch. II, §3], [14, Ch. IV]) the functions $u_{i0},\ u_{i1},\ i = 1, 2$, are continuous in $0 \le s < t \le T,\ x \in \mathbb{R}$ and satisfy the equations (1.4) in the domains $(s,x) \in [0,t) \times \mathbb{R},\ (s,x) \in [0,t) \times (D_1 \cup D_2)$, respectively, and the initial conditions

$$\lim_{s \uparrow t} u_{i0}(s,x,t) = \varphi(x), \quad x \in \mathbb{R},$$
$$\lim_{s \uparrow t} u_{i1}(s,x,t) = 0, \quad x \in D_1 \cup D_2.$$

Furthermore, for the potentials $u_{i0},\ i = 1, 2$, the following estimations are valid:

$$|D_s^r D_x^p u_{i0}(s,x,t)| \le c\|\varphi\|(t-s)^{-\frac{2r+p}{2}}, \tag{2.4}$$

where $0 \le s < t \le T,\ x, y \in \mathbb{R},\ 2r + p \le 2$.

We will also use the next lemma.

Lemma 2.1. *Let $Q_f(s),\ s \in [0,T]$ be a family of linear functionals defined on $C_b(\mathbb{R})$ such that for all $f \in C_b(\mathbb{R})$ the functions $Q_f(s)$ are Hölder continuous with the same exponent $\beta \in (0,1)$ on a closed interval $[0,T]$. Then for every $M > 0$ there exist a common constant $c > 0$ such that for all the functions $f \in C_b(\mathbb{R})$, bounded by M and for all $s,\ s' \in [0,T]$ the inequality*

$$|Q_f(s) - Q_f(s')| \le c|s - s'|^{\beta}$$

holds.

Proof. $f \mapsto |s - s'|^{-\beta}(Q_f(s) - Q_f(s'))$, for $s \ne s' \in [0,T]$ is a pointwise bounded family of linear functionals, hence it is uniformly bounded, which is the statement. □

3. Parabolic conjugation problem

In this section by the boundary integral equations method we establish the classical solvability of the problem (1.4)–(1.7).

Theorem 3.1. *Assume that the coefficients of the operators* $A_s^{(i)}$, $i=1,2$, *the function* γ *and the measure* μ *satisfy conditions 1)-3) and a), b). Then for any function* $\varphi \in C_b(\mathbb{R})$ *the problem (1.4)–(1.7) has a unique solution*

$$u(s,x,t) \in C^{1,2}([0,t) \times D_1 \cup D_2) \cap C([0,t) \times \mathbb{R}).$$

Furthermore,

$$|u(s,x,t)| \leq c\|\varphi\|, \quad 0 \leq s < t \leq T, \tag{3.1}$$

and this solution is represented as

$$u(s,x,t) = u_{i0}(s,x,t) + u_{i1}(s,x,t), \quad x \in \overline{D}_i,\ i=1,2, \quad 0 \leq s < t \leq T, \tag{3.2}$$

where a pair of functions (V_1, V_2) *in* (u_{11}, u_{21}) *is a solution of some system of Volterra integral equations of the second kind.*

Proof. We find a solution of the problem (1.4)-(1.7) of the form (3.2) with the unknown functions V_i to be determined. Without loss of generality we may assume that

$$\mu(s, D_1 \cup D_2) \equiv 1.$$

Therefore, the condition (1.7) reduces to

$$(\gamma(s)+1)u(s,0,t) - \int\limits_{D_1 \cup D_2} u(s,y,t)\mu(s,dy) = 0, \quad 0 \leq s < t \leq T. \tag{3.3}$$

If we substitute (3.2) into (3.3) then, upon using the relation (1.6), we get the following system of Volterra integral equations of the first kind for V_i:

$$\begin{aligned}\Phi_i(s,t,\varphi) = (\gamma(s)+1)\int\limits_s^t G_i(s,0,\tau,0)V_i(\tau,t,\varphi)d\tau - \\ - \sum_{j=1}^{2}\int\limits_s^t \left(\int\limits_{D_j} G_j(s,y,\tau,0)\mu(s,dy)\right) V_j(\tau,t,\varphi)d\tau, \quad i=1,2,\end{aligned} \tag{3.4}$$

where

$$\Phi_i(s,t,\varphi) = \sum_{j=1}^{2}\int\limits_{D_j} u_{j0}(s,y,t)\mu(s,dy) - (\gamma(s)+1)u_{i0}(s,0,t), \quad i=1,2.$$

Now we have to reduce (3.4) to an equivalent system of Volterra integral equations of the second kind. For this purpose we consider the Holmgren's operator

$$\mathcal{E}(s,t)F = \sqrt{\frac{2}{\pi}}\frac{d}{ds}\int_s^t (\rho - s)^{-\frac{1}{2}} F(s,t,\varphi)d\rho, \quad 0 \le s < t \le T$$

and apply it to the both sides of each equation in (3.4). After some straightforward simplifications, we get

$$\mathcal{E}(s,t)\Phi_i = -\frac{V_i(s,t,\varphi)}{\sqrt{b_i(s,0)}} + \sqrt{\frac{2}{\pi}}\frac{d}{ds}\int_s^t \left(I_i^{(1)}(s,\tau) + \sqrt{\frac{\pi}{2b_i(\tau,0)}} \cdot \gamma(s)\right) V_i(\tau,t,\varphi)d\tau -$$
$$- \sqrt{\frac{2}{\pi}}\frac{d}{ds}\sum_{j=1}^{2}\int_s^t I_j^{(2)}(s,\tau)V_j(\tau,t,\varphi)d\tau, \quad i = 1,2, \tag{3.5}$$

where

$$I_i^{(1)}(s,\tau) = \frac{1}{\sqrt{2\pi b_i(\tau,0)}}\int_s^\tau (\rho - s)^{-\frac{1}{2}}(\tau-\rho)^{-\frac{1}{2}}(\gamma(\rho)-\gamma(s))d\rho +$$
$$+ \int_s^\tau (\rho - s)^{-\frac{1}{2}}(\gamma(\rho)+1)Z_{i1}(\rho,0,\tau,0)\Big] d\rho, \quad i = 1,2,$$

$$I_i^{(2)}(s,\tau) = \int_s^\tau (\rho - s)^{-\frac{1}{2}} d\rho \int_{D_i} G_i(s,y,\tau,0)\mu(\rho,dy), \quad i = 1,2.$$

In view of the properties a), b) of the function γ and the measure μ, respectively, as well as the inequalities (2.1), (2.2), it is easy to verify that

$$\lim_{s\uparrow\tau} I_i^{(1)}(s,\tau) = 0, \quad \lim_{s\uparrow\tau} I_i^{(2)}(s,\tau) = 0, \quad i = 1,2.$$

Hence (3.5) can be reduced to the following system of Volterra integral equations of the second kind:

$$V_i(s,t,\varphi) = \sum_{j=1}^{2}\int_s^t K_{ij}(s,\tau)V_j(\tau,t,\varphi)d\tau + \Psi_i(s,t,\varphi), \quad i = 1,2, \tag{3.6}$$

where

$$K_{ii}(s,\tau) = \frac{r_i(s)}{2\sqrt{2\pi b_i(\tau,0)}}\int_s^\tau (\rho - s)^{-\frac{3}{2}}(\tau-\rho)^{-\frac{1}{2}}(\gamma(\rho)-\gamma(s))d\rho +$$

$$+ r_i(s)\frac{d}{ds}\int_s^\tau (\rho - s)^{-\frac{1}{2}}\Big[(\gamma(\rho)+1)Z_{i1}(\rho,0,\tau,0) - \int_{D_i} G_i(\rho,y,\tau,0)\mu(\rho,dy)\Big]d\rho, i = 1,2,$$

$$K_{ij}(s,\tau) = -r_i(s)\frac{d}{ds}\int_s^\tau (\rho - s)^{-\frac{1}{2}}d\rho\int_{D_j} G_j(\rho,y,\tau,0)\mu(\rho,dy), \quad i,j = 1,2,\ i \neq j,$$

$$\Psi_i(s,t,\varphi) = -r_i(s)\sqrt{\frac{\pi}{2}}\mathcal{E}(s,t)\Phi_i, \quad r_i(s) = \frac{1}{\gamma(s)+1}\sqrt{\frac{2b_i(s,0)}{\pi}}, \quad i = 1,2.$$

Let us show that there exist a solution of the system of equations (3.6) which can be obtained by the method of successive approximations

$$V_i(s,t,\varphi) = \sum_{k=0}^{\infty} V_i^{(k)}(s,t,\varphi), \quad 0 \le s < t \le T, \quad i = 1,2, \tag{3.7}$$

where

$$V_i^{(0)}(s,t,\varphi) = \Psi_i(s,t,\varphi),$$
$$V_i^{(k)}(s,t,\varphi) = \sum_{j=1}^{2}\int_s^t K_{ij}(s,\tau)V_j^{(k-1)}(\tau,t,\varphi)d\tau, \quad k = 1,2,\ldots.$$

For this purpose, we have first to estimate the functions Ψ_i and the kernels K_{ij} in (3.6).

Consider the functions $\Psi_i(s,t,\varphi)$. Calculating the derivatives on the right side of their expressions, we obtain ($i = 1,2$):

$$\Psi_i(s,t,\varphi) = r_i(s)\Phi_i(s,t,\varphi)(t-s)^{-\frac{1}{2}} - \frac{r_i(s)}{2}\int_s^t (\rho - s)^{-\frac{3}{2}}\left(\Phi_i(\rho,t,\varphi) - \Phi_i(s,t,\varphi)\right)d\rho. \tag{3.8}$$

Denote by Ψ_{i1} and Ψ_{i2} the first and second terms in (3.8), respectively. Using the estimation

$$|\Phi_i(s,t,\varphi)| \le c\|\varphi\|, \tag{3.9}$$

that follows easily from the inequalities (2.4) (when $r = p = 0$), we find that

$$|\Psi_{i1}(s,t,\varphi)| \le c\|\varphi\|(t-s)^{-\frac{1}{2}}. \tag{3.10}$$

In order to estimate $\Psi_{i1}(s,t,\varphi)$ we consider first the increments $\Phi_i(\rho,t,\varphi) - \Phi_i(s,t,\varphi)$ and write them in the form

$$\Phi_i(\rho,t,\varphi) - \Phi_i(s,t,\varphi) = N_{i1}(s,\rho,t,\varphi) + N_2(s,\rho,t,\varphi),$$

where

$$N_{i1} = \sum_{j=1}^{2} \int\limits_{D_j} [u_{j0}(\rho, y, t) - u_{j0}(s, y, t)]\mu(\rho, dy) - (\gamma(s) + 1)[u_{i0}(\rho, 0, t) - u_{i0}(s, 0, t)],$$
$$\tag{3.11}$$
$$N_2 = \sum_{j=1}^{2} \int\limits_{D_j} u_{j0}(s, y, t)(\mu(\rho, dy) - \mu(s, dy)).$$

Expressing by the Lagrange formula the increments $u_{j0}(\rho, y, t) - u_{j0}(s, y, t)$, $j = 1, 2$, and $u_{i0}(\rho, 0, t) - u_{i0}(s, 0, t)$ in (3.11) in terms of the values of their derivatives at the intermediate points and then using the inequalities (2.4), after some straightforward simplifications, we deduce that

$$|N_{i1}(s, \rho, t, \varphi)| \leq c\|\varphi\|(t - \rho)^{-1}(\rho - s), \quad 0 \leq s < \rho < t \leq T. \tag{3.12}$$

Let us now estimate N_2. Note that $u_{j0}(s, y, t)$, $j = 1, 2$, as functions of y, belong to a class $C_b(\mathbb{R})$ and are bounded by $M = \|\varphi\|$. Hence, by Lemma 1,

$$\left| \int\limits_{D_j} u_{j0}(s, y, t)(\mu(\rho, dy) - \mu(s, dy)) \right| \leq c\|\varphi\|(\rho - s)^{\frac{1+\alpha}{2}},$$

and hence,

$$|N_2(s, \rho, t, \varphi)| \leq c\|\varphi\|(\rho - s)^{\frac{1+\alpha}{2}}, \quad 0 \leq s < \rho < t \leq T. \tag{3.13}$$

Combining (3.12) and (3.13), we obtain

$$|\Phi_i(\rho, t, \varphi) - \Phi_i(s, t, \varphi)| \leq c\|\varphi\| \left[(t - \rho)^{-1}(\rho - s) + (\rho - s)^{\frac{1+\alpha}{2}}\right]. \tag{3.14}$$

Further, using the inequalities (3.9) and (3.14), we get

$$|\Psi_{i2}(s, t, \varphi)| \leq c\|\varphi\| \int\limits_{s}^{\frac{s+t}{2}} \left[\left(t - \frac{s+t}{2}\right)^{-1} (\rho - s)^{-\frac{1}{2}} + (\rho - s)^{-1+\frac{\alpha}{2}} \right] d\rho$$
$$+ c\|\varphi\| \int\limits_{\frac{s+t}{2}}^{t} (\rho - s)^{-\frac{3}{2}} d\rho \leq c\|\varphi\|(t - s)^{-\frac{1}{2}}. \tag{3.15}$$

Combining (3.10) and (3.15), we conclude that

$$|\Psi_i(s, t, \varphi)| \leq c_0\|\varphi\|(t - s)^{-\frac{1}{2}}, \quad 0 \leq s < t \leq T. \tag{3.16}$$

Proceeding by the same considerations[2] as ones leading to the estimation (3.16) we can also investigate the kernels $K_{ij}(s,\tau)$ in (3.6). We find the following result: the kernels $K_{ij}(s,\tau)$, $i,j=1,2$, can be represented as

$$K_{ij}(s,\tau)=\widetilde{K}_{ij}(s,\tau)+\overline{K}_{ij}(s,\tau),\quad 0\le s<\tau<t\le T, \tag{3.17}$$

where

$$\widetilde{K}_{ij}(s,\tau)=-r_i(s)\sqrt{\frac{\pi b_j(\tau,0)}{2}}\int\limits_{D_{j,\delta}}\frac{\partial Z_{j0}}{\partial y}(s,y,\tau,0)\mu(s,dy),$$

and $K^{(2)}_{ij}(s,\tau)$ satisfy the inequality

$$\left|\overline{K}_{ij}(s,\tau)\right|\le h(\delta)(\tau-s)^{-1+\frac{\alpha}{2}}. \tag{3.18}$$

Here δ, $h(\delta)$ are any positive number and some constant depending on δ, respectively; $D_{j,\delta}=\{y\in D_j:\ |y|<\delta\}$. It is seen that K_{ij} have non-integrable singularity, which is caused by $\widetilde{K}_{ij}$, and therefore we do not know yet whether a solution of (3.6) exists, i.e., whether the series (3.7) converges. For this reason, using (3.16) and (3.17), we try to estimate each term $V_i^{(k)}$ of series (3.7) and then to prove the convergence of (3.7).

Consider first the functions $V_i^{(1)}$. We can write

$$\begin{aligned}V_i^{(1)}(s,t,\varphi)&=\sum_{j=1}^{2}\int\limits_s^t K_{ij}(s,\tau)V_i^{(0)}(\tau,t,\varphi)d\tau=\sum_{j=1}^{2}\int\limits_s^t \widetilde{K}_{ij}(s,\tau)\Psi_i(\tau,t,\varphi)d\tau\\&+\sum_{j=1}^{2}\int\limits_s^t \overline{K}_{ij}(s,\tau)\Psi_i(\tau,t,\varphi)d\tau=V_{i1}^{(1)}+V_{i2}^{(1)}.\end{aligned} \tag{3.19}$$

Using (3.16) and (3.18), we get

$$\left|V_{i2}^{(1)}(s,t,\varphi)\right|\le 2c_0h(\delta)\|\varphi\|\frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+\alpha}{2}\right)}(t-s)^{-\frac{1-\alpha}{2}}, \tag{3.20}$$

where c_0 and $h(\delta)$ are the constants in (3.16) and (3.18), respectively.

For the functions $V_{i1}^{(1)}$ we have

$$\begin{aligned}&\left|V_{i1}^{(1)}(s,t,\varphi)\right|\le\\&\le c_0\|\varphi\|r_i(s)\sqrt{\frac{\pi}{2}}\sum_{j=1}^{2}\int\limits_s^t(t-\tau)^{-\frac12}\sqrt{b_j(\tau,0)}d\tau\int\limits_{D_{j,\delta}}\left|\frac{\partial Z_{j0}}{\partial y}(s,y,\tau,0)\right|\mu(s,dy)\le\end{aligned}$$

[2]For further details cf. [9]

$$\leq c_0\|\varphi\|\frac{r_i(s)}{2b}\sum_{j=1}^{2}\int_{s}^{t}(t-\tau)^{-\frac{1}{2}}(\tau-s)^{-\frac{3}{2}}d\tau\int_{D_{j,\delta}}|y|e^{-\frac{y^2}{2B(\tau-s)}}\mu(s,dy)=$$

$$=c_0\|\varphi\|\frac{r_i(s)}{2b}\sum_{j=1}^{2}\int_{D_{j,\delta}}|y|e^{-\frac{y^2}{2B(t-s)}}\mu(s,dy)\int_{s}^{t}(t-\tau)^{-\frac{1}{2}}(\tau-s)^{-\frac{3}{2}}e^{-\frac{y^2}{2B(t-s)}\cdot\frac{t-\tau}{\tau-s}}d\tau. \tag{3.21}$$

The change of variables $z=\frac{t-\tau}{\tau-s}$ in the inner integral in the last relation in (3.21) leads to

$$\left|V_{i1}^{(1)}(s,t,\varphi)\right|\leq$$

$$\leq c_0\|\varphi\|\frac{r_i(s)}{2b}(t-s)^{-1}\sum_{j=1}^{2}\int_{D_{j,\delta}}|y|e^{-\frac{y^2}{2B(t-s)}}\mu(s,dy)\int_{0}^{\infty}z^{-\frac{1}{2}}e^{-\frac{y^2}{2B(t-s)}z}dz\leq$$

$$\leq c_0\|\varphi\|\frac{B}{b}(t-s)^{-\frac{1}{2}}\sum_{j=1}^{2}\int_{D_{j,\delta}}e^{-\frac{y^2}{2B(t-s)}}\mu(s,dy)\leq$$

$$\leq c_0\|\varphi\|\frac{B}{b}(t-s)^{-\frac{1}{2}}\max_{s\in[0,T]}\mu\left(s,D_{1,\delta}\cup D_{2,\delta}\right). \tag{3.22}$$

Combining (3.20) and (3.22), we arrive at the inequality

$$\left|V_{i}^{(1)}(s,t,\varphi)\right|\leq$$

$$\leq c_0\|\varphi\|(t-s)^{-\frac{1}{2}}\left(\frac{2h(\delta)T^{\frac{\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+\alpha}{2}\right)}+\frac{B}{b}\max_{s\in[0,T]}\mu\left(s,D_{1,\delta}\cup D_{2,\delta}\right)\right).$$

Next, by mathematical induction method, we prove that the terms $V_i^{(k)}$ of series (3.7) satisfy the inequalities

$$\left|V_{i}^{(k)}(s,t,\varphi)\right|\leq c\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{n=0}^{k}C_k^n\cdot a^{(k-n)}m(\delta)^n,\quad k=0,1,2, \tag{3.23}$$

where

$$a^{(n)}=\frac{\left(2h(\delta)T^{\frac{\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right)\right)^n\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+n\alpha}{2}\right)},\quad n=0,1,2,\ldots,k,$$

$$m(\delta)=\frac{B}{b}\max_{s\in[0,T]}\mu\left(s,D_{1,\delta}\cup D_{2,\delta}\right).$$

Let us fix $\delta=\delta_0$ such that, $m(\delta_0)<1$. Then in view of (3.23), we have

$$\sum_{k=0}^{\infty}\left|V_{i}^{(k)}(s,t,\varphi)\right|\leq c_0\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{\infty}\sum_{n=0}^{k}C_k^n a^{(k-n)}m(\delta_0)^n=$$

$$= c_0\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{\infty}a^{(k)}\sum_{n=0}^{\infty}C_{k+n}^{n}m(\delta_0)^n =$$
$$= c_0\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{a^{(k)}}{(1-m(\delta_0))^{k+1}} =$$
$$= c_0\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{\left(\frac{h(\delta_0)}{1-m(\delta_0)}T^{\frac{\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right)\right)^k}{\Gamma\left(\frac{1+k\alpha}{2}\right)}\cdot\frac{\Gamma(\frac{1}{2})}{1-m(\delta_0)}\ . \tag{3.24}$$

The estimation (3.24) ensures the absolute and uniform convergence of series (3.7). This means that the functions $V_i(s,t,\varphi)$, $i=1,2$, exist. Furthermore, they are continuous in $0\le s<t\le T$ and satisfy the inequality

$$|V_i(s,t,\varphi)|\le c\|\varphi\|(t-s)^{-\frac{1}{2}}, \quad 0\le s<t\le T. \tag{3.25}$$

Using estimations (2.1), (2.4) and (3.25) we derive the existence of a solution $u(s,x,t)$, $0\le s<t\le T$ of conjugation problem (1.4)-(1.7) which is of the form (3.2), satisfies inequality (3.1) and belongs to $C^{1,2}([0,t)\times D_1\cup D_2)\cap C([0,t)\times\mathbb{R})$.

Thus, in order to complete the proof of the theorem it remains to prove the uniqueness of the solution of the conjugation problem (1.4)-(1.7). For this purpose, it suffices to note that the constructed function $u(s,x,t)$ in each of two domains $0\le s<t\le T$, $x\in\overline{D}_1$ and $0\le s<t\le T$, $x\in\overline{D}_2$ can be treated as a unique solution to the following first boundary-value parabolic problem:

$$\frac{\partial\omega(s,x,t)}{\partial s}+A_s^{(i)}\omega(s,x,t)=0, \quad 0\le s<t\le T,\ x\in D_i,\ i=1,2,$$
$$\lim_{s\uparrow t}\omega(s,x,t)=\varphi(x), \quad x\in D_i,\ i=1,2,$$
$$\omega(s,0,t)=\frac{1}{\gamma(s)+1}\int\limits_{D_1\cup D_2}u(s,y,t)\mu(s,dy), \quad 0\le s<t\le T.$$

The proof of Theorem 1 is now complete. □

Remark 3.2. Let, in addition to the conditions of Theorem 1, the fitting condition

$$L_t\varphi(0)=0,$$

holds, then the solution u of the problem (1.4)-(1.7) constructed in Theorem 1 belongs to

$$C^{1,2}([0,t)\times D_1\cup D_2)\cap C([0,t]\times\mathbb{R}).$$

4. Process with absorptions and jumps

Suppose that the conditions of Theorem 1 hold and consider the two-parameter family of linear operators T_{st}, $0\le s<t\le T$, acting on the function $\varphi\in C_b(\mathbb{R})$ by

the formula

$$T_{st}\varphi(x) = \int\limits_{\mathbb{R}} G_i(s,x,t,y)\varphi(y)dy + \int\limits_{s}^{t} G_i(s,x,\tau,0)V_i(\tau,t,\varphi)d\tau, \tag{4.1}$$

where the pair of functions $(V_1,\ V_2)$ is the solution of (3.6).

We introduce the subspace $C_L(\mathbb{R})$ of $C_b(\mathbb{R})$ which consists of all $\varphi \in C_b(\mathbb{R})$ with $L_t\varphi(0) = 0$. It is easily seen that the space $C_L(\mathbb{R})$ is closed in $C_b(\mathbb{R})$, and so it is a Banach space. Furthermore, it is invariant under the operators T_{st}, i.e.,

$$\varphi \in C_L(\mathbb{R}) \implies T_{st}\varphi \in C_L(\mathbb{R}).$$

Let us study properties of the operator family T_{st} in the space $C_L(\mathbb{R})$. First we note that

$$\lim_{n\to\infty} T_{st}\varphi_n(x) = T_{st}\varphi(x), \quad 0 \le s < t \le T, \quad x \in \mathbb{R},$$

for every sequence of functions $\varphi_n \in C_L(\mathbb{R})$ such that

$$\sup_n \|\varphi_n\| < \infty \quad \text{and} \quad \lim_{n\to\infty} \varphi_n(x) = \varphi(x), \quad x \in \mathbb{R}.$$

This property easily follows from Lebesgue bounded convergence theorem and it allows us to make all the following considerations, without loss of generality, under the condition that the function φ has compact support.

Now we prove that the cone of nonnegative functions remains invariant under the operators $T_{st},\ 0 \le s < t \le T$.

Lemma 4.1. *If $\varphi \in C_L(\mathbb{R})$ and $\varphi(x) \ge 0$ for all $x \in \mathbb{R}$, then $T_{st}\varphi(x) \ge 0$ for all $0 \le s < t \le T,\ x \in \mathbb{R}$.*

Proof. Let φ be any nonnegative function in $C_L(\mathbb{R})$ with compact support. If $\varphi \equiv 0$, then the assertion of the lemma is obvious. Consider now the case where the function φ not everywhere equals zero. Denote by m a minimum of the function $T_{st}\varphi(x)$ in the domain $(s,x) \in [0,t] \times \mathbb{R}$ and assume that $m < 0$. By the minimum principle ([15, Ch. II]), the value m can be attained only when $(s,x) \in [0,t] \times \{0\}$. Fix $s_0 \in [0,t]$ such that $T_{s_0t}\varphi(0) = m$. Then the following inequalities hold:

$$\gamma(s_0)T_{s_0t}\varphi(0) \le 0, \qquad \int\limits_{D_1\cup D_2} [T_{s_0t}\varphi(0) - T_{s_0t}\varphi(y)]\mu(s,dy) < 0.$$

Consequently,

$$L_{s_0}T_{s_0t}\varphi(0) < 0.$$

Since, however, the condition (1.7) holds, we get a contradiction. This completes the proof of the lemma. □

By similar considerations to those in proof of Lemma 2, it can be easily verified that the operators T_{st} are contractive, i.e.,

$$\|T_{st}\| \leq 1, \quad 0 \leq s < t \leq T.$$

Finally, we show that the operator family T_{st} has a semigroup property

$$T_{st} = T_{s\tau} T_{\tau t}, \quad 0 \leq s < \tau < t \leq T.$$

This property is a consequence of the assertion of uniqueness of the solution of the problem (1.4)–(1.7) which we have already established above. Indeed, to find $u(s, x, t)$ when $\lim_{s \uparrow t} u(s, x, t) = \varphi(x)$, the problem (1.4)–(1.7) can be solved first in the time interval $[\tau, t]$, and then with the "initial" function $u(\tau, x, t) = T_{\tau t}\varphi(x)$, we derived, it can be solved in the time interval $[s, \tau]$. In other words, $T_{st}\varphi(x) = T_{s\tau}(T_{\tau t}\varphi)(x)$, $\varphi \in C_b(\mathbb{R})$, i.e., $T_{st} = T_{s\tau}T_{\tau t}$.

The properties of the operator family T_{st}, proved above, implies (see [5, Ch. II, §1]) the next theorem.

Theorem 4.2. *Let the conditions of Theorem 1 hold. Then the two-parameter semigroup of operators T_{st}, $0 \leq s < t \leq T$, defined by formula (4.1) describes the inhomogeneous Feller process in $\mathbb{R}$, such that in D_1 and D_2 it coincides with the diffusion processes generated by $A_s^{(1)}$ and $A_s^{(2)}$, respectively, and its behavior on $S = \{0\}$ is determined by the boundary condition (1.3).*

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